International Journal of Pure and Applied Mathematics
Volume 107 No. 2 2016, 407-414
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i2.10

ijpam.eu

EXISTENCE AND UNIQUENESS OF WEAK SOLUTION OF A NONLINEAR NEUMANN PROBLEM

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Abstract: This paper deals with the equation

$$-\Delta_p u + a(x)|u|^{p-2}u = f(x,u)$$

in bounded domain $\Omega \in \mathbb{R}^N$. Relying on Browder theorem, under conditions of the monotonous function f. We obtained the existence and uniqueness of weak solutions for the weighted p-laplacian boundary value of the form

$$(P) \qquad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega \end{cases}$$

in bounded domain $\Omega \in \mathbb{R}^N$.

Key Words: weak solutions, p-Laplacian operator, Neumann problem

1. Introduction

In this paper, we are concerned with the existence and uniqueness of weak solution for a weighted p-laplacian boundary value of the form:

$$(P) \qquad \begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega. \end{cases}$$

Let Ω be a bounded domain in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodry (CAR) Satisfies the following:

Received: January 4, 2016
Published: April 9, 2016

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(H_1) $f(x, s_1) \leq f(x, s_2)$ for a.e. $x \in \Omega$ and $\forall s_1, s_2 \in \mathbb{R}$, $s_1 \geq s_2$.

(H_2) $|f(x, s)| \leq f_0(x) + c|s|^{p-1}$ p.p. $x \in \Omega$ $s \in \mathbb{R}$ that there exists $f_0 \in L^{p'}(\Omega)$, and $c > 0$.

(H_3) $0 < \alpha \leq a(x) \leq \beta < +\infty$.

This paper is organized as follows: Section 2 contains some basic definitions concerning the nonlinear operators that will be used throughout the paper. Also, we introduce the space setting of the problem and give some basic characteristics, as the equivalent norm and embedding results. In Section 3 we state the main result on the existence and uniqueness of weak solutions of the problem (P).

2. Preliminaries and Space Setting

First, we introduce some basic definitions concerning the nonlinear operators which we use extensively in this paper (0.3).

Definition 2.1. (see [7]) Let $A : V \to V'$ be an operator on a real Banach space V. We say that the operator A is:

(i) bounded iff it maps bounded sets into bounded i.e. for each $r > 0$ there exists $M > 0$ (M depending on r) such that

$$\|u\| \leq r \Rightarrow \|A(u)\| \leq M, \forall u \in V;$$

(ii) coercive: iff

$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u \rangle}{\|u\|} = \infty;$$

(iii) monotone iff $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0, \forall u_1, u_2 \in V$;

(iv) strictly monotone iff

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle > 0, \text{ for all } u_1, u_2 \in V, u_1 \neq u_2;$$

(v) strongly monotone iff there exists $k > 0$,

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq k\|u_1 - u_2\|, \text{ for all } u_1, u_2 \in V, u_1 \neq u_2;$$

(vi)continuous iff $u_n \to u$ implies $A(u_n) \to A(u)$, for all $u_n, u \in V$;

(vii) strongly continuous iff $u_n \rightharpoonup u$ implies $A(u_n) \to A(u)$, for all $u_n, u \in V$;

(viii) demicontinuous iff $u_n \to u$ implies $A(u_n) \rightharpoonup A(u)$, for all $u_n, u \in V$.

Theorem 2.1. (Browder, see [7]) *Let A be a reflexive real Banach space. Moreover let $A : V \to V'$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space V. Then, the equation $A(u) = f$ has at least one solution $u \in V$ for each $f \in V'$.*

If moreover, A is strictly monotone operator, then the equation (P) has precisely one solution $u \in V$ for every $f \in V'$.

3. Existence and Uniqueness Results

In this section, using Browder theorem, we prove the existence and uniqueness of weak solution for equation (P).

Definition 3.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution to equation (P) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx = \int_\Omega f(x,u)\varphi dx, \ \forall\varphi \in W^{1,p}(\Omega).$$

Our main results concerning problem (P) is the following theorem.

Theorem 3.1. *Let $p \geq 2$,and $f(x,u) \in CAR(\Omega, \mathbb{R})$ satisfy (H_1), (H_2) and (H_3). Then problem (P) has a unique weak solution.*

Proof. We define for the operator $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$, as $A = J - F$, where the Operators $J : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$, and $F : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ are given by

$$\langle J(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx$$

and

$$\langle F(u), \varphi\rangle = \int_\Omega f(x,u)\varphi dx$$

for all $u, \varphi \in W^{1,p}(\Omega)$. Thus, to find a weak solution of (P) is equivalent to finding $W^{1,p}(\Omega)$ which satisfies the operator equation $A(u) = 0$. Now, we have the following properties of the operators J and F.

a) J and F are well defined. Using Holders inequality, we have

$$\langle J(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_\Omega a(x)|u|^{p-2}u\varphi dx,$$

$$|\langle J(u),\varphi\rangle| \leq \int_\Omega |\nabla u|^{p-1}|\nabla\varphi|dx + \int_\Omega a(x)|u|^{p-1}|\varphi|$$
$$\leq \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\nabla\varphi|^p\right)^{\frac{1}{p}} + \beta\left(\int_\Omega |u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} < \infty.$$

Moreover
$$\langle F(u),\varphi\rangle = \int_\Omega f(x,u)\varphi dx$$
and
$$|\langle F(u),\varphi\rangle| \leq \int_\Omega |f(x,u)\varphi|dx \leq \int_\Omega (f_0(x)+c|u|^{p-1})|\varphi|dx.$$
By Hölder's inequality,
$$|\langle F(u),\varphi\rangle| \leq \left(\int_\Omega |f_0(x)|^{p'}\right)^{\frac{1}{p'}}\left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} + c\left(\int_\Omega |u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}} < \infty.$$
Hence J, F are well defined.

b) J and F are bounded operators. Indeed, for every u such that $\|u\|_{W^{1,p}(\Omega)} \leq M$,
$$\|J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} |\langle J(u),\varphi\rangle|$$
$$\leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1}\left(\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\nabla\varphi|^p\right)^{\frac{1}{p}} + \beta\left(\int_\Omega |u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}}\right)$$
$$= \sup_{\|\varphi\|\leq 1}\left(\|\nabla u\|_{L^p}^{\frac{p}{p'}}\|\varphi\|_{L^p} + \beta\|u\|_{L^p}^{\frac{p}{p'}}\|\varphi\|_{L^p}\right)$$
$$\leq \|\nabla u\|_{L^p}^{\frac{p}{p'}} + \beta\|u\|_{L^p}^{\frac{p}{p'}} \leq \|u\|^{\frac{p}{p'}} + \beta\|u\|^{\frac{p}{p'}}$$
$$2\beta\|u\|^{\frac{p}{p'}} \leq 2\beta M^{\frac{p}{p'}}.$$
Also, we get
$$\|F(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1} |\langle F(u),\varphi\rangle| \leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1}\int_\Omega (f_0(x)+c|u|^{p-1})|\varphi|$$
$$\leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)}\leq 1}[\left(\int_\Omega |f_0(x)|^{p'}\right)^{\frac{1}{p'}} + \left(\int_\Omega |u|^{(p-1)p'}\right)^{\frac{1}{p'}}]\left(\int_\Omega |\varphi|^p\right)^{\frac{1}{p}}.$$

$$\leq k(\|f_0\|_{L^{p'}(\Omega)} + k\|u\|_{W^{1,p}(\Omega)}^{\frac{p}{p'}})$$

$$\leq k(\|f_0\|_{L^{p'}(\Omega)} + kM^{\frac{p}{p'}})$$

Where k is the constant of the embedding of $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$,.

c) J and F are continuous operators if $u_n \to u$, in $W^{1,p}(\Omega)$. Then, we have

$$\|u_n - u\|_{W^{1,p}(\Omega)} \to 0, \ \|\nabla u_n - \nabla u\|_{L^p(\Omega)} \to 0, \|u_n - u\|_{L^p} \to 0$$

Applying Dominated Convergence Theorem, we obtain

$$\|(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\|_{L^p(\Omega)} \to 0$$

and

$$\|(|u_n|^{p-2}u_n - |u|^{p-2}u)\|_{L^p(\Omega)} \to 0$$

Hence

$$\|J(u_n) - J(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} |\langle J(u_n) - J(u), \varphi\rangle|$$

$$\leq \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} \left(\left(\int_\Omega ([|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u])^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right.$$

$$\left. +\beta \left(\int_\Omega (([|u_n|^{p-2}u_n - |u|^{p-2}u)])^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\varphi|^p \right)^{\frac{1}{p}} \right) \to 0 \text{ as } n \to \infty.$$

Similarly, we have

$$\|F(u_n) - F(u)\|_{(W^{1,p}(\Omega))'} = \sup_{\|\varphi\|_{W^{1,p}(\Omega)} \leq 1} |\langle F(u_n) - F(u), \varphi\rangle|$$

$$\leq k \left(\int_\Omega |f(x, u_n) - f(x, u)|^{p'} \right)^{\frac{1}{p'}} \to 0 \text{ as } n \to \infty.$$

d) Let $p \geq 2$, $\forall x_1, x_2 \in \mathbb{R}^{\mathbb{N}}$, we have the following inequality

$$|x_2|^p \geq |x_1|^p + p|x_1|^{p-2}x_1(x_2 - x_1) + \frac{|x_2 - x_1|^p}{2^{p-1} - 1}. \tag{1}$$

Now,

$$\langle J(u) - J(\varphi), u - \varphi\rangle = \int_\Omega \left[|\nabla u|^{p-2}\nabla u - |\nabla\varphi|^{p-2}\nabla\varphi\right](\nabla u - \nabla\varphi)$$

$$
\begin{aligned}
&+\int_\Omega a(x)\left[|u|^{p-2}u-|\varphi|^{p-2}\varphi\right](u-\varphi)\\
&=\int_\Omega[|\nabla u|^{p-2}\nabla u(\nabla u-\nabla\varphi)]-\int_\Omega[|\nabla\varphi|^{p-2}\nabla\varphi(\nabla u-\nabla\varphi)]\\
&+\int_\Omega a(x)[|u|^{p-2}u(u-\varphi)]-\int_\Omega a(x)[|\varphi|^{p-2}\varphi(u-\varphi)]=I_1+I_2.
\end{aligned}
$$

Using (1), we get

$$
\begin{aligned}
I_1+I_2 &\geq \frac{2}{p(2^{p-1}-1)}\int_\Omega|\nabla u-\nabla\varphi|^p dx+\frac{2}{p(2^{p-1}-1)}\int_\Omega a(x)|u-\varphi|^p dx\\
&\geq \alpha c(p)\left(\|\nabla u-\nabla\varphi\|^p_{L^p(\Omega)}+\|u-\varphi\|^p_{L^p(\Omega)}\right)\\
&=\alpha c(p)\|u-\varphi\|^p_{W^{1,p}(\Omega)},\ \text{for } p\geq 2.
\end{aligned}
$$

So

$$\langle J(u)-J(\varphi),u-\varphi\rangle \geq \alpha c(p)\|u-\varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p\geq 2. \tag{2}$$

Also, we get

$$\langle F(u)-F(\varphi),u-\varphi\rangle=\int_\Omega[f(x,u)-f(x,\varphi)](u-\varphi).$$

Since f is decreasing with respect to the second variable, we have

$$[f(x,u)-f(x,\varphi)](u-\varphi)\leq 0.$$

Consequently

$$\langle F(u)-F(\varphi),u-\varphi\rangle=\int_\Omega[f(x,u)-f(x,\varphi)](u-\varphi)\leq 0. \tag{3}$$

Equations (2) and (3) imply that

$$\langle A(u)-A(\varphi),u-\varphi\rangle\geq c(p)\|u-\varphi\|^p_{W^{1,p}(\Omega)} \text{ for } p\geq 2. \tag{4}$$

So A is strongly monotone. Now, to apply Browder theorem, it remains to prove that A is a coercive operator. From (4), we have

$$\langle A(u),u\rangle\geq\langle A(0),u\rangle+c(p)\|u\|^p_{W^{1,p}(\Omega)}.$$

On the other hand

$$\langle A(0),u\rangle=\langle J(0),u\rangle-\langle F(0),u\rangle$$

$$= -\int_\Omega f(x,0)u$$

$$\geq -(\int_\Omega [f_0(x)]^{p'})^{\frac{1}{p'}}(\int_\Omega |u|^p)^{\frac{1}{p}}$$

$$\geq -k\|f_0\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)}.$$

Then

$$\langle A(u), u\rangle \geq c(p)\|u\|^p_{W^{1,p}(\Omega)} - k\|f_0\|_{L^{p'}(\Omega)}\|u\|_{W^{1,p}(\Omega)}.$$

So

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty} \frac{\langle A(u), u\rangle}{\|u\|_{W^{1,p}(\Omega)}} = \infty.$$

This proves the coercivity condition and so, the existence of weak solution for (P). The uniqueness of weak solution of (P) is a direct consequence of (4). Suppose that u, φ be a weak solutions of (P) such that $u \neq \varphi$. Now, from (4), we have

$$0 = \langle A(u) - A(\varphi), u - \varphi\rangle \geq c(p)\|u - \varphi\|^p_{W^{1,p}(\Omega)} \geq 0.$$

Therefore $u = \varphi$ This completes the proof. □

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