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GRACEFUL LABELING OF NON-SYMMETRIC MODIFIED THETA GRAPHS

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Abstract: A class of graphs called non-symmetric modified theta graphs with three internally disjoint paths (3, j, j) for j is even or odd and (3, j, j + 1) for j is even or odd are defined and those graphs are proved graceful.

AMS Subject Classification: 05C78
Key Words: graceful labeling, non-symmetric modified theta graph

1. Introduction

Graphs considered in this paper are simple, finite and undirected. In general G (V, E) denotes the graph G with vertex set V (G) and edge set E (G), such that $|V(G)|$ = p, $|E(G)|$ = q. If ϕ is an injection from the vertices of G to the set {0, 1, . . . , q} such that, when each edge xy is induced by the label $\phi^{\square}(xy)$ = $|\phi(x) - \phi(y)|$ the resulting edge labels are distinct.

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The concept of graceful labeling was introduced Rosa[1] with the name β-valuation. Gallian[2] given the extensive survey of the contributions to graceful labeling of variety of graphs.

Definition 1. A theta graph is the union of three internally disjoint paths of length 2 between two vertices.

2. Main Result

A non-symmetric modified theta graph NSθ (3, j, (j or j + 1)) results from joining a vertex u to a diametrically opposite pair of vertices of a cycle C_n. The vertex u is called supporting vertex of NSθ (3, j, (j or j + 1)). The pair of diametrically opposite vertices of cycle C_n of NSθ (3, j, (j or j + 1)), to which the supporting vertex of the non-symmetric modified theta graph is joined, are also called supporting vertices of that non-symmetric theta graph. They are denoted by s_1 and s_2. As the support vertices of NSθ (3, j, (j or j + 1)) form a part of the cycle C_n, without loss of generality it can be taken that $s_1 = c_1$ and $s_2 = c_{1+m}$, where n $\in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is set of all positive integers.

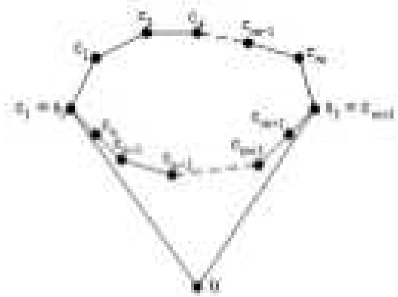

Figure 1: General form of NSθ (3, j, (j or j + 1))

Theorem. *All non-symmetric modified theta graphs are graceful.*

Proof. Proof of this theorem consists of 8 cases and in each case steps are given to vertex labeling of NSθ (3, j, (j or j + 1)), j = 1, 2, . . . , n − 1.

Let E(A) be the collection of supporting edge($\phi^{\square}(us_i)$) values from supporting vertex, E(B), k = 2, 2m or 2m + 1 be incident edge($\phi^{\square}(s_1c_k)$) values with a vertex s_1, E(D), k = m or m + 2 be incident edge($\phi^{\square}(s_2c_k)$) values with a vertex s_2, $1 \leq i \leq 2$. Let E(F) be the collection of edge($\phi^{\square}(c_jc_{j+1})$) values in C_n, $1 \leq j \leq n - 1$. Now, E(G) = {E(A) $\bigcup$ E(B) $\bigcup$ E(D) $\bigcup$ E(F)}.

Case 1: The cycle C_n, (n = 2m = 6, 14, 22, 30, …, 8i – 2, 1 ≤ i ≤ n), in which, m (4i – 1) is odd.

The vertex labeling for case 1 is as follows:

Step 1 : ϕ (u) = 2m + 2. **Step 2**: ϕ (s_1) = 0.

Step 3 : ϕ (c_{2j+1}) = j, 1 ≤ j ≤ m – 1. **Step 4**: ϕ (c_{2j}) = 2m + 2 – j, 1 ≤ j ≤ $\frac{m-1}{2}$. **Step 5** : ϕ (s_2) = $\frac{3(m+1)}{2}$.

Step 6 : ϕ ($c_{\frac{3(m+1)}{2}+2(j-1)}$) = ϕ ($c_{\frac{3(m-1)}{2}}$) – 2 – (j – 1), 1 ≤ j ≤ $\frac{m-3}{4}$.

Step 7 : ϕ (c_{2m}) = m.

Now, induced labeling of edges for case 1 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 2. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 1.

3 : $\phi^{\square}(c_{j+1}c_{j+2})$ = 2m + 1 – j, 1 ≤ j ≤ m. **4**: $\phi^{\square}(c_{2m}s_1)$ = m.

5 : $\phi^{\square}(c_{m+1+j}c_{m+j+2})$ = m – j, 1 ≤ j ≤ $\frac{(m-3)}{2}$. **6**: $\phi^{\square}(us_2)$ = $\frac{m+1}{2}$.

7 : $\phi^{\square}(c_{2m-\frac{m+1}{2}+j}c_{2m-\frac{m+1}{2}+j+1})$ = $\frac{(m+1)}{2}$ – j, 1 ≤ j ≤ $\frac{(m-1)}{2}$.

The resultant edge values E(G) are distinct.

An example of Case 1 for m = 7 is given in Figure 2

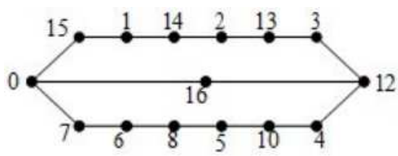

Figure 2: Graceful labeling of NSθ (3, 7, 7)

Case 2: The cycle C_n(n = 2m = 10, 18, 26, …, 8i + 2, 1 ≤ i ≤ n), in which m (4i + 1) is odd.

The vertex labeling for case 2 is as follows:

Step 1 : ϕ (u) = 2m + 2. **Step 2**: ϕ (s_1) = 0. **Step 3**: ϕ (c_{2j+1}) = j, 1 ≤ j ≤ $\frac{3m-3}{4}$. **Step 4**: ϕ (c_{2j}) = 2m + 2 – j, 1 ≤ j ≤ $\frac{m-1}{2}$.

Step 5 : ϕ (s_2) = m + 1. **Step 6**: ϕ (c_{2m}) = m + 2.

Step 7 : ϕ (c_{m+3}) = ϕ (c_{m-1}) – 2. **Step 8**:ϕ (c_{m+3+2j}) = ϕ (c_{m+1+2j}) – j, 1 ≤ j ≤ $\frac{m-5}{4}$.

The remaining $(\frac{m-1}{2})$ edge values are grouped in the form of even and odd integer values separately as follows:

Step 9 : ϕ $(c_{\frac{3m+3}{2}}) = \phi$ $(c_{\frac{3m+2}{2}}) - |\phi(c_{m+2}) - \phi(s_2)| - 1$. ϕ $(c_{\frac{3m+5}{2}}) = \phi$ $(c_{\frac{3m+3}{2}})$ $+ \left|\phi(c_{\frac{3m}{2}+1}) - \phi(c_{\frac{m-2}{2}})\right| - 3$. (This process continues up to $\frac{m-1}{4}$ vertices and up to edge value 2 as even edge values are reduced).

Step 10 : ϕ $(c_{2m-1}) = \phi$ $(c_{2m}) - |\phi(c_{m+2}) - \phi(s_2)| - 2$. ϕ $(c_{2m-2}) = \phi$ (c_{2m-1}) $+ |\phi(c_{m+2}) - \phi(s_2)| - 4$. (This process continues up to $\frac{m-1}{4}$ vertices and up to edge value 1 as odd edge values are reduced.)

Now, induced labeling of edges for case 2 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 2. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 1. **3**: $\phi^{\square}(c_{j+1}c_{j+2})$ = 2m + 1 – j, 1 ≤ j ≤ (m – 2). **4**: $\phi^{\square}(c_{2m}s_1)$ = m + 2.

5 : $\phi^{\square}(us_2)$ = m + 1. **6**: $\phi^{\square}(c_{m+1+j}c_{m+j+2})$ = m + 1 – j, 1 ≤ j ≤ $\frac{(m-3)}{2}$. **7**: $\phi^{\square}(c_ms_2) = \frac{m+3}{2}$. $\phi^{\square}(c_{m+2}s_2) = \phi^{\square}(c_{m+2}s_2) = \frac{m+1}{2}$.

The remaining $(\frac{m-1}{2})$ even number of edges in the form of even and odd integer valued edges are as follows:

8 : $\phi^{\square}$ $(c_{\frac{3m+3}{2}}\ c_{\frac{3m+2}{2}}) = \frac{m-1}{2}$. **9**: $\phi^{\square}$ $(c_{\frac{3m+2+j+1}{2}}\ c_{\frac{3m+2+j}{2}}) = \frac{m-1}{2}$ – 2j, 1 ≤ j ≤ $\frac{m-1}{4}$. (This process continues up to edge value 2 as even edge values are reduced).

10 :$\phi^{\square}$ $(c_{2m-1}\ c_{2m}) = \phi^{\square}(c_{m+2}s_2) - 2$. **11**: $\phi^{\square}$ $(c_{2m-1-j}\ c_{2m-j}) = \frac{m+1}{2}$ – 2j, 1 ≤ j ≤ $\frac{m-1}{4}$. (This process continues up to edge value 1 as odd edge values are reduced.)

The resultant edge values E(G) are distinct.

An example of Case 2 for m = 9 is given in Figure 3

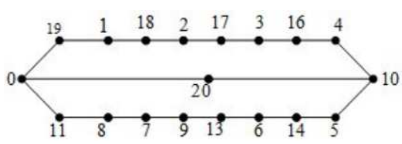

Figure 3: Graceful labeling of NSθ (3, 9, 9)

Case 3: The cycle C_n, (n = 2m = 8, 16, 24,. . ., 8i, 1 ≤ i ≤ n), in which m (4i) is even.

The vertex labeling for case 3 is as follows:

Step 1 : ϕ (u) = 2m + 2. **Step 2**: ϕ (s_1) = 0.

Step 3 : ϕ (c_{2j+1}) = j, $1 \leq j \leq \frac{m-2}{2}$. **Step 4**: ϕ (c_{2j}) = 2m + 2 – j, $1 \leq j \leq \frac{3m}{4}$. **Step 5** : ϕ (s_2) = m + 1.

Step 6 : ϕ ($c_{m+3+2(j-1)}$) = ϕ ($c_{m+(j-1)}$) – j, for j = $1 \leq j \leq \frac{m-4}{4}$.

Step 7 : ϕ (c_{2m}) = m + 2. ϕ (c_{2m-1}) = ϕ (c_{m-1}) + 1.

The remaining edges from {1, 2,. . ., ($\frac{m-2}{2}$) } are obtained by substituting the edge value by comparing the remaining vertex values.

Step 8 :ϕ (c_{2m-2}) = ϕ (c_{2m-1}) + $\frac{m-2}{2}$. ϕ (c_{2m-3}) = ϕ (c_{2m-2}) + $\frac{m-4}{2}$. ϕ (c_{2m-3-j}) = ϕ (c_{2m-2-j}) – $\frac{m-4}{2}$ – j, $1 \leq j \leq \frac{m-6}{2}$.

The induced edge values for case 3 is as follows:

1 : $\phi^{*}(us_1)$ = 2m + 2. **2**: $\phi^{*}(s_1c_2)$ = 2m + 1. **3**: $\phi^{*}(c_{j+1}c_{j+2})$ = 2m + 1 – j, $1 \leq j \leq$ (m – 2). **4**: $\phi^{*}(c_{2m}s_1)$ = m + 2.

5 : $\phi^{*}(us_2)$ = m + 1. **6**: $\phi^{*}(c_{m+1+j}c_{m+j+2})$ = m + 1 – j, $1 \leq j \leq \frac{(m-4)}{2}$.

7 : $\phi^{*}(c_{2m}c_{2m-1}) = \frac{m+4}{2}$. $\phi^{*}(c_m s_2) = \frac{m+2}{2}$. $\phi^{*}(c_{m+2}s_2) = \frac{m}{2}$.

The remaining edge values {1, 2,. . ., ($\frac{m-2}{2}$) } are results from the following.

8 : ϕ^{*} ($c_{2m-2}c_{2m-1}$) = $\frac{m-2}{2}$. ϕ^{*} ($c_{2m-3}c_{2m-2}$) = $\frac{m-4}{2}$. ϕ^{*} ($c_{2m-3-j}c_{2m-2-j}$) = $\frac{m-4}{2}$ – j, $1 \leq j \leq \frac{m-6}{2}$.

The resultant edge values E(G) are distinct.

An example of Case 3 for m = 8 is given in Figure 4

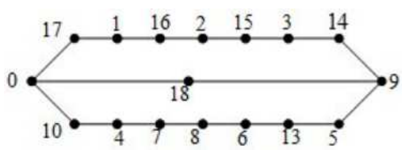

Figure 4: Graceful labeling of NSθ (3, 8, 8)

Case 4: The cycle C_n, (n = 2m = 12, 20, 28,. . ., 8i + 4, $1 \leq i \leq n$), in which m (4i + 2) is even.

The vertex labeling for case 4 is as follows:

Step 1 : ϕ (u) = 2m + 2. **Step 2**: ϕ (s_1) = 0.

Step 3 : ϕ (c_{2j+1}) = j, $1 \leq j \leq \frac{m-2}{2}$.

Step 4 : ϕ (c_{2j}) = 2m + 2 – j, $1 \leq j \leq \lfloor \frac{2m+2}{3} \rfloor$. **Step 5** : ϕ (s_2) = m + 1. **Step 6**: ϕ (c_{2m}) = m + 2.

Step 7 : ϕ (c_{2m-1}) = ϕ (c_{m-1}) + 1. **Step 8**: ϕ (c_{m+3}) = ϕ (c_{m-1}) + 2. ϕ (c_{m+3+2j}) = ϕ (c_{m+3}) + j, for j = $1 \leq j \leq \frac{m-6}{2}$.

The remaining edge values {1, 2,. . ., ($\frac{m-2}{2}$) } are grouped in the form even and odd integer separately as follows: (edges with odd values are called odd edges and edges with even values are called even edges)

Step 9 : ϕ (c_{2m-2}) = ϕ (c_{2m-1}) – $|\phi(c_{m+2}) - \phi(s_2)|$ – 1. ϕ ($c_{2m-2-2j}$) = ϕ (c_{2m-2}) – 2j, $1 \leq j \leq \frac{(m-2)}{2}$. (even edge values)

Step 10 : ϕ ($c_{\frac{3m+2}{2}}$) = ϕ ($c_{\frac{3m+2}{2}-1}$) – $|\phi(c_{m+2}) - \phi(s_2)|$ + 2. ϕ ($c_{\frac{3m+2}{2}+2j}$) = ϕ ($c_{\frac{3m+2}{2}}$) – 2j, $1 \leq j \leq \frac{(m-2)}{2}$. (odd edge values)

The induced edge values for case 4 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 2. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 1.

3 : $\phi^{\square}(c_{j+1}c_{j+2})$ = 2m + 1 – j, $1 \leq j \leq$ (m – 2).

4 : $\phi^{\square}(c_{2m}s_1)$ = m + 2. **5**: $\phi^{\square}(us_2)$ = m + 1.

6 : $\phi^{\square}(c_{m+1+j}c_{m+j+2})$ = m + 1 – j, $1 \leq j \leq \frac{(m-4)}{2}$. $\phi^{\square}(c_{2m}c_{2m-1}) = \frac{(m+4)}{2}$. **7**: $\phi^{\square}(c_{m-1}s_2) = \frac{m+2}{2}$. $\phi^{\square}(c_{m+2}s_2) = \frac{m}{2}$.

8 : $\phi^{\square}(c_{2m-2}c_{2m-1}) = \frac{m-2}{2}$. $\phi^{\square}(c_{2m-2-2j}c_{2m-2-2j+1}) = \frac{m-2}{2}$ – 2j, $1 \leq j \leq \frac{(m-2)}{4}$. (even edge values).

9 : $\phi^{\square}(c_{\frac{3m+2}{2}}c_{\frac{3m+2}{2}-1}) = \frac{(m-4)}{2}$. $\phi^{\square}$ ($c_{\frac{3m+2}{2}+2j}$ $c_{\frac{3m+2}{2}+2j-1}$) = $\frac{(m-4)}{2}$ – 2j, $1 \leq j \leq \frac{(m-4)}{2}$. (odd edge values).

The resultant edge values E(G) are distinct.

An example of Case 4 for m = 10 is given in Figure 5

Case 5: The cycle C_n (n = 2m + 1 = 9, 17, 25, . . ., 8i + 1, $1 \leq i \leq n$), in which m (4i) is even.

The vertex labeling for case 5 is as follows:

Step 1 : ϕ (u) = 2m + 3. **Step 2**: ϕ (s_1) = 0.

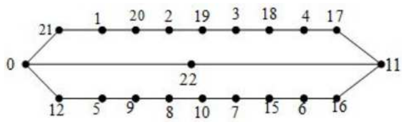

Figure 5: Graceful labeling of NSθ (3, 10, 10)

Step 3 : ϕ (c_{2j+1}) = j, $1 \leq j \leq \frac{m}{4}$.

Step 4 : ϕ (c_{2j}) = 2m + 3 – j, $1 \leq j \leq m$. **Step 5**: ϕ (s_2) = $\frac{m+4}{2}$.

Step 6 : ϕ (c_{2m}) = m + 2. **Step 7**: ϕ ($c_{\frac{m}{2}+2j+1}$) = ϕ ($c_{\frac{m}{2}+2j-1}$) + 1 + j, $1 \leq j \leq \frac{m-4}{4}$. **Step 8**: ϕ (c_{m+3}) = ϕ (s_2) + 1.

Step 9 : ϕ ($c_{m+3+2(j-1)}$) = ϕ (c_{m+3}) + j, $1 \leq j \leq \frac{m-4}{2}$.

The induced edge values for case 5 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 3. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 2. **3**: $\phi^{\square}(c_jc_{j+1})$ = 2m + 2 – j, $1 \leq j \leq \frac{m}{2}$. **4** $\phi^{\square}(us_2) = \frac{3m+2}{2}$. **5**: $\phi^{\square}(c_{\frac{m}{2}+1+j}c_{\frac{m}{2}+j+2}) = \frac{3m+2}{2}$ – j, $1 \leq j \leq \frac{(m-4)}{2}$. **6**: $\phi^{\square}(c_{2m}s_1)$ = m + 2.

7 : $\phi^{\square}(c_{m-1+j}c_{m+j})$ = m + 2 – j, $1 \leq j \leq m + 1$.

The resultant edge values E(G) are distinct.

An example of Case 5 for m = 8 is given in Figure 6

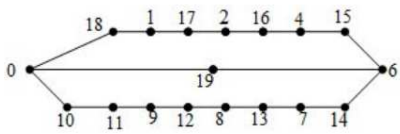

Figure 6: Graceful labeling of NSθ (3, 8, 9)

Case 6: The cycle C_n (n = 2m + 1 = 13, 21, 29, …, 8i + 5, $1 \leq i \leq n$), in which m (4i + 2) is even.

The vertex labeling for case 6 is as follows:

Step 1 : ϕ (u) = 2m + 3. **Step 2**: ϕ (s_1) = 0. **Step 3**: ϕ (c_{2j+1}) = j, $1 \leq j \leq \frac{m-2}{4}$. **Step 4**: ϕ (c_{2j}) = 2m + 3 – j, $1 \leq j \leq \frac{m}{2}$.

Step 5 : ϕ (c_{2m}) = m + 1. **Step 6**: ϕ (c_{2j+1}) = $\frac{m-2}{4}$ + 1 + j, $\frac{m-2}{4} + 1 \leq j \leq m$. **Step 7**: ϕ (c_{m+2j}) = ϕ (c_{m-2+2j}) – 1 – j, $1 \leq j \leq \frac{m}{2}$.

Now, induced edge values for case 6 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 3. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 2. **3**: $\phi^{\square}(c_jc_{j+1})$ = 2m + 2 – j, 1 ≤ j ≤ $\frac{m-2}{2}$. **4**: $\phi^{\square}(us_2) = \frac{3m+4}{2}$.

5 : $\phi^{\square}(c_{\frac{m+2}{2}+j-1}c_{\frac{m+2}{2}+j}) = \frac{3m+4}{2}$ – j, 1 ≤ j ≤ $\frac{m-2}{2}$

6 : $\phi^{\square}(c_ms_2)$ = m + 2. $\phi^{\square}(c_{2m}s_1)$ = m + 1.

7 : $\phi^{\square}(c_{m+j}c_{m+j+1})$ = m + 1 – j, 1 ≤ j ≤ m.

The resultant edge values E(G) are distinct.

An example of Case 6 for m = 10 is given in Figure 7

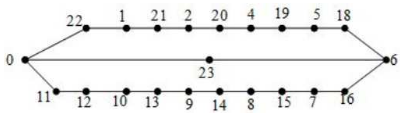

Figure 7: Graceful labeling of NSθ (3, 10, 11)

Case 7: The cycle C_n, (n = 2m + 1 = 7, 15, 23, ..., 8i – 1, 1 ≤ i ≤ n) in which m (4i – 1) is odd.

The vertex labeling for case 7 is as follows:

Step 1 : ϕ (u) = 2m + 3. **Step 2**: ϕ (s_1) = 0. **Step 3**: ϕ (c_{2j+1}) = j, 1 ≤ j < $\frac{m+1}{2}$. **Step 4**: ϕ (c_{2j}) = 2m + 3 – j, 1 ≤ j ≤ $\frac{3(m+1)}{4}$.

Step 5 ϕ (c_{m+2+2j}) = ϕ (c_{m+2}) + 2 – (j – 1), for j = 1 ≤ j ≤ $\frac{m-3}{4}$.

Step 6 : ϕ (c_{2m-1}) = $\frac{m+3}{2}$. **Step 7**: ϕ (c_{2m}) = m.

Step 8 : ϕ (c_{2m-1}) = ϕ (c_{2m}) – $|\phi(u) - \phi(s_2)|$ – 1. ϕ (c_{2m-1}) = ϕ (c_{2m}) – $|\phi(u) - \phi(s_2)|$ – 2. (This process continues up to edge value 1 except the edge value $|\phi(c_{2m+1}) - \phi(c_{2m})|$).

The induced edge values for case 7 is as follows:

1 : $\phi^{\square}(us_1)$ = 2m + 3. **2**: $\phi^{\square}(s_1c_2)$ = 2m + 2. **3**: $\phi^{\square}(c_jc_{j+1})$ = 2m + 2 – j, 1 ≤ j ≤ m + 1. **4**: $\phi^{\square}(us_2) = \frac{m+1}{2}$. $\phi^{\square}(c_{2m+1}s_1)$ = m.

5 : $\phi^{\square}$ ($c_{m+3+(j-1)}$ c_{m+3+j}) = m – j, for j = 1 ≤ j ≤ $\frac{m-3}{2}$.

6 : $\phi^{\square}(c_{2m+1}c_{2m}) = \lfloor \frac{m}{2} \rfloor - 1$.

7 : $\phi^{*}(c_{2m-1}c_{2m-2}) = \frac{m-1}{2}$. $\phi^{*}(c_{2m-1-j}c_{2m-2-j}) = \frac{m-1}{2} - j$, $1 \le j \le \frac{m-3}{2}$. (This process continues up to edge value 1 except the edge value $\phi^{*}(c_{2m+1}c_{2m})$).

The resultant edge values E(G) are distinct.

An example of Case 7 for m = 7 is given in Figure 8

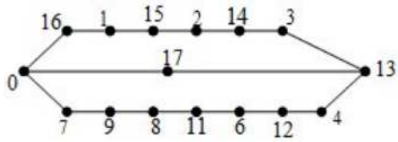

Figure 8: Graceful labeling of NSθ (3, 7, 8)

Case 8: The cycle C_n, (n = 2m + 1 = 11, 19, 27, ..., 8i + 3, $1 \le i \le n$), in which m (4i + 1) is odd.

The vertex labeling for case 8 is as follows:

Step 1 : ϕ (u) = 2m + 3. **Step 2**: ϕ (s_1) = 0.

Step 3 : ϕ (c_{2j+1}) = j, $1 \le j \le \frac{3m+1}{4}$.

Step 4 : ϕ (c_{2j}) = 2m + 3 – j, $1 \le j \le \frac{m-1}{2}$. **Step 5** : ϕ (s_2) = m.

Step 6 : ϕ (c_{m+3}) = ϕ (c_{m-1}) – 2. ϕ ($c_{m+5+2(j-1)}$) = ϕ ($c_{m+3+2(j-1)}$) – j, $1 \le j \le \frac{m-3}{2}$. **Step 7**: ϕ (c_{2m+1}) = m + 2.

Step 8 : ϕ ($c_{2m-1-2(j-1)}$) = ϕ ($c_{2m+1-2(j-1)}$) – j, $1 \le j \le \frac{m-5}{4}$.

The induced labeling of edges for case 8 is as follows:

1 : $\phi^{*}(us_1)$ = 2m + 3. **2**: $\phi^{*}(s_1c_2)$ = 2m + 2.

3 : $\phi^{*}(c_{j+1}c_{j+2})$ = 2m + 2 – j, $1 \le j \le m - 2$. **4**: $\phi^{*}(us_2)$ = m + 3. $\phi^{*}(c_{2m+1}s_1)$ = m + 2.

5 : ϕ^{*} ($c_{m+2+(j-1)}$ c_{m+2+j}) = m + 2 – j, for j = $1 \le j \le \frac{m+1}{2}$.

6 : $\phi^{*}(s_2c_m) = \frac{m+1}{2}$. $\phi^{*}(s_2c_{m+2}) = \frac{m-1}{2}$.

7 : $\phi^{*}(c_{2m-\frac{m-3}{2}+j}c_{2m-\frac{m-3}{2}+j+1}) = \frac{m-1}{2} - j$, $1 \le j \le \frac{m-3}{2}$.

The resultant edge values E(G) are distinct. □

An example of Case 8 for m = 9 is given in Figure 9

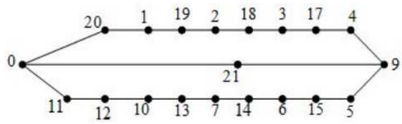

Figure 9: Graceful labeling of NSθ (3, 9, 10)

3. Conclusion

All non-symmetric modified theta graphs are graceful.

References

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