

NONLINEAR INVARIANCE OF FLAT CONNECTION IN GAUGE GRAVITY

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Abstract: At this paper one studies the spinning gravitational field formed by a torsion free and flat connection. In existing such a connection i.e the Maurer - Cartan 1 - form ω on a tangent bundle, the nonlinear invariance of the term $\text{tr}[\omega \wedge * \omega]$ gives the spin current form. Also one sees that the action integral of the gravitational field is bounded such that $\Lambda(\sim \frac{1}{G}) \leq \int_M (\mathcal{L}_{Mat} + \mathcal{L}_{Gr}) < \infty$ depending of the vacuum energy density Λ_0 and gravitational constant G .

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1. Introduction

In the Einstein' s theory of the gravity the curvature of a space - time manifold is created by the energy momentum. However, there is an algebraic relation between the energy - momentum and the curvature and if the energy - momentum vanishes (also the vacuum energy density is zero), the curvature continues propagating, that is the vanishing energy - momentum does not imply zero curvature, i.e the Schwarzschild solution and gravitational radiation. The solution to the Einstein equation over asymptotically locally flat space gives the plane gravitational wave and this kind of the plane waves is a quadruple. However this solution is with respect to the metric tensor which is the source potential of the gravitational field.

In the general theory of the relativity, the gravitation is predicted as a scalar (spin 0) - tensor (spin 2) field theory, i.e Brans - Dicke theory [2]. As well known that the spin 2 - structure is via Einstein' s metric theory of the gravity since the metric is a second rank tensor field. This field creates the source potential of the gravitational field as a Levi - Civita connection, which is symmetric and torsion free. Besides, the Utiyama' s ansatz [12] has shown that the gravitational interaction isn't a gauge theory like Yang - Mills type [13] theories, because the affine connection which is the source of the interaction depends on the Riemannian metric of a space - time manifold. However, in the Kibble' s formalism [10], the affine connection which is the source of the gravity isn't symmetric and it is an asymmetric affine connection defined by Eddington [4] and its antisymmetric part is Cartan' s torsion interpreted as the spin density in the continuum mechanics. Therefore, the source potential may only be the metric tensor (that is its Levi - Civita connection) i.e metric - affine theories of the gravity [8].

2. Geometrical Notations

Let M be an real n manifold endowed by a Riemannian metric and with boundary, also oriented. The local coordinates over this manifold are $x^\mu \in \mathbb{R}^n$, ($\mu^2 = 1, \dots, n$). Given a tangent and cotangent bundles together with local coordinate bases by $TM : \{\partial_\mu\}$ and $T^*M : \{dx^\mu\}$, where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Show by $\Lambda^r(M)$ the bundle of the r - exterior forms on this manifold. The Hodge duality operator for any r - form is $*$: $\Lambda^r(M) \rightarrow \Lambda^{n-r}(M)$. The exterior derivative operator over the base manifold M is also defined by the map $d : \Lambda^r(M) \rightarrow \Lambda^{r+1}(M)$. For an exterior form $\alpha \in \Lambda^r(M)$ the norm is written as $\|\alpha\|^2 = \alpha \wedge *\alpha$.

Choose a frame $\{e_A\} \in TM$ and its co-frame $\{e^A\} \in T^*M$. They are associated to the coordinates bases such that $e_A = e_A^\mu \partial_\mu$ and $e^A = e_\mu^A dx^\mu$ via n - bein fields e_A^μ and e_μ^A (we will call tetrad them over 4 dimension). Therefore, any TM and $\text{End}(TM)$ valued - $p \leq n$ - forms on this bundle is written as follow

$$X^A = e_\sigma^A (X^\sigma)_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad (1)$$

$$Y_B^A = e_\sigma^A e_B^\rho (Y_\rho^\sigma)_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \quad (2)$$

Also the metric on the tangent bundle TM defined by the frames $\{e_A\}$ is given by $g_{AB} = \langle e_A, e_B \rangle$ together with the volume element

$$*1 = e^1 \wedge \dots \wedge e^n = \sqrt{|\text{dete}_\mu^A|} d\text{Vol}_M, \quad (3)$$

where

$$dVol_M = dx^1 \wedge \dots \wedge dx^n. \tag{4}$$

We show by $\Lambda^r(TM)$ the bundle of the r - forms on the tangent bundle spanned by the co-frames $\{e^A\}$. A connection on this bundle is given by the map $\nabla : \Lambda^0(TM) \rightarrow \Lambda^1(TM)$ and we give its covariant and exterior covariant derivatives, respectively, by

$$\nabla = d + \Gamma, \quad d^\nabla = d + \frac{1}{2}[\Gamma, \bullet]_\wedge, \tag{5}$$

where Γ is connection 1 - form defined by $\nabla e_A = \Gamma^C_A e_C$, $\Gamma^C_A \in \Lambda^1(\text{End}(TM))$. The curvature 2 - form of this connection is

$$R^A_B = d^\nabla \Gamma^A_B = d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B \in \Lambda^2(\text{End}(TM)) \tag{6}$$

Following [9] the torsion 2 - form belonging to this connection is given as

$$T^A = d^\nabla e^A = de^A + \Gamma^A_B \wedge e^B \in \Lambda^2(TM). \tag{7}$$

3. Traditional Gravitational Lagrangian

The simplest gravitational Lagrangian of the Einstein - Hilbert action (together with cosmological term) can be written follow

$$\mathcal{L}(\epsilon, \Gamma) = \mathcal{L}_{Mat}(\epsilon, \Gamma) - \frac{1}{16\pi G} \text{tr}[R_{AB} \wedge * \epsilon^{AB}] + \frac{\Lambda}{8\pi G} * 1, \tag{8}$$

where G is the gravitational constant, Λ the energy density of the vacuum, i.e cosmological constant, \mathcal{L}_{Mat} a matter Lagrangian interacting to gravitational field and $R_{AB} = g_{AC} R^C_B$. Also we use a presentation

$$\epsilon^{AB\dots} = \epsilon^A \wedge \epsilon^B \wedge \dots. \tag{9}$$

This Lagrangian reads the energy - momentum and spin current equations follow

$$-\mathfrak{T}^A = \frac{1}{8\pi G} R_{BC} \wedge * \epsilon^{ABC} - \frac{\Lambda}{8\pi G} * \epsilon^A, \tag{10}$$

$$-\mathfrak{S}^{AB} = -\frac{1}{8\pi G} d^\nabla (* \epsilon^{AB}), \tag{11}$$

where the energy - momentum ($n - 1$) and spin current ($n - 1$) forms, respectively, are defined from [10] and [7] follow

$$\mathfrak{T}^A = \frac{\delta \mathcal{L}_{Mat}}{\delta \epsilon^A}, \quad \mathfrak{S}_B^A = \frac{\delta \mathcal{L}_{Mat}}{\delta \Gamma_B^A}. \tag{12}$$

As seen that the energy - momentum equation corresponds to the Einstein equation with cosmological constant. The energy momentum equation is the set of the nonlinear first order partial differential equations to the connection while the spin equation is the non - linear second order ones.

The term in $d^\nabla(*\epsilon^{AB})$ in the spin equation given in eq. (11) relates to the torsion of the connection. So that, simply, if one consider a real 4 dimensional manifold, then the term $*\epsilon^{AB}$ is again a 2 - form, and so we can mention a(n) (anti -) self dual base 1 - form

$$*\epsilon^{AB} = \pm \epsilon^{AB}. \tag{13}$$

The similar self dual 2 - form ansats for the gravity is given in [5], [3] via Ashtekar variables [1]. Then, for $*\epsilon^{AB} = \pm \epsilon^{AB}$

$$d^\nabla(\epsilon^{AB}) = T^A \wedge \epsilon^B - \epsilon^A \wedge T^B. \tag{14}$$

For a torsion free connection we have

$$d^\nabla(*\epsilon^{AB}) = 0. \tag{15}$$

Thus for the torsion free connection the spin current form vanishes: $-\mathfrak{S}^{AB} = 0$. However, we aim a formalism for the spinning gravity even though the torsion free connection. The way for such a formalism we will handle the Maurer - Cartan 1 - form of a local group as the gauge gravitational connection.

4. Flat Connection and Its Nonlinear Invariance

The connection of any fiber bundle transforms as inhomogeneous under a local gauge transformation $g_B^A \in \text{End}(TM)$ such that $\omega' = g^{-1}\omega g + g^{-1}dg$, where the term violating the homogenous is Maurer - Cartan form

$$\omega_B^A = (g^{-1})_C^A dg_B^C \in \Lambda^1(\text{End}(TM)) \tag{16}$$

and it satisfies the Maurer - Cartan equation

$$d\omega_B^A + \omega_C^A \wedge \omega_B^C = 0. \tag{17}$$

It is seen that if the Maurer - Cartan equation form ω would be a connection 1 - form, the curvature of this connection would vanish.

In contrast to the connection 1 - form, the curvature 2 - form transforms as homogenous under a local gauge transformation such that $\mathcal{R} \rightarrow g^{-1}\mathcal{R}g$ and it serves a gauge invariance term, i.e. Yang - Mills invariance $\mathcal{R} \wedge *\mathcal{R}$. However, since the Maurer - Cartan 1 - form is a invariant form, it becomes also a flat connection, and so the quadratic term

$$\text{tr}[\omega_C^A \wedge *\omega_B^C] \tag{18}$$

has a gauge invariance term.

More generally a Lagrangian formulation can be investigated. Such a formalism is maden to adding some nonlinear terms of the invariance terms into Lagrangian, i.e in Einstein' s theory including some quadratic nonlinear terms (in proper Ricci scalar) of curvature relating the semi - quantum gravity and the accelerated cosmological models [6]. However, another example of such an approach is given by Krasnov [11]. Here we will investigate a formalism using the nonlinear of the invariance term $\text{tr}[\omega_C^A \wedge *\omega_B^C]$ of the flat connection ω . We know that $\mathcal{C}^\infty(M) = \Lambda^0(M) \cong \Lambda^n(M)$. Therefore we define a smooth scalar follow

$$\mathbf{u} = \text{tr}[\omega_C^A \wedge *\omega_B^C] \in \mathcal{C}^\infty(M), \tag{19}$$

$$\mathcal{L} = (\Lambda_0 + f(\mathbf{u})) * 1, \tag{20}$$

where Λ_0 is the vacuum energy density and

$$\begin{aligned} f(\mathbf{u}) &= \text{Nonlinear Terms of } \mathbf{u}, \text{ i.e} \\ &= -\frac{1}{16\pi G}\mathbf{u} + a\mathbf{u}^2 + \frac{b}{\mathbf{u}} + \dots \end{aligned} \tag{21}$$

The action integral is such that

$$S[\epsilon, \omega] = \int_M (\mathcal{L}_{Gr} + \mathcal{L}_{Mat}). \tag{22}$$

Hence, the energy - momentum and spin equations are such that, respectively,

$$-\mathfrak{T}^A = (F + \Lambda_0) * \epsilon^A, \tag{23}$$

$$-\mathfrak{D}_B^A = 2(f' + \Lambda_0) * \omega_B^A, \tag{24}$$

where

$$f' = \frac{df}{d\mathbf{u}}, \tag{25}$$

$$F(\mathbf{u}) = f(\mathbf{u}) - \mathbf{u}f'. \quad (26)$$

Thus, this model with torsion free serves a spin current form controlled by the $f(\mathbf{u})$ and Λ_0 .

5. Conclusion

The equation eq. (23) emphasises an importance for the parameter Λ . Even though this parameter is different zero, we cannot knowledge about energy - momentum behavior of the spinning matter. Therefore, the nonlinear diffeomorphism approach gets fell its own importance. Defining a parameter follow

$$\Lambda(\mathbf{u}) = F(\mathbf{u}) + \Lambda_0 \quad (27)$$

and considering the vacuum energy density Λ and coupling constant G given in eqs. (23) and (24), respectively, the energy - momentum and spin equations are rewritten as

$$-\mathfrak{T}^A = \Lambda(\mathbf{u}) * \epsilon^A \quad (28)$$

$$-\mathfrak{D}_B^A = \frac{1}{8\pi G(\mathbf{u})} * \omega_B^A, \quad (29)$$

where

$$\frac{1}{16\pi G(\mathbf{u})} = (f' + \Lambda_0). \quad (30)$$

If one chooses as $f(\mathbf{u}) = \mathbf{u} + Constant$, i.e Einstein - Hilbert Lagrangian with cosmological constant $\mathcal{R} - 2\Lambda$, then $F(\mathbf{u}) = 0$. In addition to this, one may consider a (or some) limit value(s) of the \mathbf{u} which makes $F(\mathbf{u}) = 0$, that is $\mathbf{u}_0 \in \{\mathbf{u} \mid f(\mathbf{u}) - \mathbf{u}f' = 0\}$ for any $f(\mathbf{u})$. Therefore, under such conditions being $F(\mathbf{u}) = 0$, the parameter $\Lambda(\mathbf{u})$ approaches to the vacuum value, $\Lambda(\mathbf{u}) \rightarrow \Lambda_0$. On the other hand, we say $G(\mathbf{u}) \rightarrow G$ and the $\frac{1}{16\pi G(\mathbf{u})} \rightarrow (1 + \Lambda_0)$. Therefore the gravitational constant is proportional to the energy density of the vacuum:

$$\frac{1}{16\pi G} \sim \Lambda_0. \quad (31)$$

In Krasnov [11] the constant curvature is proportional by the square of a dimensionful parameter M_0 with dimensions of mass. The fourth power of this parameter is proportional by the cosmological constant Λ and by the gravitational coupling constant as $\frac{1}{G}$, that is we write $M_0^4 \sim \Lambda \sim \frac{1}{G}$. Thus we can see such a similar case in eq. (31).

Since the term $\|\omega\|^2 = \text{tr}[\omega_C^A \wedge * \omega_B^C]$ relates also to the norm of the connection 1 - form ω , the stable value of the functional $\int_M \frac{1}{16\pi G} \text{tr}[\omega_C^A \wedge * \omega_B^C]$ becomes proportional to $\Lambda \sim \frac{1}{G}$. Then the actions integral of the total Lagrangians are bounded follow

$$\Lambda(\sim \frac{1}{G}) \leq \int_M (\mathcal{L}_{Mat} + \mathcal{L}_{Gr}) < \infty.$$

As seen from this case that the gauge invariance term $\text{tr}[\omega_C^A \wedge * \omega_B^C]$ of a flat connection which created by a tangent bundle endomorphism can enough to investigate the theory of the spinning matter gravitationally interacting, without torsion and / or an extra symmetry group with translation part, i.e Poincaré group.

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