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NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS BY USING HYBRID BLOCK-PULSE FUNCTIONS AND BERNSTEIN POLYNOMIALS

M. Tavassoli Kajani

Department of Mathematics
Islamic Azad University
Isfahan (Khorasgan) Branch, IRAN

Abstract: In this paper the hybrid block-pulse function and Bernstein polynomials are introduced to approximate the solution of linear Volterra integral equations. Both second and first kind integral equations, with regular, as well as weakly singular kernels, have been considered. Numerical examples are given to demonstrate the applicability of the proposed method. The obtained results show that the hybrid block-pulse function and Bernstein polynomials are more accurate that Bernstein polynomials.

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1. Introduction

Bernstein polynomials (**BP**) have been recently used for the solution of some nonlinear integro-differential equations, both BVP, by Yuzbasi [1] and Islam & Hossain [2]. Also these have been used to solve some classes of mathematical equations by [3, 4, 5, 6, 7, 8]. These were further used to solve Falkner-Skan equation by Tavassoli Kajani *et al.* [9]. Tavassoli Kajani *et al.* [7] employed rational second kind Chebyshev functions to solve integro-differential equations. There are some applications of Bernstein polynomials, for example see [10], and extend them for hybrid Bernstein polynomials and block-pulse functions.

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Volterra integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. Various methods are available in the literature concerning their numerical solutions. Recently Legendre polynomials were used by Maleki & Tavassoli Kajani [11] and Tavassoli Kajani [12] to solve certain Volterras population growth model with fractional order and Volterras population growth model.

In this paper we have developed a simple method, based on approximation of the unknown function on the hybrid block-pulse function and Bernstein polynomials basis, for the solution of Volterra integral equations with regular kernels, as well as weakly singular kernels, that is Abels integral equation. Abels integral equations possess weakly singular kernels of the type $(x-t)^{\alpha},\ 0<\alpha<1$. Although analytical solution of Abels integral is very well known, yet the numerical solution is not so well pronounced due to some computational difficulties which arise due to the presence of the differential operator in the solution ([13], p. 27). However, the present method avoids any such computational difficulty, and uses a very direct algorithm for computation of the unknown function.

1.1. Definition

Bernstein polynomials can be defined on some interval $[a,b]$ by:

$$B_{j,n}(x)=\binom{n}{j}\frac{(x-a)^j(b-x)^{n-j}}{(b-a)^n},\quad j=0,1,2,\ldots,n. \tag{1}$$

These polynomials form a partition of unity, that is $\sum_{j=0}^{n}B_{j,n}(x)=1$, and can be used for approximating any function continuous in $[a,b]$.

A set of block-pulse functions $b_i(x)$, $i=1,2,\ldots,m$ on the interval $[a,b)$ are defined as follows:

$$b_i(x)=\begin{cases}1, & a_{i-1}\leqslant x<a_i,\\ 0, & \text{otherwise},\end{cases} \tag{2}$$

where $a_i=\frac{(m-i)a+ib}{m},\quad i=0,1,2,,\ldots,m.$

The block-pulse functions on $[a,b)$ are disjoint, that is for $i=1,2,\ldots,m$, $j=1,2,\ldots,m$ we have: $b_i(x)b_j(x)=\delta_{ij}b_i(x)$, also these functions have the property of orthogonality on $[0,1)$.

For $i=1,2,\ldots,m$ and $j=0,1,2,\ldots,n$ the hybrid Bernstein polynomials and block-pulse functions (**HBB**) are defined as:

$$Bb_{i,j}(x)=\begin{cases}\binom{n}{j}\frac{(x-a_{i-1})^j(a_i-x)^{n-j}}{(a_i-a_{i-1})^n}, & a_{i-1}\leqslant x<a_i,\\ 0, & \text{otherwise},\end{cases} \tag{3}$$

this functions satisfy the following

$$\sum_{j=0}^{n} Bb_{i,j}(x) = \begin{cases} 1, & a_{i-1} \leqslant x < a_i. \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall x \in [a,b]: \quad \sum_{i=1}^{n} \sum_{j=0}^{n} Bb_{i,j}(x) = 1,$$

$$\forall k \neq l: \quad Bb_{k,j}(x) Bb_{l,j}(x) = 0.$$

Our proposed functions show the piecewise polynomials accurately while Bernstein polynomials do not have this property.

2. The General Method

2.1. Volterra Equations with Regular Kernels

We consider the integral equation of the first kind given by,

$$\int_a^x k(x,t)\phi(t)dt = f(x), \qquad a < x < b. \tag{4}$$

Here $\phi(t)$ is the unknown function to be determined, $k(x,t)$, the kernel, is a continuous and square integrable function, $f(x)$ being the known function satisfying $f(a) = 0$.

To determine an approximate solution of (4), $\phi(t)$ is approximated in the **HBB** basis on $[a,b]$ as

$$\phi(t) = \sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} Bb_{ij}(t). \tag{5}$$

Here a_{ij} $(i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, n)$ are unknown constants to be determined substituting (5) in (4) we obtain,

$$\sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} \alpha_{ij}(x) = f(x), \tag{6}$$

where

$$\alpha_{ij}(x) = \int_a^x k(x,t) Bb_{ij}(t)dt. \tag{7}$$

Let us set $x = x_{kl}$ $(k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n)$ in (6) x_{kl}'s being chosen as suitable distinct points in (a_{k-1}, a_k), and x_{k0} is taken near a_{k-1} and x_{kn} near a_k such that $a_{k-1} < x_{k0} < x_{k1} < \cdots < x_{kn} < a_k$. Putting $x = x_{kl}$ we obtain the linear system

$$\sum_{i=1}^{m}\sum_{j=0}^{n} a_{ij}\alpha_{ijkl} = f_{kl}, \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n. \tag{8}$$

where

$$\alpha_{ijkl} = \alpha_{ij}(x_{kl}), \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n$$

and

$$f_{kl} = f(x_{kl}), \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n.$$

The linear system (8) can be easily solved by standard methods for the unknown constants a_{ij}'s. These a_{ij} $(i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, n)$ are then used in (5) to obtain the unknown function $\phi(t)$ approximately.

We now consider the second kind Volterra integral equation, given by,

$$c(x)\phi(x) + \int_a^x k(x,t)\phi(t)dt = f(x), \qquad a < x < b. \tag{9}$$

where $k(x,t)$ is a regular kernel, $c(x)$, $f(x)$ are known functions, then applying the same procedure as described above, we obtain

$$\sum_{i=1}^{m}\sum_{j=0}^{n} a_{ij}\beta_{ij}(x) = f(x) \tag{10}$$

where

$$\beta_{ij}(x) = c(x)Bb_{ij}(x) + \int_a^b k(x,t)Bb_{ij}(t)dt$$

Choosing x_{kl}'s $(k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n)$ as described we obtain the linear system

$$\sum_{i=1}^{m}\sum_{j=0}^{n} a_{ij}\beta_{ijkl} = f_{kl}, \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n \tag{11}$$

where

$$\beta_{ijkl} = \beta_{ij}(x_{kl}), \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n$$

and

$$f_{kl} = f(x_{kl}), \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n$$

The system (11) is solved to obtain the unknown constants a_{ij} $(i = 1, 2, \dots, m,\ j = 0, 1, \dots, n)$ which are then used to obtain the unknown function $\phi(t)$.

Also we select x_{kl}'s $(k = 1, 2, \dots, m,\ l = 0, 1, \dots, n)$ points as follows:

$$\begin{cases} x_{k0} = 10^{-10} + a_{k-1} & k = 1, 2, \dots, m \\ x_{kl} = x_{k0}\dfrac{(a_k - a_{k-1})l}{n+1} & l = 1, 2, \dots, n, \qquad k = 1, 2, \dots, m \end{cases}$$

2.2. Volterra Equations with Weakly Singular Kernels

We consider the weakly singular integral equation of the first kind, that is the Abel's integral equation given by,

$$\int_a^x \frac{\phi(t)}{(x-t)^\alpha} dt = f(x), \qquad a < x < b. \tag{12}$$

with $0 < \alpha < 1$ and $f(a) = 0$. This is a Volterra integral equation with a weakly singular kernel. The exact solution of (12) is well known [14] and is given by

$$\phi(x) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dx} \left[\int_a^x \frac{f(t)}{(x-t)^{(1-\alpha)}} dt \right]. \tag{13}$$

Since $f(a) = 0$, (13) can be simplified to the form

$$\phi(x) = \frac{\sin(\alpha\pi)}{\pi} \int_a^x \frac{f'(t)}{(x-t)^{(1-\alpha)}} dt \tag{14}$$

Because $Bb_{ij}(x),\ (i = 1, 2, \dots, m,\ j = 0, 1, \dots, n)$ are polynomial on the interval (a_{i-1}, a_i), then we assume

$$Bb_{ij}(x) = \begin{cases} \sum_{q=0}^{n} d_q^{i,j} x^q & a_{i-1} < x < a_i \\ 0 & \text{otherwise} \end{cases}$$

where $d_q^{i,j}$ depends on a_{i-1}, a_i and j

By substituting (5) in (12) we have

$$\sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} \int_a^x \frac{Bb_{ij}(t)}{(x-t)^\alpha} dt = f(x).$$

Choosing x_{kl}'s $(k = 1, 2, \dots, m,\ l = 0, 1, \dots, n)$ as described above we obtain the linear system

$$\sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} \int_a^{x_{kl}} \frac{Bb_{ij}(t)}{(x_{kl}-t)^\alpha} dt = f(x_{kl}), \tag{15}$$

where

$$\int_a^{x_{kl}} \frac{Bb_{ij}(t)}{(x_{kl}-t)^\alpha}dt = \begin{cases} \sum_{q=0}^{n} d_q^{i,j} \int_{a_{i-1}}^{x_{kl}} \frac{t^q}{(x_{kl}-t)^\alpha}dt & a_{i-1} \leqslant x_{kl} \leqslant a_i \\ 0 & x_{kl} < a_{i-1} \\ \binom{n}{j} \int_{a_{i-1}}^{a_i} \frac{(t-a_{i-1})^j (a_i-t)^{n-j}}{(a_i-a_{i-1})^n (x_{kl}-t)^\alpha}dt & a_i < x_{kl} \end{cases}$$

The linear system (15) can be solved to obtain the unknown constants a_{ij}, $(i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, n)$, which are then used to approximate the unknown function $\phi(t)$.

We next consider a Volterra integral equation of the second kind with weakly singular kernel given by

$$\phi(x) + \lambda \int_a^x \frac{\phi(t)}{(x-t)^\alpha}dt = f(x), \qquad 0 < \alpha < 1, \quad a < x < b, \tag{16}$$

λ being a constant.

Approximating $\phi(t)$ as before we obtain

$$\sum_{i=1}^{m}\sum_{j=0}^{n} a_{ij}\alpha_{ij}(x) = f(x)$$

where $\alpha_{ij}(x) = Bb_{ij}(x) + \lambda \int_a^x \frac{Bb_{ij}(t)}{(x-t)^\alpha}dt$.

For suitably chosen $x = x_{kl}$'s $(k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n)$, we get the linear system

$$\sum_{i=1}^{m}\sum_{j=0}^{n} a_{ij}\alpha_{ijkl} = f_{kl}, \quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n \tag{17}$$

where $\alpha_{ijkl} = \alpha_{ij}(x_{kl})$, $\quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n$ and $f_{kl} = f(x_{kl})$, $\quad k = 1, 2, \ldots, m,\ l = 0, 1, \ldots, n$.

Solving the linear system (17) for a_{ij} $(i = 1, 2, \ldots, m,\ j = 0, 1, \ldots, n)$ we obtain the unknown function $\phi(t)$ ultimately.

3. Quadrature Formulae

3.1. General Idea

We often want to calculate the integral in (7) when we use our method for integral equations. We give a method of construction of quadrature formulae for the calculation of integral in (7). The idea of quadrature formulae is to find weights $\omega_i^{(n,m)}$ and abscissae $t_i^{(n,m)}$ such that:

$$\begin{aligned}
\int_a^b f(t)Bb_{ij}(t)dt = & \int_{a_{i-1}}^{a_i} f(t) \binom{n}{j} \frac{(t-a_{i-1})^j (a_i - t)^{n-j}}{(a_i - a_{i-1})^n} \, dt \\
= & \frac{a_i - a_{i-1}}{b-a} \int_a^b f\left(\frac{a_i - a_{i-1}}{b-a}(t-a) + a_{i-1} \right) B_{j,n}(t) \, dt \\
& \simeq Q_r^{(i,j)}[f(t)] := \sum_{k=0}^{r-1} \omega_k^{(i,j)} f(t_k^{(i,j)}). \qquad (18)
\end{aligned}$$

Set

$$\mathcal{M}_p^{(j,n)} = \int_a^b t^p B_{j,n}(t)dt \qquad (19)$$

Then, we have:

$$\begin{aligned}
\int_a^b t^p Bb_{i,j}(t)dt = & \int_{a_{i-1}}^{a_i} t^p Bb_{ij}(t)dt \\
= & \frac{a_i - a_{i-1}}{b-a} \int_a^b \left(\frac{a_i - a_{i-1}}{b-a}(t-a) + a_{i-1} \right)^p B_{j,n}(t)dt \\
= & \left(\frac{a_i - a_{i-1}}{b-a} \right)^{p+1} \int_a^b \left(t + \frac{ba_{i-1} - aa_i}{a_i - a_{i-1}} \right)^p B_{j,n}(t)dt \\
= & \left(\frac{a_i - a_{i-1}}{b-a} \right)^{p+1} \int_a^b \sum_{l=0}^{p} \binom{p}{l} \left(\frac{ba_{i-1} - aa_i}{a_i - a_{i-1}} \right)^{p-l} t^l B_{j,n}(t)dt \\
= & \left(\frac{a_i - a_{i-1}}{b-a} \right)^{p+1} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{ba_{i-1} - aa_i}{a_i - a_{i-1}} \right)^{p-l} \mathcal{M}_l^{(j,n)} \qquad (20)
\end{aligned}$$

For $j = 0, 1, \cdots, n$ and $i = 1, 2, \cdots, m$ taking $\left\{ t_k^{(i,j)} \right\}_{k=0}^{s}$ such that $t_k^{(i,j)} \in [a_{i-1}, a_i]$, by (18) and (20), for $p = 0, 1, \cdots, s$, we can solve the following linear equations:

$$\sum_{k=0}^{s} \omega_k^{(i,j)} (t_k^{(i,j)})^p = \left(\frac{a_i - a_{i-1}}{b-a} \right)^{p+1} \sum_{l=0}^{p} \binom{p}{l} \left(\frac{ba_{i-1} - aa_i}{a_i - a_{i-1}} \right)^{p-l} \mathcal{M}_l^{(j,n)} \qquad (21)$$

to find $\omega_i^{(n,m)}$.

So, we can get nm quadrature formulae whose degree of accuracy is s. More efficient quadrature formulae can be constructed by also treating the abscissae $\left\{ t_k^{(i,j)} \right\}_{k=0}^{s}$ as unknowns, cf. Gauss quadrature formulae.

3.2. Calculation of $\mathcal{M}_p^{(j,n)}$

We know that the Bernstein polynomials satisfy the following conditions:

$B_{j,n}(a) = B_{j,n}(b) = 0,\ j = 1,2,\dots,n-1.$

$B_{j,n}(t) = \dfrac{b-t}{b-a}B_{j-1,n}(t) + \dfrac{t-a}{b-a}B_{j-1,n-1}(t),\ n = 1,2,\dots,\ j = 1,2,\dots,n-1.$

$\dfrac{dB_{j,n}(t)}{dt} = \dfrac{n}{b-a}(B_{j-1,n-1}(t) - B_{j,n-1}(t)),\ n = 1,2,\dots,\ j = 1,2,\dots,n-1.$

So, we have:

$$\begin{aligned}\mathcal{M}_p^{(j,n)} &= \int_a^b t^p \left(\frac{b-t}{b-a}B_{j-1,n}(t) + \frac{t-a}{b-a}B_{j-1,n-1}(t)\right) dt\\ &= \frac{b}{b-a}\int_a^b t^p B_{j-1,n}(t)dt - \frac{1}{b-a}\int_a^b t^{p+1}B_{j-1,n}(t)dt\\ &\quad + \frac{1}{b-a}\int_a^b t^{p+1}B_{j-1,n-1}(t)dt - \frac{a}{b-a}\int_a^b t^p B_{j-1,n-1}(t)dt\\ &= \frac{b}{b-a}\mathcal{M}_p^{(j-1,n)} - \frac{1}{b-a}\mathcal{M}_{p+1}^{(j-1,n)} + \frac{1}{b-a}\mathcal{M}_{p+1}^{(j-1,n-1)}\\ &\quad - \frac{a}{b-a}\mathcal{M}_p^{(j-1,n-1)},\end{aligned}$$

and

$$\mathcal{M}_{p-1}^{(j,n)} = \frac{n}{p(b-a)}\big(\mathcal{M}_p^{(j,n-1)} - \mathcal{M}_p^{(j-1,n-1)}\big). \tag{22}$$

4. Illustrative Examples

Here we illustrate the above mentioned methods using eight illustrative examples, which include two first kind and two second kind Volterra integral equations with regular kernels and three first kind and one second kind Volterra integral equation with weakly singular kernels.

We solve examples 1, 2 and 4 using **BP** with $n = 7$, and **HBB** with $m = 4$ and $n = 2$, example 3 using **BP** with $n = 13$, and **HBB** with $m = 4$ and $n = 3$, examples 5a and 7 using **BP** with $n = 10$, and **HBB** with $m = 4$ and $n = 3$ and examples 5b and 6 using **BP** with $n = 5$, and **HBB** with $m = 4$ and $n = 2$. Table 1 shows the obtained results.

Example 1. We consider the Volterra integral equation of the first kind given by,

$$\int_0^x \frac{\phi(t)}{x^2+t^2}dt = x, \qquad 0 < x < 1, \tag{23}$$

which has the exact solution $\phi(x) = \frac{4}{4-\pi}x^2$.

Example 2. We consider another first kind integral equation with a regular kernel given by,

$$\int_0^x \frac{\phi(t)}{\sqrt{x^2+t^2}}dt = x, \qquad 0 < x < 1, \tag{24}$$

whose exact solution is $\phi(x) = \frac{x}{\sqrt{2}-1}$.

Example 3. We consider the second kind Volterra integral equation,

$$\phi(x) + \int_0^x (t^2 - 3x^3)\phi(t)dt = \frac{1}{4}[4x^3 + x - 1]\exp(-2x) + \frac{3}{8}[1 - 2x^2], \qquad 0 < x < 1, \tag{25}$$

which has the exact solution $\phi(x) = xe^{-2x}$. ([15]).

Example 4. We consider another Volterra integral equation of the second kind given by,

$$\phi(x) + \int_0^x \frac{1+x}{1+t}\phi(t)dt = 1 - x - \frac{3}{2}x^2 + \frac{x^3}{2}, \qquad 0 < x < 1, \tag{26}$$

having the exact solution $\phi(x) = 1 - x^2$. ([15]).

Example 5. Here we consider the Abel integral equation given by,

$$\int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}}dt = x^r, \qquad 0 < x < 1, \tag{27}$$

where r is any positive number. This is a first kind Volterra integral equation with weak singularity. The exact solution of the integral equation (3.9) is given by,

$$\phi(x) = \frac{2^{2r-1}}{\pi}r\frac{(\Gamma(r))^2}{\Gamma(2r)}x^{r-\frac{1}{2}}.$$

(a) For $r = 5$ the exact solution is $\phi(x) = \frac{1280}{315\pi}x^{\frac{9}{2}}$.

(b) For $r = \frac{3}{2}$ exact solution for $\phi(x)$ is obtained as $\phi(x) = \frac{3}{4}x$.

Example 6. Here we consider the integral equation given by

$$\int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}}dt = \frac{x^{\frac{1}{2}}}{2}[{}_1\Gamma_1(1, \frac{3}{2}; ix) +_1 \Gamma_1(1, \frac{3}{2}; -ix)], \qquad 0 < x < 1, \tag{28}$$

i here is imaginary unit, and ${}_1\Gamma_1(a, b; z)$ is the hypergeometric function. Then (3.11) has the exact solution given by $\phi(x) = \cos(x)$. ([16], pp. 424).

Example 7. In this example we consider a second kind weakly singular Volterra integral equation given by,

$$\phi(x) - \int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}} dt = x^7 (1 - \frac{4096}{6435} x^{\frac{1}{2}}), \qquad 0 < x < 1, \tag{29}$$

which has the exact solution $\phi(x) = x^7$.

	$\|\phi(x) - \phi_B(x)\|_\infty$	$\|\phi(x) - \phi_{Bb}(x)\|_\infty$
Ex. 1	5×10^{-10}	5×10^{-13}
Ex. 2	4×10^{-11}	5×10^{-14}
Ex. 3	2×10^{-12}	6×10^{-14}
Ex. 4	3×10^{-9}	8×10^{-12}
Ex. 5a	7×10^{-7}	3×10^{-10}
Ex. 5b	8×10^{-14}	5×10^{-16}
Ex. 6	2×10^{-7}	9×10^{-9}
Ex. 7	4×10^{-7}	2×10^{-10}

Table 1: B is **BP** and Bb is **HBB**.

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