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# FOURTH ORDER DIAGONALLY IMPLICIT MULTISTEP BLOCK METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

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**Abstract:** A fourth order diagonally implicit multistep block method is introduced to approximate the solution of fuzzy differential equations (FDEs). The problem is interpreted by using Seikkala's derivative. This method approximates two points simultaneously in a block along the interval. The Lagrange interpolating polynomial is applied in the formation of the formulas. The stability and convergence of this method at each computation points are given. Numerical solutions of this method are compared with the Runge-Kutta method of order four (RK(4)). The numerical results are given to highlight the performance of the proposed method when solving FDEs.

**Key Words:** block method, fuzzy differential equations, lower triangular matrix

## 1. Introduction

The first-order linear FDEs occurred in many real-world applications such as engineering [1], medicine (see [2], [3]), finance [4] and population models [5]. The problems are permeating with uncertainty. FDEs is a theory of differential equations (DEs) that consist initial values of fuzzy numbers. This leads to a fuzzy initial-value problem (FIVP). Kaleva [6] and Seikkala [7] handle the FIVP. The FIVP provides initial condition which is the solution to the FDEs. There are a few derivatives of fuzzy functions to the FIVP whereby Buckley

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and Feuring [8] have generalized it. Among widely used derivative is Seikkala's derivative. Under this interpretation, the existence and uniqueness of the solutions are considered. For more information, refer to [6] and [7].

Since the exact solution is complicated to obtain, a method for approximating the solutions are used. The earliest numerical method for solving this problem is the Euler method. Ma et al. [9] introduce it and has contributed significantly in numerical area. The FDEs is solved numerically whereby a new system of ordinary differential equations is formed after substituted by its parametric form. Meanwhile, Abbasbandy and Viranloo [10] apply the fourth order RK method to solve FDEs while Palligkinis et al. [11] propose RK method for a generalized problem and show the convergence of the method. Ghazanfari and Shakerami [12] present the extended RK-like formulae to increase the accuracy of the solutions. Meanwhile, Allahviranloo et al. (see [13], [14], [15]) use predictor-corrector method for solving FDEs by considering the Adams methods. Mehrkanoon et al. [16] and Zawawi et al. [17] propose block methods to solve FDEs. However, the convergences of the methods are not being discussed.

In Bede [18], the FDEs is able to be converted into a ODEs system by using characterization theorem. This theorem shows that the FDEs and the ODEs are similar under specific conditions. Therefore, any numerical method to solve the ODEs can be used.

Recent works with the block methods to solve ODEs are done by Majid and Suleiman [19], and Ibrahim et al. [20]. In this paper, a combination of predictor and corrector formulas in the form of block is emphasized. A 2-point 1 block fourth order diagonally implicit multistep method is proposed to solve the FDEs based on Seikkala's derivative. A constant step size is being considered. The aim is the implementation of the block method in order to obtain accurate approximate solutions under this interpretation. This method has advantages such as less function evaluations number, total steps and execution times. The convergent of the block method based on FDEs is proven.

The paper is organized as follows: In Section 2, some basic definitions are reviewed while in Section 3, the FIVP is defined. The derivation of the block method is shown in Section 4 and the implementation of the block method for FIVP is presented Section 5. The results of this numerical method are discussed in section 6. The final section is the conclusion.

## 2. Preliminaries

In this section, some definitions for the fuzzy numbers are reviewed. Further information; refer Xu et al. [21].

**Definition 1.** Let $R$ denotes the set of all real numbers. A fuzzy number is a fuzzy set $u : R \to [0,1]$ with the following properties:

a. $u$ is upper semi continuous,

b. $u$ is fuzzy convex, i.e., $u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}$ for all $x_1, x_2 \in R, \lambda \in [0,1]$,

c. $u$ is normal, i.e., $\exists x \in R$ for which $u(x) = 1$,

d. supp $u = \{x \in R | u(x) > 0\}$ is the support of the $u$, and its closure $cl$(supp $u$) is compact.

**Definition 2.** Let $E$ be the set of all fuzzy numbers on $R$. The $r$-level set of a fuzzy number $u \in E, 0 \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R | u(x) > 0\} & \text{if } 0 < r \leq 1, \\ cl(\text{supp } u) & \text{if } r = 0. \end{cases}$$

The $r$-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r), \overline{u}(r)$ refer to the lower bound and the upper bound of $[u]_r$.

On behalf of $u, v \in E$ and $\lambda \in R$, the sum $u + v$ and the product $\lambda \odot u$ are denoted by $[u+v]_r = [u]_r + [v]_r; [\lambda \odot u]_r = \lambda [u]_r, \forall r \in [0,1]$ where $[u]_r + [v]_r$ represents the addition of two intervals (subsets) of $R$ and $\lambda [u]_r$ is the usual product between a scalar and a subset of $R$.

The distance between two fuzzy numbers is known as the Hausdorff distance, given by $D : E \times E \to R_+ \bigcup 0$,

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}.$$

$D$ is a metric in $E$ and has the following conditions [22]:

a. $D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in E,$

b. $D(k \odot u, k \odot v) = |k| D(u,v), \quad \forall k \in R, \quad u, v \in E,$

c. $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \quad \forall u, v, w, e \in E,$

d. $(D, E)$ is a complete metric space.

**Definition 3.** [23] Let $f : R \to E$ is called a fuzzy function. If for arbitrary fixed $t_0 \in R$ and $\varepsilon > 0, \delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon,$$

$f(t)$ is said to be continuous.

**Definition 4.** [12] A mapping $y : I \to E$ is called a fuzzy process. The parametric form of $y(t)$ is denoted by

$$|y(t)|_r = [\underline{y}(t; r), \overline{y}(t; r)], t \in I, r \in (0, 1],$$

where $I$ is a real interval. The Seikkalas derivative of $y'(t)$ is defined

$$|y'(t)|_r = [\underline{y'}(t; r), \overline{y'}(t; r)], t \in I, r \in (0, 1],$$

provided that this equation defines a fuzzy number $y'(t) \in E$.

A first-order FIVP is shown below

$$\begin{aligned} y'(t) &= f(t, y(t)), t \in [t_0, T], \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

where $y$ indicates a fuzzy function of $t$, $y_0$ is fuzzy number, $y'$ is the fuzzy derivative of $y$ while $f(t, y)$ is a fuzzy function of crisp variable $t$ and fuzzy variable of $y$. By characterization theorem, the FDEs is able to be translated into ODEs system.

**Theorem 5.** *(Characterization Theorem) Consider the FIVP (1), where $f : [a, b] \times E \to E$ such that*

a. $[f(y)]_r = [\underline{f}(y(t; r)), \overline{f}(y(t; r))]$,

b. *$\underline{f}$ and $\overline{f}$ are equicontinuous (that is, for any $\varepsilon > 0$ and any $(t, y) \in [a, b] \times E$ such that $|\underline{f}(y(t; r)) - \underline{f}(y(t_1; r))| < \varepsilon$ and $|\overline{f}(y(t; r)) - \overline{f}(y(t_1; r))| < \varepsilon$ for all $r \in [0, 1]$, whenever $(t, y), (t_1, y_1) \in [a, b] \times R^2$ and $||(t, y) - (t_1, y_1)|| < \delta$ and uniformly bounded on any bounded set,*

c. *there exits an $L > 0$ such that*

$$|\underline{f}(t; y) - \underline{f}(t_1; y_1)| \le L|y - y_1| \text{ for all } r \in [0, 1],$$
$$|\overline{f}(t; y) - \overline{f}(t_1; y_1)| \le L|y - y_1| \text{ for all } r \in [0, 1].$$

*Then the FIVP and the systems of ODEs are equivalent. The equivalence means that each solution of FDEs is a system of ODEs and vice versa.*

$$\begin{cases} \underline{y}'(t;r) = \underline{f}(y(t,r)), \\ \overline{y}'(t;r) = \overline{f}(y(t,r)), \\ \underline{y}(0;r) = \underline{y}_0(r), \\ \overline{y}(0;r) = \overline{y}_0(r) \end{cases} \tag{2}$$

See Bede [18] and Dizicheh et al. [24] for further details.

## 3. Fuzzy Initial Value Problem

Based in [12], the FIVP (1) and Zadehs extension principle cause to the following definition of $f(t, y(t))$ when $y = y(t)$ is a fuzzy number

$$f(t,y)(s) = \sup\{y(\tau)\backslash s = f(t,\tau)\}, s \in R.$$

That is

$$[f(y(t))]_r = [\underline{f}(y;r), \overline{f}(y;r)], r \in (0,1],$$

where

$$\begin{aligned} \underline{f}(y;r) &= \min\{f(u)|u \in [\underline{y}(t;r), \overline{y}(t;r)]\}, \\ \overline{f}(y;r) &= \max\{f(u)|u \in [\underline{y}(t;r), \overline{y}(t;r)]\}. \end{aligned} \tag{3}$$

The mapping of $f(y)$ is a fuzzy function. Meanwhile the Seikkala's derivative is defined by

$$[f'(y(t))]_r = [\underline{f}'(y;r), \overline{f}'(y;r)], t \in I, r \in (0,1],$$

whereby this validate the fuzzy number $f'(y) \in E$, such that

$$\begin{aligned} \underline{f}'(y;r) &= \min\{f'(u)|u \in [\underline{y}(t;r), \overline{y}(t;r)]\}, \\ \overline{f}'(y;r) &= \max\{f'(u)|u \in [\underline{y}(t;r), \overline{y}(t;r)]\}. \end{aligned} \tag{4}$$

$f$ satisfies the Lipschitz condition since it is the condition for the existence of a unique solution to (1).

$$||f(y) - f(z)|| \le L||y - z||, L > 0. \tag{5}$$

Therefore the FIVP (1) has a unique solution. The proof can be found in [7].

## 4. Derivation of Block Method

The proposed block method is a numerical method that is based on multistep formulas. The block method has the advantage to estimate solutions more than one point at a time. According to [19], the IVP consists of interval $[a, b]$

$$y' = f(t, y), y(a) = y_0, t \in [a, b] \tag{6}$$

is able to split into a series of block. The proposed method is a two-point one block diagonally implicit multistep method; hence, each block contains two points. The two-point one block method estimated the solution for $y_{n+1}$ and $y_{n+2}$ at the points $t_{n+1}$ and $t_{n+2}$ through moving two points in a single block. As a multistep method, it requires more than one back values. The formulas for

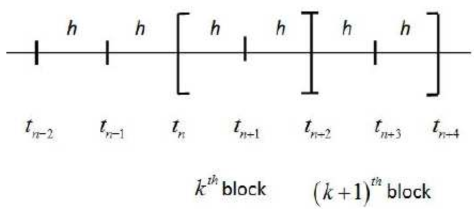


Figure 1: 2-point Block Method

diagonally implicit multistep block method are formulated by using Lagrange interpolating polynomial. The values for $y_{n+1}$ at interpolation points of $(t_{n-1}, f_{n-1})$, $(t_n, f_n)$, $(t_{n+1}, f_{n+1})$ and $y_{n+2}$ at interpolation points of $(t_{n-1}, f_{n-1})$, $(t_n, f_n)$, $(t_{n+1}, f_{n+1})$, $(t_{n+2}, f_{n+2})$ are interpolated. Based from (6), $y' = f(t, y)$ is integrated and will produce

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y)dt. \tag{7}$$

Lagrange interpolating polynomial is used to replace the function $f(t, y)$ while the integral is evaluated with $s = \dfrac{t - t_{n+1}}{h}$, $dt = hds$ and taking the integration limit from $-1$ to $0$ in (7). This will generate corrector formula for $y_{n+1}$,

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}) \tag{8}$$

Meanwhile, by considering $s = \dfrac{t - t_{n+2}}{h}$, $dx = hds$ and taking the integration limit from $-2$ to $0$ in (7) will generate corrector formula for $y_{n+2}$,

$$y_{n+2} = y_n + \frac{h}{90}(29f_{n+2} + 124f_{n+1} + 24f_n + 4f_{n-1} - f_{n-2}). \tag{9}$$

The derivation for predictor formulas are the same to that of corrector formulas and the order is one less. The formulas (8) and (9) are writable in matrix form such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h\begin{bmatrix} \frac{9}{24} & 0 \\ \frac{124}{90} & \frac{29}{90} \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$
$$+ h\begin{bmatrix} -\frac{5}{24} & \frac{19}{24} \\ \frac{4}{90} & \frac{24}{90} \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h\begin{bmatrix} 0 & \frac{1}{24} \\ 0 & -\frac{1}{90} \end{bmatrix}\begin{bmatrix} f_{n-3} \\ f_{n-2} \end{bmatrix} \tag{10}$$

From (10), the lower triangular matrix is formed, therefore the method is called diagonally implicit.

Based on (8), the error coefficient for $y_{n+1}$ is

$$\begin{aligned} C_q &= \sum_{j=0}^{k}\left(\frac{j^q\alpha_j}{q!} - \frac{j^{q-1}\beta_j}{(q-1)!}\right) \\ C_1 &= (0) \\ &\vdots \\ C_5 &= \left(-\frac{19}{720}\right). \end{aligned} \tag{11}$$

Since $C_5 \neq 0$, the $y_{n+1}$ is order four. Meanwhile, based on (9), the error coefficient for $y_{n+2}$ is

$$\begin{aligned} C_q &= \sum_{j=0}^{k} \left( \frac{j^q \alpha_j}{q!} - \frac{j^{q-1}\beta_j}{(q-1)!} \right) \\ C_1 &= (0) \\ &\vdots \\ C_6 &= \left(-\frac{1}{90}\right). \end{aligned} \tag{12}$$

Given that $C_6 \neq 0$, the $y_{n+2}$ is order five. Further information, refer Lambert [25]. Consequently, the block method is order four since it will take the smallest order.

## 5. Implementation of Block Method for Solving FIVP

In this section, the implementation of the block method for FIVP is given. Consider the FIVP (1), where $f$ is a continuous mapping from $E$ into $E$ and $y_0 \in E$ by means of $r$-level sets

$$[y_0]_r = [\underline{y}(0;r), \overline{y}(0;r)], r \in (0,1]. \tag{13}$$

The interval $[0,T]$ is replaced by a set of grid points $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$. The exact solution which is

$$[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)] \tag{14}$$

is approximated by

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]. \tag{15}$$

The grid points in which the solutions are calculated are $h = \dfrac{T - t_0}{N}$, $t_n = t_0 + nh$ where $(0 \leq n \leq N)$. The exact and approximate solutions at $t_n$ are defined by

$$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)] \tag{16}$$

and

$$[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]. \tag{17}$$

From the block method formulas (8) and (9), the fuzzy two-point one block fourth order diagonally implicit multistep method for the exact and approximate solutions are able to be denoted as

$$\begin{aligned}\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\underline{Y}(t_{n+1};r)) \\ &+ 19f(t_n,\underline{Y}(t_n;r)) - 5f(t_{n-1},\underline{Y}(t_{n-1};r)) \\ &+ f(t_{n-2},\underline{Y}(t_{n-2};r))],\end{aligned} \tag{18}$$

$$\begin{aligned}\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\overline{Y}(t_{n+1};r)) \\ &+ 19f(t_n,\overline{Y}(t_n;r)) - 5f(t_{n-1},\overline{Y}(t_{n-1};r)) \\ &+ f(t_{n-2},\overline{Y}(t_{n-2};r))],\end{aligned} \tag{19}$$

$$\begin{aligned}\underline{Y}(t_{n+2};r) = \underline{Y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\underline{Y}(t_{n+2};r)) \\ &+ 124f(t_{n+1},\underline{Y}(t_{n+1};r)) + 24f(t_n,\underline{Y}(t_n;r)) \\ &+ 4f(t_{n-1},\underline{Y}(t_{n-1};r)) - f(t_{n-2},\underline{Y}(t_{n-2};r))],\end{aligned} \tag{20}$$

$$\begin{aligned}\overline{Y}(t_{n+2};r) = \overline{Y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\overline{Y}(t_{n+2};r)) \\ &+ 124f(t_{n+1},\overline{Y}(t_{n+1};r)) + 24f(t_n,\overline{Y}(t_n;r)) \\ &+ 4f(t_{n-1},\overline{Y}(t_{n-1};r)) - f(t_{n-2},\overline{Y}(t_{n-2};r))],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\underline{y}(t_{n+1};r)) \\ &+ 19f(t_n,\underline{y}(t_n;r)) - 5f(t_{n-1},\underline{y}(t_{n-1};r)) \\ &+ f(t_{n-2},\underline{y}(t_{n-2};r))],\end{aligned} \tag{22}$$

$$\begin{aligned}\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\overline{y}(t_{n+1};r)) \\ &+ 19f(t_n,\overline{y}(t_n;r)) - 5f(t_{n-1},\overline{y}(t_{n-1};r)) \\ &+ f(t_{n-2},\overline{y}(t_{n-2};r))],\end{aligned} \tag{23}$$

$$\begin{aligned}\underline{y}(t_{n+2};r) = \underline{y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\underline{y}(t_{n+2};r)) \\ &+ 124f(t_{n+1},\underline{y}(t_{n+1};r)) + 24f(t_n,\underline{y}(t_n;r)) \\ &+ 4f(t_{n-1},\underline{y}(t_{n-1};r)) - f(t_{n-2},\underline{y}(t_{n-2};r))],\end{aligned} \tag{24}$$

$$\begin{aligned}\overline{y}(t_{n+2};r) = \overline{y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\overline{y}(t_{n+2};r)) \\ &+ 124f(t_{n+1},\overline{y}(t_{n+1};r)) + 24f(t_n,\overline{y}(t_n;r)) \\ &+ 4f(t_{n-1},\overline{y}(t_{n-1};r)) - f(t_{n-2},\overline{y}(t_{n-2};r))].\end{aligned} \tag{25}$$

The following algorithm is based on using the two-point one block diagonally implicit method.

**Algorithm.**

To approximate the solution of the FIVP

$$\begin{aligned}&y'(t) = f(t,y), a \le t \le b,\\ &y(a) = \eta.\end{aligned}$$

An arbitrary positive $N$ is chosen. The initial value, $y_0 = \eta_0$ is obtained from the FIVP and starting point, $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ are obtained by RK(4) where $t_n = a + nh$.

*Step 1.* Set $h = \dfrac{b-a}{N}$,
$w(t_0) = \eta_0, \quad w(t_1) = \eta_1, \quad w(t_2) = \eta_2, \quad w(t_3) = \eta_3,$
$w(t_4) = \eta_4, \quad w(t_5) = \eta_5, \quad w(t_6) = \eta_6.$
*Step 2.* Set $n = 1$.
*Step 3.* Set $t_{n+1} = t_0 + nh$ and $t_{n+2} = t_0 + nh$
*Step 4.* Set

$$\begin{aligned}\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\underline{y}(t_{n+1};r)) + 19f(t_n,\underline{y}(t_n;r)) \\ &- 5f(t_{n-1},\underline{y}(t_{n-1};r)) + f(t_{n-2},\underline{y}(t_{n-2};r))],\end{aligned}$$

$$\begin{aligned}\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{h}{24}[&9f(t_{n+1},\overline{y}(t_{n+1};r)) + 19f(t_n,\overline{y}(t_n;r)) \\ &- 5f(t_{n-1},\overline{y}(t_{n-1};r)) + f(t_{n-2},\overline{y}(t_{n-2};r))],\end{aligned}$$

$$\begin{aligned}\underline{y}(t_{n+2};r) = \underline{y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\underline{y}(t_{n+2};r)) + 124f(t_{n+1},\underline{y}(t_{n+1};r)) \\ &+ 24f(t_n,\underline{y}(t_n;r)) + 4f(t_{n-1},\underline{y}(t_{n-1};r)) - f(t_{n-2},\underline{y}(t_{n-2};r))],\end{aligned}$$

$$\begin{aligned}\overline{y}(t_{n+2};r) = \overline{y}(t_n;r) + \frac{h}{90}[&29f(t_{n+2},\overline{y}(t_{n+2};r)) + 124f(t_{n+1},\overline{y}(t_{n+1};r)) \\ &+ 24f(t_n,\overline{y}(t_n;r)) + 4f(t_{n-1},\overline{y}(t_{n-1};r)) - f(t_{n-2},\overline{y}(t_{n-2};r))].\end{aligned}$$

*Step 5.* $n = n + 2$.

*Step 6.* If $t_n < T - h$, go to *Step 3.*

*Step 7.* Set

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{h}{24}[9f(t_{n+1},\underline{y}(t_{n+1};r)) + 19f(t_n,\underline{y}(t_n;r)) - 5f(t_{n-1},\underline{y}(t_{n-1};r)) + f(t_{n-2},\underline{y}(t_{n-2};r))],$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{h}{24}[9f(t_{n+1},\overline{y}(t_{n+1};r)) + 19f(t_n,\overline{y}(t_n;r)) - 5f(t_{n-1},\overline{y}(t_{n-1};r)) + f(t_{n-2},\overline{y}(t_{n-2};r))],$$

*Step 8.* End.

In order to show the convergence of these approximates

$$\lim_{h\to 0} \underline{y}(t;r) = \underline{Y}(t;r),$$
$$\lim_{h\to 0} \overline{y}(t;r) = \overline{Y}(t;r),$$

the following lemma will be considered.

**Lemma 6.** [13] Let a sequence of numbers $\{w_n\}_{n=0}^{N}$ satisfy:

$$|w_{n+1}| \le A|w_n| + B|w_{n-1}| + C, \quad 0 \le n \le N-1$$

for some given positive constants $A$, $B$, and $C$. Then

$$|w_n| \le \left(A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \ldots + \beta_s B^{\left[\frac{n}{2}\right]}\right)|w_1| + \left(A^{n-2}B + \gamma_1 A^{n-4}B^2 + \ldots + \gamma_t AB^{\left[\frac{n}{2}\right]}\right)|w_0| + \left(A^{n-2} + A^{n-3} + \ldots + 1\right)C + \left(\delta_1 A^{n-4} + \delta_2 A^{n-5} + \ldots + \delta_m A + 1\right)BC + \left(\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \ldots + \zeta_l A + 1\right)B^2C + \left(\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \ldots + \lambda_p A + 1\right)B^3C + \ldots,$$

when $n$ is odd and

$$|w_n| \le \left(A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \ldots + \beta_s AB^{\left[\frac{n}{2}\right]-1}\right)|w_1| + \left(A^{n-2}B + \gamma_1 A^{n-4}B^2 + \ldots + \gamma_t B^{\left[\frac{n}{2}\right]}\right)|w_0| + \left(A^{n-2} + A^{n-3} + \ldots + 1\right)C + \left(\delta_1 A^{n-4} + \delta_2 A^{n-5} + \ldots + \delta_m A + 1\right)BC + \left(\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \ldots + \zeta_l A + 1\right)B^2C + \left(\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \ldots + \lambda_p A + 1\right)B^3C + \ldots,$$

when $n$ is even, where $\beta_s, \gamma_t, \delta_m, \zeta_l, \lambda_p$ are constants for all $s$, $t$, $m$, $l$ and $p$.

The proof can be done by mathematical induction.

**Theorem 7.** *For arbitrary fixed $r : 0 \leq r \leq 1$, the corrector of the block method approximates of (22) and (23) converge to the exact solutions $\underline{Y}(t;r), \overline{Y}(t;r)$ for $\underline{Y}, \overline{Y} \in C^5[t_0, T]$.*

*Proof.* It is ample to show

$$\lim_{h\to 0} \underline{y}(t;r) = \underline{Y}(t;r),$$
$$\lim_{h\to 0} \overline{y}(t;r) = \overline{Y}(t;r).$$

With the exact value, the following results will be yielded

$$\begin{aligned}\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) &+ \frac{9h}{24} f(t_{n+1}, \underline{Y}(t_{n+1};r)) + \frac{19h}{24} f(t_n, \underline{Y}(t_n;r)) \\ &- \frac{5h}{24} f(t_{n-1}, \underline{Y}(t_{n-1};r)) + \frac{h}{24} f(t_{n-2}, \underline{Y}(t_{n-2};r)) - \frac{19}{720} h^5 \underline{Y}^5(\xi_n),\end{aligned}$$

$$\begin{aligned}\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) &+ \frac{9h}{24} f(t_{n+1}, \overline{Y}(t_{n+1};r)) + \frac{19h}{24} f(t_n, \overline{Y}(t_n;r)) \\ &- \frac{5h}{24} f(t_{n-1}, \overline{Y}(t_{n-1};r)) + \frac{h}{24} f(t_{n-2}, \overline{Y}(t_{n-2};r)) - \frac{19}{720} h^5 \overline{Y}^5(\xi_n),\end{aligned}$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$. Consequently

$$\begin{aligned}\underline{Y}(t_{n+1};r) - \underline{y}(t_{n+1};r) = \underline{Y}(t_n;r) &- \underline{y}(t_n;r) \\ &+ \frac{9h}{24}\{f(t_{n+1}, \underline{Y}(t_{n+1};r)) - f(t_{n+1}, \underline{y}(t_{n+1};r))\} \\ &+ \frac{19h}{24}\{f(t_n, \underline{Y}(t_n;r)) - f(t_n, \underline{y}(t_n;r))\} \\ &- \frac{5h}{24}\{f(t_{n-1}, \underline{Y}(t_{n-1};r)) - f(t_{n-1}, \underline{y}(t_{n-1};r))\} \\ &+ \frac{h}{24}\{f(t_{n-2}, \underline{Y}(t_{n-2};r)) - f(t_{n-2}, \underline{y}(t_{n-2};r))\} \\ &- \frac{19}{720} h^5 \underline{Y}^5(\xi_n),\end{aligned}$$

$$\begin{aligned}\overline{Y}(t_{n+1};r) - \overline{y}(t_{n+1};r) &= \overline{Y}(t_n;r) - \overline{y}(t_n;r)\\ &+ \frac{9h}{24}\{f(t_{n+1},\overline{Y}(t_{n+1};r)) - f(t_{n+1},\overline{y}(t_{n+1};r))\}\\ &+ \frac{19h}{24}\{f(t_n,\overline{Y}(t_n;r)) - f(t_n,\overline{y}(t_n;r))\}\\ &- \frac{5h}{24}\{f(t_{n-1},\overline{Y}(t_{n-1};r)) - f(t_{n-1},\overline{y}(t_{n-1};r))\}\\ &+ \frac{h}{24}\{f(t_{n-2},\overline{Y}(t_{n-2};r)) - f(t_{n-2},\overline{y}(t_{n-2};r))\}\\ &- \frac{19}{720}h^5\overline{Y}^5(\xi_n).\end{aligned}$$

Denote $w_n = \underline{Y}(t_n;r) - \underline{y}(t_n;r), v_n = \overline{Y}(t_n;r) - \overline{y}(t_n;r)$. Then

$$\begin{aligned}|w_{n+1}| &\le \left(1 + \frac{19h}{24}L_1\right)|w_n| + \frac{h}{24}L_2|w_{n-2}| - \frac{5h}{24}L_3|w_{n-1}|\\ &+ \frac{9h}{24}L_4|w_{n+1}| - \frac{19}{720}h^5\underline{M},\\ |v_{n+1}| &\le \left(1 + \frac{19h}{24}L_5\right)|v_n| + \frac{h}{24}L_6|v_{n-2}| - \frac{5h}{24}L_7|v_{n-1}|\\ &+ \frac{9h}{24}L_8|v_{n+1}| - \frac{19}{720}h^5\overline{M},\end{aligned}$$

where $\underline{M} = \max_{t_0 \le t \le T} |\underline{Y}^5(t_n;r)|$ and $\overline{M} = \max_{t_0 \le t \le T} |\overline{Y}^5(t_n;r)|$.

Take into account

$$L = \max\{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8\} < \frac{24}{h},$$

then

$$\begin{aligned}|w_{n+1}| &\le \left(\frac{24 + 19hL}{24 - 9hL}\right)|w_n| - \left(\frac{5hL}{24 - 9hL}\right)|w_{n-1}| + \left(\frac{hL}{24 - 9hL}\right)|w_{n-2}|\\ &- \left(\frac{19}{720 - 270hL}\right)h^5\underline{M},\\ |v_{n+1}| &\le \left(\frac{24 + 19hL}{24 - 9hL}\right)|v_n| - \left(\frac{5hL}{24 - 9hL}\right)|v_{n-1}| + \left(\frac{hL}{24 - 9hL}\right)|v_{n-2}|\\ &- \left(\frac{19}{720 - 270hL}\right)h^5\overline{M},\end{aligned}$$

are yielded, where $|u_n| = |w_n| + |v_n|$, followed and modified Lemma 6, as $w_0 =$

$v_0 = 0$, $w_1 = v_1 = 0$ and $w_2 = v_2 = 0$;

$$|u_n| \leq \frac{\left(\frac{24+19hL}{24-9hL}\right)^{n-1} - 1}{\frac{28hL}{24-9hL}} \times \frac{-19}{720-270hL} h^5 \left(\underline{M} + \overline{M}\right)$$
$$+ \left\{ \delta_1 \left(\frac{24+19hL}{24-9hL}\right)^{n-4} + \delta_2 \left(\frac{24+19hL}{24-9hL}\right)^{n-5} + \ldots + \delta_m \left(\frac{24+19hL}{24-9hL}\right) + 1 \right\}$$
$$\times \left(\frac{-5hL}{24-9hL}\right) \left(\frac{-19}{720-270hL} h^5 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \left\{ \zeta_1 \left(\frac{24+19hL}{24-9hL}\right)^{n-6} + \zeta_2 \left(\frac{24+19hL}{24-9hL}\right)^{n-7} + \ldots + \zeta_n \left(\frac{24+19hL}{24-9hL}\right) + 1 \right\}$$
$$\times \left(\frac{hL}{24-9hL}\right) \left(\frac{-19}{720-270hL} h^5 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \left\{ \lambda_1 \left(\frac{24+19hL}{24-9hL}\right)^{n-8} + \lambda_2 \left(\frac{24+19hL}{24-9hL}\right)^{n-9} + \ldots + \lambda_p \left(\frac{24+19hL}{24-9hL}\right) + 1 \right\}$$
$$\times \left(\frac{-5hL}{24-9hL}\right)^2 \left(\frac{-19}{720-270hL} h^5 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \ldots$$

are produced. If $h \to 0$ then $w_n \to 0$, $v_n \to 0$ which concludes proof. □

**Remarks 8.** From the theorem results, the convergence order is $O\left(h^6\right)$.

**Theorem 9.** *For arbitrary fixed $r : 0 \leq r \leq 1$, the corrector of the block method approximates of (24) and (25) converge to the exact solutions $\underline{Y}(t;r), \overline{Y}(t;r)$ for $\underline{Y}, \overline{Y} \in C^6[t_0, T]$.*

*Proof.* Similar to Theorem 7.

With the exact value, the following results will be yielded

$$\begin{aligned}
\underline{Y}(t_{n+2};r) &= \underline{Y}(t_n;r) + \frac{29h}{90} f(t_{n+2}, \underline{Y}(t_{n+2};r)) + \frac{124h}{90} f(t_{n+1}, \underline{Y}(t_{n+1};r)) \\
&+ \frac{24h}{90} f(t_n, \underline{Y}(t_n;r)) + \frac{4h}{90} f(t_{n-1}, \underline{Y}(t_{n-1};r)) - \frac{h}{90} f(t_{n-2}, \underline{Y}(t_{n-2};r)) \\
&- \frac{1}{90} h^6 \underline{Y}^6(\xi_n),
\end{aligned}$$

$$\begin{aligned}\overline{Y}(t_{n+2};r) = \overline{Y}(t_n;r) &+ \frac{29h}{90}f(t_{n+2},\overline{Y}(t_{n+2};r)) + \frac{124h}{90}f(t_{n+1},\overline{Y}(t_{n+1};r)) \\ &+ \frac{24h}{90}f(t_n,\overline{Y}(t_n;r)) + \frac{4h}{90}f(t_{n-1},\overline{Y}(t_{n-1};r)) - \frac{h}{90}f(t_{n-2},\overline{Y}(t_{n-2};r)) \\ &- \frac{1}{90}h^6\overline{Y}^6(\xi_n),\end{aligned}$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$. Consequently

$$\begin{aligned}\underline{Y}(t_{n+2};r) - \underline{y}(t_{n+2};r) = {} & \underline{Y}(t_n;r) - \underline{y}(t_n;r) \\ &+ \frac{29h}{90}\{f(t_{n+2},\underline{Y}(t_{n+2};r)) - f(t_{n+2},\underline{y}(t_{n+2};r))\} \\ &+ \frac{124h}{90}\{f(t_{n+1},\underline{Y}(t_{n+1};r)) - f(t_{n+1},\underline{y}(t_{n+1};r))\} \\ &+ \frac{24h}{90}\{f(t_n,\underline{Y}(t_n;r)) - f(t_n,\underline{y}(t_n;r))\} \\ &+ \frac{4h}{90}\{f(t_{n-1},\underline{Y}(t_{n-1};r)) - f(t_{n-1},\underline{y}(t_{n-1};r))\} \\ &- \frac{h}{90}\{f(t_{n-2},\underline{Y}(t_{n-2};r)) - f(t_{n-2},\underline{y}(t_{n-2};r))\} \\ &- \frac{1}{90}h^6\underline{Y}^6(\xi_n),\end{aligned}$$

$$\begin{aligned}\overline{Y}(t_{n+2};r) - \overline{y}(t_{n+2};r) = {} & \overline{Y}(t_n;r) - \overline{y}(t_n;r) \\ &+ \frac{29h}{90}\{f(t_{n+2},\overline{Y}(t_{n+2};r)) - f(t_{n+2},\overline{y}(t_{n+2};r))\} \\ &+ \frac{124h}{90}\{f(t_{n+1},\overline{Y}(t_{n+1};r)) - f(t_{n+1},\overline{y}(t_{n+1};r))\} \\ &+ \frac{24h}{90}\{f(t_n,\overline{Y}(t_n;r)) - f(t_n,\overline{y}(t_n;r))\} \\ &+ \frac{4h}{90}\{f(t_{n-1},\overline{Y}(t_{n-1};r)) - f(t_{n-1},\overline{y}(t_{n-1};r))\} \\ &- \frac{h}{90}\{f(t_{n-2},\overline{Y}(t_{n-2};r)) - f(t_{n-2},\overline{y}(t_{n-2};r))\} \\ &- \frac{1}{90}h^6\overline{Y}^6(\xi_n).\end{aligned}$$

Denote $w_n = \underline{Y}(t_n;r) - \underline{y}(t_n;r), v_n = \overline{Y}(t_n;r) - \overline{y}(t_n;r)$. Then

$$\begin{aligned}
|w_{n+2}| &\leq \frac{29h}{90}L_1|w_{n+2}| + \frac{124h}{90}L_2|w_{n+1}| + \left(1 + \frac{24h}{90}L_3\right)|w_n| \\
&+ \frac{4h}{90}L_4|w_{n-1}| - \frac{h}{90}L_5|w_{n-2}| - \frac{1}{90}h^6\underline{M}, \\
|v_{n+2}| &\leq \frac{29h}{90}L_6|v_{n+2}| + \frac{124h}{90}L_7|v_{n+1}| + \left(1 + \frac{24h}{90}L_8\right)|v_n| \\
&+ \frac{4h}{90}L_9|v_{n-1}| - \frac{h}{90}L_{10}|v_{n-2}| - \frac{1}{90}h^6\overline{M},
\end{aligned}$$

where $\underline{M} = \max_{t_0 \leq t \leq T} |\underline{Y}^6(t_n;r)|$ and $\overline{M} = \max_{t_0 \leq t \leq T} |\overline{Y}^6(t_n;r)|$.

Take into account

$$L = \max\{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}\} < \frac{90}{h},$$

then

$$\begin{aligned}
|w_{n+2}| &\leq \left(\frac{124hL}{90 - 29hL}\right)|w_{n+1}| + \left(\frac{90 + 24hL}{90 - 29hL}\right)|w_n| + \left(\frac{4hL}{90 - 29hL}\right)|w_{n-1}| \\
&- \left(\frac{hL}{90 - 29hL}\right)|w_{n-2}| - \left(\frac{1}{90 - 29hL}\right)h^6\underline{M}, \\
|v_{n+2}| &\leq \left(\frac{124hL}{90 - 29hL}\right)|v_{n+1}| + \left(\frac{90 + 24hL}{90 - 29hL}\right)|v_n| + \left(\frac{4hL}{90 - 29hL}\right)|v_{n-1}| \\
&- \left(\frac{hL}{90 - 29hL}\right)|v_{n-2}| - \left(\frac{1}{90 - 29hL}\right)h^6\overline{M},
\end{aligned}$$

are yielded, where $|u_n| = |w_n| + |v_n|$, followed and modified Lemma 6, as $w_0 =$

$v_0 = 0$, $w_1 = v_1 = 0$ and $w_2 = v_2 = 0$;

$$|u_n| \leq \frac{\left(\frac{124hL}{90-29hL}\right)^{n-1} - 1}{\frac{153hL-90}{90-29hL}} \times \frac{-1}{90-29hL} h^6 \left(\underline{M} + \overline{M}\right)$$
$$+ \left\{ \delta_1 \left(\frac{124hL}{90-29hL}\right)^{n-4} + \delta_2 \left(\frac{124hL}{90-29hL}\right)^{n-5} + \ldots + \delta_m \left(\frac{124hL}{90-29hL}\right) + 1 \right\}$$
$$\times \left(\frac{90+24hL}{90-29hL}\right) \left(\frac{-1}{90-29hL} h^6 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \left\{ \zeta_1 \left(\frac{124hL}{90-29hL}\right)^{n-6} + \zeta_2 \left(\frac{124hL}{90-29hL}\right)^{n-7} + \ldots + \zeta_n \left(\frac{124hL}{90-29hL}\right) + 1 \right\}$$
$$\times \left(\frac{4hL}{90-29hL}\right) \left(\frac{-1}{90-29hL} h^6 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \left\{ \lambda_1 \left(\frac{124hL}{90-29hL}\right)^{n-8} + \lambda_2 \left(\frac{124hL}{90-29hL}\right)^{n-9} + \ldots + \lambda_p \left(\frac{124hL}{90-29hL}\right) + 1 \right\}$$
$$\times \left(\frac{-hL}{90-29hL}\right) \left(\frac{-1}{90-29hL} h^6 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \left\{ \mu_1 \left(\frac{124hL}{90-29hL}\right)^{n-10} + \mu_2 \left(\frac{124hL}{90-29hL}\right)^{n-11} + \ldots + \mu_q \left(\frac{124hL}{90-29hL}\right) + 1 \right\}$$
$$\times \left(\frac{90+24hL}{90-29hL}\right)^2 \left(\frac{-1}{90-29hL} h^6 \left(\underline{M} + \overline{M}\right)\right)$$
$$+ \ldots$$

are obtained. If $h \to 0$ then $w_n \to 0$, $v_n \to 0$ which concludes proof. □

**Remarks 10.** From the theorem results, the convergence order is $O\left(h^7\right)$.

**Theorem 11.** *The block method is zero-stable.*

*Proof.* For the block method, the characteristic polynomial $\rho(r) = R^2 - R$ satisfies $|R| \leq 1$, therefore it is zero-stable. □

As a result, this method is convergent and stable.

## 6. Numerical Results

By using $PE(CE)^m$ mode, the fuzzy block method is adapted and $m$ is repeated twice for every step. Meanwhile, RK(4) is used as the initial point. The block method is examined numerically on FIVP. Two values of step sizes were used in the numerical results i.e. $h = 0.1$ and $h = 0.01$. There are three types of fuzzy numbers were tested such that triangular, trapezium and parallelogram in order to validate the performances of the proposed diagonally block method.

*Problem 1* [9]

$$y'(t) = y(t),$$
$$y(0) = (0.75 + 0.25r, 1.125 - 0.125r).$$

The exact solution at $t = 1$ is given by

$$Y(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], 0 \leq r \leq 1.$$

By Theorem 5, this is equivalent to the systems of ODEs

$$\begin{cases} \underline{y}'(t; r) = \underline{y}(t; r), t \in [0, 1] \\ \underline{y}(0; r) = 0.75 + 0.25r, 0 \leq r \leq 1 \\ \overline{y}'(t; r) = \overline{y}(t; r), t \in [0, 1] \\ \overline{y}(0; r) = 1.125 - 0.125r, 0 \leq r \leq 1 \end{cases}$$

*Problem 2* [26]

$$y'(t) = y(t),$$
$$y(0) = (0.8 + 0.125r, 1.1 - 0.1r).$$

The exact solution at $t = 1$ is given by

$$Y(1; r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], 0 \leq r \leq 1.$$

By Theorem 5, this is equivalent to the systems of ODEs

$$\begin{cases} \underline{y}'(t; r) = \underline{y}(t; r), t \in [0, 1] \\ \underline{y}(0; r) = 0.8 + 0.125r, 0 \leq r \leq 1 \\ \overline{y}'(t; r) = \overline{y}(t; r), t \in [0, 1] \\ \overline{y}(0; r) = 1.1 - 0.1r, 0 \leq r \leq 1 \end{cases}$$

*Problem 3* [17]

$$y'(t) = ty(t),$$
$$y(0) = (1.01 + 0.1r\sqrt{e}, 1.5 - 0.1r\sqrt{e}).$$

The exact solution at $t = 1$ is given by

$$Y(1;r) = \left[(1.01 + 0.1r\sqrt{e})e^{\frac{1}{2}}, (1.5 - 0.1r\sqrt{e})e^{\frac{1}{2}}\right], 0 \le r \le 1.$$

By Theorem 5, this is equivalent to the systems of ODEs

$$\begin{cases} \underline{y}'(t;r) = t\underline{y}(t;r), t \in [0,1] \\ \underline{y}(0;r) = 1.01 + 0.1r\sqrt{e}, 0 \le r \le 1 \\ \overline{y}'(t;r) = t\overline{y}(t;r), t \in [0,1] \\ \overline{y}(0;r) = 1.5 - 0.1r\sqrt{e}, 0 \le r \le 1 \end{cases}$$

Below are the notations used in the tabulated results:

| | | |
|---|---|---|
| $h$ | : | Step size |
| $m$ | : | Iteration numbers in $PE(CE)^m$ mode |
| $r$ | : | Fuzzy numbers with bounded $r$-level intervals |
| $\underline{y}(t;r), \overline{y}(t;r)$ | : | Lower bounded approximated solution, upper bounded approximated solution |
| TFE | : | Total number of function evaluations |
| TS | : | Total number of steps |
| RK(4) | : | Fourth order Runge-Kutta method |
| 2P1D1(4) | : | Fourth order two-point one block diagonally implicit multistep method |

Both block method and RK method are being studied. The absolute error is defined such that

$$|y(x_n) - y_n|$$

where $y(x_n)$ is the exact solution while $y_n$ is the approximated solution. Numerical results are presented and compared. The program is written in C language by using Microsoft Visual C++ platform.

In Table 1, at $h = 0.1$ and $h = 0.01$, the results for RK(4) is slightly better compared to 2P1DI(4). However, when the number of iteration increased to two, at both step sizes, the 2P1DI(4) yielded accurate results than the RK(4). The accuracy of 2P1DI(4) decreased in terms of error as the step size getting smaller. Furthermore, the 2P1DI(4) needs less number of function evaluations and less number of steps than the RK(4). The results showed that the 2P1DI(4) managed to reduce by almost half of the number of steps of the RK(4). Thus, the 2P1DI(4) is less in computational cost. From Figure 2, the results of the 2P1DI(4) when $m = 2$ are very close to the exact solutions.

Table 1: Comparison between 2P1DI(4) and RK(4) for Problem 1.

| | | | $h = 0.1$ | | | | $h = 0.01$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Method** | $\boldsymbol{m}$ | $\boldsymbol{r}$ | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** |
| RK(4) | 1 | 0.0 | 1.5632e-006, 2.3449e-006 | 400 | 100 | 0.1470 | 1.6848e-010, 2.5272e-010 | 4000 | 1000 | 0.4514 |
| | | 0.2 | 1.6675e-006, 2.2928e-006 | | | | 1.7972e-010, 2.4711e-010 | | | |
| | | 0.4 | 1.7717e-006, 2.2406e-006 | | | | 1.9095e-010, 2.4149e-010 | | | |
| | | 0.6 | 1.8759e-006, 2.1885e-006 | | | | 2.0218e-010, 2.3588e-010 | | | |
| | | 0.8 | 1.9801e-006, 2.1364e-006 | | | | 2.1341e-010, 2.3026e-010 | | | |
| | | 1.0 | 2.0843e-006, 2.0843e-006 | | | | 2.2464e-010, 2.2464e-010 | | | |
| 2P1DI(4) | 1 | 0.0 | 7.4595e-006, 1.1189e-005 | 230 | 70 | 0.1292 | 1.3327e-010, 1.9990e-010 | 1580 | 520 | 0.2404 |
| | | 0.2 | 7.9568e-006, 1.0941e-005 | | | | 1.4215e-010, 1.9545e-010 | | | |
| | | 0.4 | 8.4541e-006, 1.0692e-005 | | | | 1.5103e-010, 1.9101e-010 | | | |
| | | 0.6 | 8.9514e-006, 1.0443e-005 | | | | 1.5992e-010, 1.8657e-010 | | | |
| | | 0.8 | 9.4487e-006, 1.0195e-005 | | | | 1.6880e-010, 1.8213e-010 | | | |
| | | 1.0 | 9.9460e-006, 9.9460e-006 | | | | 1.7769e-010, 1.7769e-010 | | | |
| 2P1DI(4) | 2 | 0.0 | 3.3931e-008, 5.0897e-008 | 300 | 70 | 0.1446 | 4.4085e-012, 6.6152e-012 | 2550 | 520 | 0.2988 |
| | | 0.2 | 3.6193e-008, 4.9766e-008 | | | | 4.7033e-012, 6.4677e-012 | | | |
| | | 0.4 | 3.8455e-008, 4.8634e-008 | | | | 4.9969e-012, 6.3198e-012 | | | |
| | | 0.6 | 4.0717e-008, 4.7503e-008 | | | | 5.2900e-012, 6.1728e-012 | | | |
| | | 0.8 | 4.2979e-008, 4.6372e-008 | | | | 5.5844e-012, 6.0258e-012 | | | |
| | | 1.0 | 4.5241e-008, 4.5241e-008 | | | | 5.8797e-012, 5.8797e-012 | | | |

Table 2: Comparison between 2P1DI(4) and RK(4) for Problem 2.

| | | | $h = 0.1$ | | | | $h = 0.01$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Method** | $m$ | $r$ | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** |
| RK(4) | 1 | 0.0 | 1.6675e-006, 2.2928e-006 | 400 | 100 | 0.1618 | 1.7972e-010, 2.4711e-010 | 4000 | 1000 | 0.4374 |
| | | 0.2 | 1.7196e-006, 2.2511e-006 | | | | 1.8533e-010, 2.4262e-010 | | | |
| | | 0.4 | 1.7717e-006, 2.2094e-006 | | | | 1.9095e-010, 2.3812e-010 | | | |
| | | 0.6 | 1.8238e-006, 2.1677e-006 | | | | 1.9656e-010, 2.3363e-010 | | | |
| | | 0.8 | 1.8759e-006, 2.1260e-006 | | | | 2.0218e-010, 2.2914e-010 | | | |
| | | 1.0 | 1.9280e-006, 2.0843e-006 | | | | 2.0780e-010, 2.2464e-010 | | | |
| 2P1DI(4) | 1 | 0.0 | 7.9568e-006, 1.0941e-005 | 230 | 70 | 0.1128 | 1.4215e-010, 1.9545e-010 | 1580 | 520 | 0.2048 |
| | | 0.2 | 8.2055e-006, 1.0742e-005 | | | | 1.4659e-010, 1.9190e-010 | | | |
| | | 0.4 | 8.4541e-006, 1.0543e-005 | | | | 1.5103e-010, 1.8835e-010 | | | |
| | | 0.6 | 8.7028e-006, 1.0344e-005 | | | | 1.5548e-010, 1.8479e-010 | | | |
| | | 0.8 | 8.9514e-006, 1.0145e-005 | | | | 1.5992e-010, 1.8124e-010 | | | |
| | | 1.0 | 9.2001e-006, 9.9460e-006 | | | | 1.6436e-010, 1.7769e-010 | | | |
| 2P1DI(4) | 2 | 0.0 | 3.6193e-008, 4.9766e-008 | 300 | 70 | 0.1164 | 4.7033e-012, 6.4677e-012 | 2550 | 520 | 0.2790 |
| | | 0.2 | 3.7324e-008, 4.8861e-008 | | | | 4.8508e-012, 6.3496e-012 | | | |
| | | 0.4 | 3.8455e-008, 4.7956e-008 | | | | 4.9969e-012, 6.2297e-012 | | | |
| | | 0.6 | 3.9586e-008, 4.7051e-008 | | | | 5.1430e-012, 6.1142e-012 | | | |
| | | 0.8 | 4.0717e-008, 4.6146e-008 | | | | 5.2900e-012, 5.9943e-012 | | | |
| | | 1.0 | 4.1848e-008, 4.5241e-008 | | | | 5.4374e-012, 5.8797e-012 | | | |

Table 3: Comparison between 2P1DI(4) and RK(4) for Problem 3.

| | | | $h = 0.1$ | | | | $h = 0.01$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Method** | $\boldsymbol{m}$ | $\boldsymbol{r}$ | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** | **Error** $\underline{y}(1;r), \overline{y}(1;r)$ | **TFE** | **TS** | **Time (sec)** |
| RK(4) | 1 | 0.0 | 2.6628e-007, 3.9547e-007 | 400 | 100 | 0.1574 | 2.3535e-011, 3.4952e-011 | 4000 | 1000 | 0.4364 |
| | | 0.2 | 2.7498e-007, 4.0416e-007 | | | | 2.4303e-011, 3.5722e-011 | | | |
| | | 0.4 | 2.8367e-007, 4.1286e-007 | | | | 2.5071e-011, 3.6490e-011 | | | |
| | | 0.6 | 2.9236e-007, 4.2155e-007 | | | | 2.5841e-011, 3.7258e-011 | | | |
| | | 0.8 | 3.0106e-007, 4.3024e-007 | | | | 2.6608e-011, 3.8024e-011 | | | |
| | | 1.0 | 3.0975e-007, 4.3894e-007 | | | | 2.7377e-011, 3.8794e-011 | | | |
| 2P1DI(4) | 1 | 0.0 | 2.9785e-005, 4.4235e-005 | 230 | 70 | 0.1064 | 6.2545e-010, 9.2889e-010 | 1580 | 520 | 0.2018 |
| | | 0.2 | 3.0758e-005, 4.5208e-005 | | | | 6.4587e-010, 9.4931e-010 | | | |
| | | 0.4 | 3.1730e-005, 4.6180e-005 | | | | 6.6629e-010, 9.6973e-010 | | | |
| | | 0.6 | 3.2702e-005, 4.7153e-005 | | | | 6.8671e-010, 9.9015e-010 | | | |
| | | 0.8 | 3.3675e-005, 4.8125e-005 | | | | 7.0713e-010, 1.0106e-009 | | | |
| | | 1.0 | 3.4647e-005, 4.9097e-005 | | | | 7.2755e-010, 1.0310e-009 | | | |
| 2P1DI(4) | 2 | 0.0 | 9.4545e-006, 1.4041e-005 | 300 | 70 | 0.1130 | 1.5742e-010, 2.3379e-010 | 2550 | 520 | 0.3050 |
| | | 0.2 | 9.7632e-006, 1.4350e-005 | | | | 1.6256e-010, 2.3893e-010 | | | |
| | | 0.4 | 1.0072e-005, 1.4659e-005 | | | | 1.6769e-010, 2.4407e-010 | | | |
| | | 0.6 | 1.0381e-005, 1.4967e-005 | | | | 1.7283e-010, 2.4920e-010 | | | |
| | | 0.8 | 1.0689e-005, 1.5276e-005 | | | | 1.7797e-010, 2.5434e-010 | | | |
| | | 1.0 | 1.0998e-005, 1.5585e-005 | | | | 1.8311e-010, 2.5948e-010 | | | |

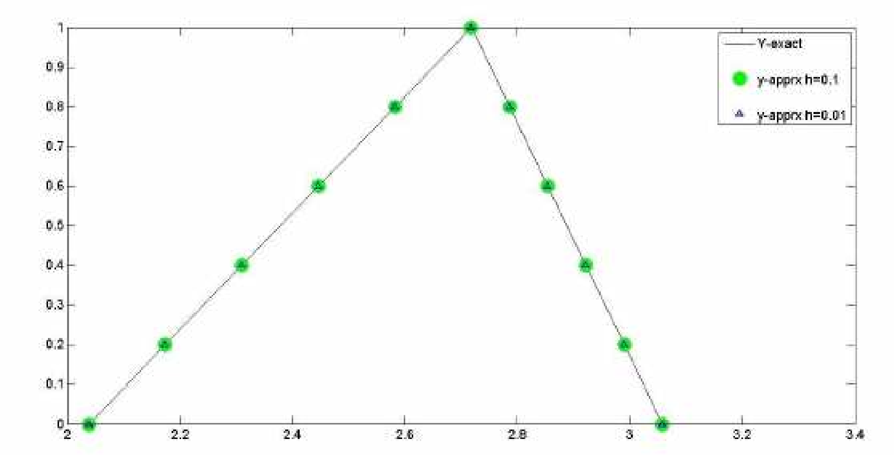


Figure 2: Problem 1 at $h = 0.1$ and $h = 0.01$

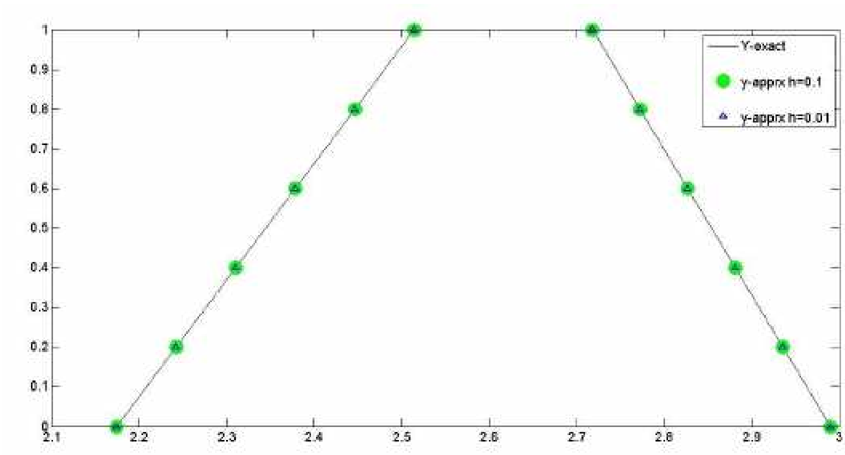


Figure 3: Problem 2 at $h = 0.1$ and $h = 0.01$

Meanwhile based on the outcome in Table 2, the 2P1DI(4) provides a better performance as compared to the RK(4) method. The 2P1DI(4) is less costly even with better accuracy. Figure 3 confirms the accuracy of the block method.

However, from Table 3, the 2P1DI(4) does not give good results as compared to the RK(4). Although $m = 2$, the 2P1DI(4) still cannot transcend the RK(4) approximations. On the other hand, the 2P1DI(4) needs less number of function evaluations and less number of steps than the RK(4). Thus, it is less

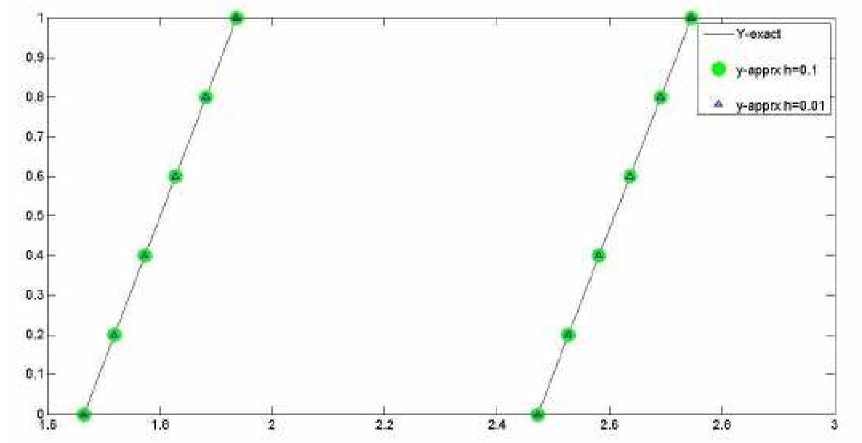


Figure 4: Problem 3 at $h = 0.1$ and $h = 0.01$

computational cost and converged fast.

## 7. Conclusions

In this research, the fourth order two-point one block diagonally implicit multi-step has been implemented for solving FDEs. It produces accurate approximation regardless of the computational cost. Therefore, it is confirmed that this method is suitable for solving first-order FDEs and provided accurate approximation only at certain problems.

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## References

[1] W. Pedrycz and F. Gomide, "Towards human-centric computing", *Fuzzy Systems Enginering*, John Willey and Sons, Hoboken, New Jersey. (2007).

[2] E. Y. Deeba and A. DE Korvin, "Analysis by fuzzy difference equations of a model of $CO_2$ level in the blood", *Applied Mathematics Letters*, **12**, 33-40 (1999).

[3] M. F. Abbod, D.G. von Keyserlingk, D. A. Linkens and M. Mahfouf, “Survey of utilization of fuzzy technology in medicine and healthcare”, *Fuzzy Sets and Systems*, **120**, 331-349 (2001).

[4] K. A. Chrysafis, B. K. Papadopoulos and G. Papaschinopoulos, “On the fuzzy difference equations of finance”, *Fuzzy Sets and Systems*, **159**, 3259-3270 (2008).

[5] M. Guo. X. Xue and R. Li, “Impulsive functional differential inclusions and fuzzy population models”, *Fuzzy Sets and Systems*, **138**, 601-615 (2003).

[6] O. Kaleva, “Fuzzy differential equations”, *Fuzzy Sets and Systems*, **24**, 301-317 (1987).

[7] S. Seikkala, “On the fuzzy initial value problem”, *Fuzzy Sets and Systems*, **24**, 319-330 (1987).

[8] J. J. Buckley and T. Feuring, “Fuzzy differential equations”, *Fuzzy Sets and Systems*, **110**, 43-54 (2000).

[9] M. Ma, M. Friedman and A. Kandel, “Numerical solutions of fuzzy differential equations”, *Fuzzy Sets and Systems*, **105**, 133-138 (1999).

[10] S. Abbasbandy and T. Allah Viranloo, “Numerical solution of fuzzy differential equation by Runge-Kutta method”, *Nonlinear Studies*, **11**, 117-129 (2004).

[11] S. Ch. Palligkinis, G. Papageorgiou and I. Th. Famelis, “Runge-Kutta methods for fuzzy differential equations”, *Applied Mathematics and Computation*, **209**, 97-105 (2009).

[12] B. Ghazanfari and A. Shakerami, “Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order 4”, *Fuzzy Sets and Systems*, **189**, 74-91 (2011).

[13] T. Allahviranloo, N. Ahmady and E. Ahmady, “Numerical solution of fuzzy differential equations by predictor-corrector method”, *Information Sciences*, **177**, 1633-1647 (2007).

[14] T. Allahviranloo, S. Abbasbandy, N. Ahmady and E. Ahmady, “Erratum to “Numerical solution of fuzzy differential equations by predictor-corrector method” [Inform. Sci. 177(7)(2007) 1633-1647]”, *Information Sciences*, **178**, 1780-1782 (2008).

[15] T. Allahviranloo, N. Ahmady and E. Ahmady, “Improved predictor-corrector method for solving fuzzy initial value problems”, *Information Sciences*, **179**, 945-955 (2009).

[16] S. Mehrkanoon, M. Suleiman and Z.A. Majid, “Block method for numerical solution of fuzzy differential equations”, *International Mathematical Forum*, **4**, 2269-2280 (2009).

[17] I. S. M. Zawawi, Z. B. Ibrahim and M. Suleiman, “Diagonally implicit block backward differentiation formulas for solving fuzzy differential equations”, *AIP Conference Proceeding*, **1522**, 681-687 (2013).

[18] B. Bede, “Note on “Numerical solutions of fuzzy differential equations by predictor-corrector method” ”, *Information Sciences*, **178**, 1917-1922 (2008).

[19] Z. A. Majid and M. Suleiman, “Performance of 4-point diagonally implicit block method for solving ordinary differential equations”, *MATEMATIKA*, **22**, 137-146 (2006).

[20] Z. B. Ibrahim, M.B. Suleiman and K.I. Othman, “Implicit r-point block backward differentiation formula for solving first-order stiff ODEs”, *Applied Mathematics and Computation*, **186**, 558-565 (2007).

[21] J. Xu, Z. Liao and Z. Hu, “A class of linear differential dynamical systems with fuzzy initial condition”, *Fuzzy Sets and Systems*, **158**, 2339-2358 (2007).

[22] D. Dubois and H. Prade, "Towards fuzzy differential calculus part 3: differentiation", *Fuzzy Sets and Systems*, **8**, 225-233 (1982).

[23] Z. Guang-Quan, "Fuzzy continuous function and its properties", *Fuzzy Sets and Systems*, **43**, 159-171 (1991).

[24] A. K. Dizicheh, S. Salahshour, and F. B. Ismail, "A note on "Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order 4"", *Fuzzy Sets and Systems*, **233**, 96-100 (2013).

[25] J. D. Lambert, "Computational Methods in Ordinary Differential Equations", John Willey and Sons, New York, USA. (1973).

[26] C. Duraisamy and B. Usha, "Another approach to solution of fuzzy differential equations", *Applied Mathematical Sciences*, **16**, 777-790 (2010).