International Journal of Pure and Applied Mathematics
Volume 107 No. 1 2016, 59-68
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i1.6
ijpam.eu

ON FUZZY AUTOCATALYTIC SET

Azmirul Ashaari[1], Tahir Ahmad[2 §]

[1]Department of Mathematical Science
Faculty of Science
Universiti Teknologi Malaysia
81310 Skudai, Johor, MALAYSIA

[2]Centre for Sustainable Nanomaterials
Ibnu Sina Institute for Scientific and Industrial Research
Universiti Teknologi Malaysia
81310 UTM, Skudai, Johor, MALAYSIA

Abstract: Fuzzy autocatalytic set (FACS) is one of the branch from graph theory. This FACS is used to introduce the fuzziness on connectivity of a graph. The introduction of FACS has given more accurate concept in application according a real-life problem. In this paper, some properties on fuzzy autocatalytic set of a graph theory is presented. This result give a new insight of FACS which is then expand the development of Fuzzy autocatalytic set (FACS) in the future.

AMS Subject Classification: 03E72, 05C50
Key Words: graph theory, fuzzy graph, autocatalytic set, fuzzy autocatalytic set

1. Introduction

Graph theory is one of mathematical disciplines. It was first introduced by Swiss mathematician Leonhard Euler to solve the Seven Bridges of Knigsberg problem [1]. The problem is to find a walkthrough of the city, where islands can only be reached via bridges. The walk path must not go through each bridge twice, and every bridge cannot be crossed a half-way every time. According to Balakrishnan and Ranganathan [2] the development of graph has grown into

Received: January 22, 2016
Published: March 23, 2016

§Correspondence author

significant area of mathematical researches and into other disciplines such as physics, chemistry, psychology, sociology and computer sciences. Graph can be used to model natural or manmade system [3]. Generally, graph is defined as the networks of points or nodes that are connected by links [2]. It can be described as a set of lines that used to connect to the set of points [4]. In particular, a directed graph $G = G\,(\mathrm{V}, \mathrm{E})$ is defined by a set V of nodes and a set E of links where each link is an ordered pair of nodes [2]. The set of nodes and links can be represented as$\mathrm{V} = \{v_1, v_2, v_3, ..., v_n\}$ and $\mathrm{E} = \{e_1, e_2, e_3, ..., e_n\}$. Some fundamental definitions on graph are given below.

Definition 1. (Connected graph, see [5]) The graph is a connected graph for every pair of vertices is joined by path, and it is disconnected otherwise.

Figure 1: (a) Connected graph (b) Disconnected graph

Definition 2. (Connected graph, see [5]) If each vertices in a graph has access to every other vertices, then the graph is irreducible.

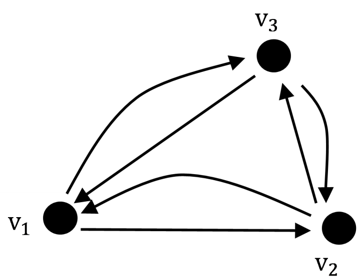

Figure 2: Irreducible graph of three vertices

The simplest irreducible subgraph is a vertices with 1-cycle. According to Jain and Krishna [5], the characteristic of an irreducible graph is, its adjacency matrices are also irreducible and vice versa. An irreducible graph is defined as strongly connected graphs. All strongly connected graphs are stated as irreducible, but not for a graph which consist only single vertex and no edges.

2. Fuzzy Graph

The introduction of fuzzy set by Zadeh [6] has led to a new way of thinking, particularly in modelling. It has been used in various disciplines and has opened a new history in graph theory. The implementation of fuzzy into graph theory was introduced by Rosenfeld [7]. The definition of fuzzy graph is presented as follow:

Definition 3. (Fuzzy Graph, see [7]) Fuzzy graph $\mathrm{G} = (\sigma, \mu)$ is a pair of functions $\sigma = \mathrm{S} \rightarrow [0,1]$ and $\mu = \mathrm{S} \times \mathrm{S} \rightarrow [0,1]$ for $\forall \mathrm{x}, \mathrm{y} \in \mathrm{S}$ where $\mu(\mathrm{x}, \mathrm{y}) \leq \sigma(\mathrm{x}) \wedge \sigma(\mathrm{y})$.

Definition 3 indicates that the vertices and edges fuzzy value lie from 0 to 1. Additionally, Yeh and Bang [8] introduced another version of fuzzy graph.

Definition 4. (Fuzzy Graph, see [8]) A Fuzzy graph $\mathrm{G} = (\mathrm{V}, R)$ is define as a pair whereby, V is set of vertices and R is a fuzzy set of edges.

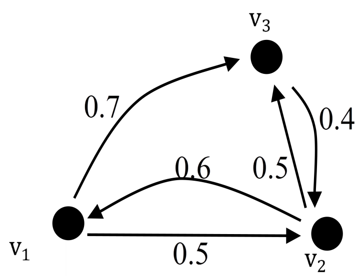

Figure 3: Three vertices of fuzzy graph

Definition 4 indicates that fuzziness only occurred on edges. As a result Blue et al. [9] introduced five types of possible fuzziness for a graph. It is known as taxonomy of fuzzy graphs. Tahir et al. [9] refined the taxonomy as below:

A fuzzy graph G_F is a graph which satisfies one of the fuzziness (G_F^i of the i^{th} type) or any of its combination:

Type 1: $G_F^1 = \{G_{1_F}, G_{2_F}, G_{3_F}, ..., G_{n_F}\}$ where fuzziness is on G_F^i for $i = 1, 2, 3, \ldots, n$.

Type 2: $G_F^2 = \{V, E_F\}$ where the edge set is fuzzy.

Type 3: $G_F^3 = \{V, E\,(t_F, h_F)\}$ where both the vertex and edge set are crispy, but the edge has fuzzy head and tail.

Type 4: $G_F^4 = \{V_F, E\}$ where the vertex set is fuzzy.

Type 5: $G_F^5 = \{V, E\,(w_F)\}$ where both the vertex and edge set are crispy, but the edge has fuzzy weights.

In the next section, a concept known as fuzzy autocatalytic set and its relationship with graph is presented.

3. Fuzzy Autocatalytic Set

An autocatalytic refers to a product or compounds used to speed up a chemical reaction known as catalyst. According to Ostwald [11], catalysis is used to accelerate the slow chemical reaction with an addition of a foreign substance, but its not consumed by the reaction. In general, an autocatalytic set is defined as a set of entities or a collection of entities where the word entities can be anything such as people, molecule or object [12, 13, 14]. Further, Jain and Krishna [5] formalized the definition of an autocatalytic set in the form of graph.

Definition 5. (An Autocatalytic Set, see [5]) An autocatalytic set (ACS) is a sub graph, each of whose nodes have at least one incoming edges from a vertices belonging to the same sub graph.

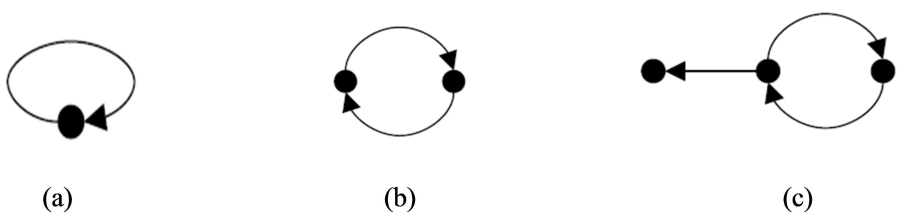

Figure 4: (a) a 1-cycle, the simplest ACS (b) a 2-cycle (c) an ACS but not irreducible graph

In [15], vertices j were presented as catalyst for vertices i. The simplest of ACS graph is a vertex with 1-cycle. Figure 1 shows some samples of ACS.

The relationship between cycle and irreducible subgraph and ACS are given in the following theorem.

Theorem 6. (see [5, 16]) *(i) All cycle are irreducible subgraphs and all irreducible subgraphs are ACSs.*

(ii) Not all ACSs are irreducible subgraphs and not all irreducible subgraph are cycle.

Furthermore, the relationship between ACS and Perron-Frobenius eigenvalue $\lambda_1(C)$is given by the following theorems.

Theorem 7. (see [16]) *An ACS should consist of a closed walk. Therefore:*

(i) $\lambda_1(C) = 0$ *if a graph has no ACS*

(ii) $\lambda_1(C) \geq 1$ *if a graph has an ACS*

Theorem 8. (see [16]) *The subgraph of any PFE of graph G is an ACS, if* $\lambda_1(C) \geq 1$.

Further, Tahir et al. [15] introduced fuzziness into an autocatalytic set. The definition for fuzzy autocatalytic set is given as below:

Definition 9. (Fuzzy Autocatalytic Set, see [15]) Fuzzy Autocatalytic Set (FACS) is a sub graph each of whose nodes have at least one incoming link with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$ from a vertices belonging to the same sub graph.

In the same paper [15], they introduced fuzzy autocatalytic set of a graph type 3.

Definition 10. (Fuzzy Autocatalytic Set of a Graph Type 3, see [15]) Let $e_i \in E$. The fuzzy head of e_i is denoted as $h(e_i)$ and the fuzzy tail $t(e_i)$ are function of e_i such that $h : E \to [0,1]$ and $t : E \to [0,1]$ for $e_i \in E$. A fuzzy edge connectivity is a tuple $(t(e_i), h(e_i))$ and the set of all fuzzy edge connectivity is denoted as $C = \{(t(e_i), h(e_i)) : e_i \in E\}$.

This development of autocatalytic set has given a great impact on graph theory. By the introduction of fuzziness into a graph of autocatalytic set, it has led into a new branch of graph theory, namely FACS. Next section discusses some properties for a graph of FACS.

4. Some Features on ACS and FACS

In this section, some new features of ACS and FACS are presented in the form of lemma, theorems and corollaries.

Theorem 11. *Let $G_{k_{FT3}}(V,E)$ be a fuzzy graph of type -3 which is autocatalytic set; i.e. FACS. Let $G_{FT3} = \{G_{k_{FT3}} : k = 1,2,\ldots,n\}$ be the finite set of all fuzzy graph of type-3 and let $M_F^{n\times n} = \left\{[a_{ij}]^{n\times n} : a_{ij} \in [0,1]\right\}$.*
Define $\theta : G_{FT3} \to M_F^{n\times n} \theta\left(G_{i_{FT3}}\right) = [a_{ij}]$. Then θ is bijective function.

Proof. (i) Let

$$G_{FT3}(V,E) = G'_{FT3}\left(V',E'\right)$$
$$\Rightarrow V = \{v_1, v_2, \ldots, v_n\} = \left\{v'_1, v'_2, \ldots, v'_n\right\} = V',$$

and

$$E = \{\mu(v_j, v_i)\}_{i,j=1,2,\ldots,n} = \left\{\mu'\left(v'_j, v'_i\right)\right\}_{i,j=1,2,\ldots,n} = E'$$
$$\Rightarrow \mu(v_j, v_i) = a_{ij} = a'_{ij} = \mu\left(v'_j, v'_i\right)$$
$$\Rightarrow [a_{ij}] = [a_{ij}]',$$

Here θ is a well-defined function.

(ii) θ is onto since for $[a_{ij}] \in M_F^{n\times n}, \exists G_{i_{FT3}} \theta\left(G_{i_{FT3}}\right) = [a_{ij}]$ and

$$a_{ij} = \mu(v_i, v_j)$$

for $(v_i, v_j) \in G_{i_{FT3}}$.

(iii) Suppose $\theta(G_1(V,E)) = \theta(G_2(V,E))$,

$$a_{ij} = a'_{ij} \Rightarrow (v_j, v_i) = \left(v'_j, v'_i\right) \Rightarrow v_i = v'_i \text{and} v_j = v'_j, \forall i,j = 1,2,\ldots,n$$
$$\Rightarrow G_1(V,E) = G_2(V,E).$$

Hence, θ is one to one.
Therefore, the inverse of θ does exist. □

From here, every FACS of fuzzy graph type-3 can be mapped to a square matrix.

Lemma 12. *If* $\mathrm{G}(\mathrm{V},\mathrm{E})$ *is an autocatalytic set and* $|\mathrm{V}| = \mathrm{n}$, *then* $|\mathrm{E}| \leq \mathrm{n}^2$.

Proof. Let $\mathrm{G}(\mathrm{V},\mathrm{E})$ be an ACS and $\mathrm{V}=\{v_1,v_2,v_3,...,v_n\}$ therefore, every vertices has incoming link and $|\mathrm{V}|=n$. The possibility of edges for G is a set

$$\mathrm{E}' = \left(\begin{array}{ccccc} (e_1,e_2) & (e_1,e_3) & (e_1,e_4) & \cdots & (e_n,e_n) \\ & (e_2,e_3) & (e_2,e_4) & \cdots & (e_2,e_n) \\ & & (e_3,e_4) & \cdots & (e_3,e_n) \\ & & & \ddots & \vdots \\ & & & & (e_{n-1},e_n] \end{array} \right]$$

$$\cup \left(\begin{array}{ccccc} & & & & (e_n,e_{n-1}) \\ & & & \ddots & \vdots \\ & & (e_n,e_{n-2}) & \cdots & (e_{n-1},e_{n-2}) \\ & (e_n,e_{n-3}) & (e_{n-1},e_{n-3}) & \cdots & (e_{n-2},e_{n-3}) \\ (e_n,e_1) & (e_{n-1},e_1) & (e_{n-2},e_1) & \cdots & (e_2,e_1) \end{array} \right]$$

$$\cup\{(e_1,e_1)(e_2,e_2)(e_3,e_3)...\}.$$

Hence

$$\begin{aligned} \left|\mathrm{E}'\right| =&2\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}2\\2\end{pmatrix}+n \\ =&2\begin{pmatrix}2\\2\end{pmatrix}+n \\ =&2\left(\frac{n!}{2!(n-2)}\right)+n \\ =&\frac{n!}{(n-2)!}+n \\ =&\frac{n(n-1)(n-2)!}{(n-2)!}+n \\ =&n(n-1)+n \\ =&n^2-n+n \\ =&n^2. \end{aligned}$$

Therefore, $|\mathrm{E}|\leq\left|\mathrm{E}'\right|=n^2$. □

Lemma 12 indicates that a finite set of vertices requires a finite set of edges.

Theorem 13. *Any autocatalytic set* $\mathrm{G}(\mathrm{V}, \mathrm{E})$ *with* n *vertices has edges at most* n^2*, i.e.* $|\mathrm{E}(n)| \leq n^2$ *where* $\mathrm{E}(n)$ *is set of edges with for n vertices.*

Proof. We will use mathematical induction to prove the theorem.

(i) Suppose $\mathrm{G}(\mathrm{V}, \mathrm{E})$ is an autocatalytic set with $\mathrm{V} = \{e_1\}$ where G is define as a loop namely (e_1, e_1). Hence $|\mathrm{E}(1)| = 1 = 1^2$.

(ii) Assume $\mathrm{G}(\mathrm{V}, \mathrm{E})$ is an autocatalytic set with n vertices such that $|\mathrm{E}(n)| \leq n^2$. This assumption is supported by Lemma 12. Now, consider $\mathrm{G}'(\mathrm{V}, \mathrm{E})$ is an autocatalytic set with $n+1$ vertices.

Therefore:

$$\begin{aligned}
\mathrm{E}(n+1) =& \mathrm{E}(\{e_1, e_2, e_3, ..., e_n, e_{n+1}\}) \\
=& \mathrm{E}(\{e_1, e_2, e_3, ..., e_n\} \cup \{e_{n+1}\}) \\
=& \mathrm{E}(\{e_1, e_2, e_3, ..., e_n\}) \cup \mathrm{E}(\{e_{n+1}\}) \\
=& \mathrm{E}(n) + \mathrm{E}(1) \\
\leq& n^2 + 1 \\
<& n^2 + 2n + 1 \\
=& (n+1)^2
\end{aligned}$$

$$|\mathrm{E}(n+1)| \leq (n+1)^2.$$

Hence, any autocatalytic set has at most n^2 edges □

We can use similar argument in Theorem 13 for a proper subgraph.

Corollary 14. *Let* $\mathrm{G}(\mathrm{V}, \mathrm{E})$ *is an autocatalytic set, with* $|\mathrm{V}| = n$ *and* $\mathrm{H}(\mathrm{V_H}, \mathrm{E_H})$ *is a proper subgraph of G, then* $|\mathrm{E_H}| < n^2$.

Proof. Let $\mathrm{G}(\mathrm{V}, \mathrm{E})$ is an autocatalytic set with $|\mathrm{V}| = n$. Supposed $\mathrm{H}(\mathrm{V_H}, \mathrm{E_H})$ is a subgraph.

Then by Theorem 13, $|\mathrm{E_H}| < |\mathrm{E}| \leq n^2$, since H is a proper subgraph of G. □

Similar results as above can also be generated for any FACS too.

Corollary 15. *If* $\mathrm{G}_F(\mathrm{V}_F, \mathrm{E}_F)$ *is FACS with* $|\mathrm{V}_F| = n$*, then* $|\mathrm{E_F}| \leq n^2$

Proof. Let $\mathrm{G}_F(\mathrm{V}_F, \mathrm{E}_F)$ is FACS and V_F is it vertices with $|\mathrm{V}_F| = n$. Then by using similar argument as in Theorem 13, $|\mathrm{E_F}| \leq n^2$. □

Corollary 16. *Let* $\mathrm{G}_F\left(\mathrm{V}_F,\mathrm{E}_F\right)$ *is FACS with* $|\mathrm{V}_F|=n$ *and* $\mathrm{H}_F\left(\mathrm{V}'_F,\mathrm{E}'_F\right)$ *is a proper fuzzy subgraph of* G_F*, then* $\left|\mathrm{E}'_F\right|<n^2$.

Proof. Let $\mathrm{G}_F\left(\mathrm{V}_F,\mathrm{E}_F\right)$ is an autocatalytic set with $|\mathrm{V}_F|=n$. Supposed $\mathrm{H}_F\left(\mathrm{V}'_F,\mathrm{E}'_F\right)$ is its fuzzy subgraph. Then by the Corollary 15,

$\left|\mathrm{E}'_F\right|<|\mathrm{E}_F|\leq n^2$, since H_F is a proper fuzzy subgraph of G_F. □

5. Conclusion

We have shown that ACS and FACS with finite vertices require finite edges. In particular, ACS or FACS with vertices has an upperbound number of edges, namely n^2. This result is also true for subgraph of ACS and FACS.

Acknowledgments

This work has been supported by Ibnu Sina Institute, MyBrain15 scholarship from Ministry of High Education Malaysia and University Teknologi Malaysia.

References

[1] Carlson,Sc., *Graph Theory*, Encyclopedia Britannica (2010).

[2] Balakrishnan R., Ranganathan K., *A textbook of Graph Theory*' Springer (2012)

[3] Harary, F., *Graph Theory*, California, USA: Addison Wesley Publishing Company (1969).

[4] Epp, S.S., *Discrete Mathematics with Applications*, Boston: PWS Publishing Company (1993).

[5] Jain, S. and Krishna, S., Autocatalytic Sets and the Growth of Complexity in an Evolutionary Model, *Physical Review Letters*, Vol.81 (1998) 5684-5687.

[6] Zadeh, L. A., Outline of a new approach to The Analysis of Complex Systems and Decision Processes, *IEEE Trans. Systems. Man and Cybernatics*, Vol.3 (1965), 28-44.

[7] Rosenfeid, A., Fuzzy Graphs'I',In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes: Proceedings of the USJapan Seminar on Fuzzy Sets and Their Applications*,Held at the University of California, Berkeley, California (1974), July 1-4, p. 77. Academic Press.

[8] Yeh, R. T. and Bang S. Y., *Fuzzy Relations, Fuzzy Graphs and Their Applications to Clustering Analysis*, New York: Academic Press (1975).

[9] Blue M. and Bush B. and Puckett J., Unified approach to fuzzy graph problem, *Fuzzy Set System*, Vol.125 (2002), 355-368.

[10] Sabariah B, Tahir. A., Khairil A.A., Graphical Presentation Of A Clinical Waste Incinerations Proses, *Paper presented at the 10th National Symposium Of Mathematical Sciences*, Johor Bahru (2002).

[11] Ostwald W., Definition der Katalyse, *Zeitschrift fr Physikalische Chemie*, Vol.15 (1894), 705706.

[12] Eigen, M., McCaskill, J. and Schuster, P., The Molecular Quasi-species, *Advanced Chemistry Physics*, Vol.75 (1989), 149-263.

[13] Kauffman, S.A., Autocatalytic Sets of Proteins, *Journal of Theoretical Biology*, Vol.119 (1983), 1-24.

[14] Rossler, O.E., A System Theoretic Model of Biogenesis, *Z. Naturforschung*, Vol.26b (1971), 741-746.

[15] Tahir A., Sabariah B. and Arshad, Khairil. A., *Modeling A Clinical Incineration Process Using Fuzzy Autocatalytic Set*, Journal of Mathematical Chemistry, Springer (2009).

[16] Jain, S. and Krishna, S., Emergence and Growth of Complex Networks in Adaptive Systems, *Computer Physics Communications*, Vol.121 (1999), 116-121.