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SOFT L-FUZZY PRE-UNIFORM SPACES INDUCED BY SOFT L-FUZZY TOPOGENOUS ORDERS

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Abstract: In this paper, we introduce the notions of soft L-fuzzy pre-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices. We study some relationships between previous spaces. As main results, we investigate the soft L-fuzzy pre-uniformities (resp. soft L-quasi-uniformities) induced by soft L-fuzzy topogenous orders (resp. soft L-fuzzy topogenous structure). We give their examples.

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1. Introduction

Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [5,7-9]. Recently, Molodtsov [11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3]. Pawlak's rough set [12,13] can be viewed as a special case of soft rough sets [3]. The topological structures of soft sets have been developed by many researchers [2,7-9,14-17].

Ko [7] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft F :

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$A \to P(U)$ where L is a complete residuated lattice. Ko [7-9] introduced the soft topological structures, L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices.

In this paper, we introduce the notions of soft L-fuzzy pre-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices. We study some relationships between previous spaces. As main results, we investigate the soft L-fuzzy pre-uniformities (resp. soft L-fuzzy quasi-uniformities) induced by soft L-fuzzy topogenous orders (resp. soft L-fuzzy topogenous structure). We give their examples.

2. Preliminaries

Definition 1. [4,5] An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \to, \oplus, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by $x \oplus y = (x^* \odot y^*)^*$ and $x^* = x \to 0$.

Lemma 2. [4,5] *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $1 \to x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$,

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(6) $x \oplus (\bigwedge_i y_i) = \bigwedge_i (x \oplus y_i)$,

(7) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,

(8) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,

(9) $x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i)$,

(10) $(\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y)$,

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(12) $x \odot (x \to y) \leq y$ and $x \to y \leq (y \to z) \to (x \to z)$,

(13) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,

(14) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,

(15) $x \to y \leq (x \odot z) \to (y \odot z)$ and $(x \to y) \odot (y \to z) \leq x \to z$,

(16) $x \odot y \odot (z \odot w) \leq (x \odot z) \oplus (y \odot w)$.

(17) $x \to y = y^* \to x^*$.

Definition 3. [7-9] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 4. [7-9] Let (F, A) and (G, A) be two fuzzy soft sets over a common universe X.

(1) (F, A) is a fuzzy soft subset of (G, A), denoted by $(F, A) \leq (G, A)$ if $F(\epsilon) \leq G(\epsilon)$, for each $\epsilon \in A$.

(2) $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(\epsilon) = F(\epsilon) \wedge G(\epsilon)$ for each $\epsilon \in A$.

(3) $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(\epsilon) = F(\epsilon) \vee G(\epsilon)$ for each $\epsilon \in A$.

(4) $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(\epsilon) = F(\epsilon) \odot G(\epsilon)$ for each $\epsilon \in A$.

(5) $(F, A)^* = (F^*, A)$ if $F^*(\epsilon) = (F(\epsilon))^*$ for each $\epsilon \in A$.

(6) $(F, A) \oplus (G, A) = (F \oplus G, A)$ if $(F \oplus G)(\epsilon) = (F^*(\epsilon) \odot G^*(\epsilon))^*$ for each $\epsilon \in A$.

Definition 5. [9] A mapping $\xi : S(X, A) \times S(X, A) \to L$ is called a soft L-fuzzy semi-topogenous order on (X, A) if it satisfies the following axioms.

(ST1) $\xi((1_X, A), (1_X, A)) = \xi((0_X, A), (0_X, A)) = 1$.

(ST2) If $\xi((F, A), (G, A)) \neq 0$, then $(F, A) \leq (G, A)$.

(ST3) If $(F_1,A) \leq (F,A)$, $(G,A) \leq (G_1,A)$, then $\xi((F,A),(G,A)) \leq \xi((F_1,A),(G_1,A))$.

A mapping ξ is called a soft strong L-fuzzy semi-topogenous order on (X,A) if it satisfies (ST1), (ST3) and the following axiom.

(S) $\xi((F,A),(G,A)) \leq S((F,A),(G,A))$ where

$$S((F,A),(G,A)) = \bigwedge_{a\in A}\bigwedge_{x\in X}(F(a)(x) \to G(a)(x)).$$

Remark 6. If ξ is a soft (resp. strong) L-fuzzy semi-topogenous order on (X,A). Define a mapping $\xi^s : S(X,A)\times S(X,A) \to L$ as $\xi^s((F,A),(G,A)) = \xi((G,A)^*,(F,A)^*)$. Then ξ^s is a soft (resp. strong) L-fuzzy semi-topogenous order on (X,A).

Definition 7. [9] A soft (resp. strong) L-fuzzy semi-topogenous order ξ is called:

(1) soft (resp. strong) L-fuzzy topogenous if (T)

$$\begin{aligned}&\xi((F_1,A)\odot(F_2,A),(G_1,A)\odot(G_2,A))\\ &\geq \xi((F_1,A),(G_1,A))\odot\xi((F_2,A),(G_2,A)).\end{aligned}$$

(2) soft (resp. strong) L-fuzzy cotopogenous if (CT)

$$\begin{aligned}&\xi((F_1,A)\oplus(F_2,A),(G_1,A)\oplus(G_2,A))\\ &\geq \xi((F_1,A),(G_1,A))\odot\xi((F_2,A),(G_2,A)),\end{aligned}$$

(3) soft (resp. strong) L-fuzzy bitopogenous if ξ are soft (resp. strong) L-fuzzy topogenous and soft (resp. strong) L-fuzzy cotopogenous.

Definition 8. [9] A soft (resp. strong) L-fuzzy topogenous (resp. cotopogenous) order ξ on (X,A) is said to be a soft (resp. strong) L-fuzzy topogenous (resp. cotopogenous) structure if $\xi\circ\xi \geq \xi$, where

$$\begin{aligned}&(\xi\circ\xi)((F,A),(H,A))\\ &= \textstyle\bigvee_{(G,A)\in S(X,A)}\xi((F,A),(G,A))\odot\xi((G,A),(H,A)).\end{aligned}$$

Definition 9. [9] A mapping $\mathcal{U} : S(X\times X,A) \to L$ is called a soft L-fuzzy pre-uniformity on X iff it satisfies the properties.

(SU1) There exists $(U,A) \in S(X\times X,A)$ such that $\mathcal{U}((U,A)) = 1$,

(SU2) If $(V,A) \leq (U,A)$, then $\mathcal{U}((V,A)) \leq \mathcal{U}((U,A))$, (SU3) For every $(U,A),(V,A)\in S(X\times X,A)$,

$$\mathcal{U}((U,A)\odot(V,A))\geq\mathcal{U}((U,A))\odot\mathcal{U}((V,A))$$

(SU4) If $\mathcal{U}((U,A))\neq 0$, then $(1_{\triangle},A)\leq(U,A)$, where

$$1_{\Delta}(x,y)=\begin{cases}1 & \text{if } x=y\\ 0 & \text{if } x\neq y,\end{cases}$$

A soft L-fuzzy pre-uniformity $\mathcal{U}$ is called a soft L-fuzzy quasi-uniformity if

(Q) $\mathcal{U}\leq\mathcal{U}\circ\mathcal{U}$, where $\mathcal{U}\circ\mathcal{U}((U,A))=\bigvee\{\mathcal{U}((V,A))\odot\mathcal{U}((W,A))\mid(V,A)\circ(W,A)\leq(U,A)\}$, for all $x,y\in X, a\in A$,

$$\begin{aligned}&(V(a)\circ W(a))(x,y)\\ &=\textstyle\bigvee_{z\in X}(V(a)(z,x)\odot W(a)(x,y)).\end{aligned}$$

The triple $(X,A,\mathcal{U})$ is called a soft L-fuzzy pre-(resp. quasi-) uniform space.

Remark 10. Let $(X,\mathcal{U})$ be a soft L-fuzzy quasi-uniform space, then by (SU1) and (SU2), we have $\mathcal{U}(1_{X\times X})=1$ because $(U,A)\leq(1_{X\times X},A)$ for all $(U,A)\in S(X\times X,A)$.

3. Soft L-Fuzzy Pre-Uniform Spaces Induced by Soft L-Fuzzy Topogenous Orders

Lemma 11. For every $(F,A),(G,A)\in S(X,A)$, we define $(U_{F,G},A)$, $(U^{-1}_{F,G},A)\in S(X\times X,A)$ by

$$U_{F,G}(a)(x,y)=F(a)(x)\rightarrow G(a)(y)$$

$$U^{-1}_{F,G}(a)(x,y)=U_{F,G}(a)(y,x).$$

Then we have the following statements:

(1) $(1_{X\times X},A)=(U_{0_X,0_X},A)=(U_{1_X,1_X},A)$,

(2) If $(F_1,A)\leq(F_2,A)$ and $(G_1,A)\leq(G_2,A)$, then $(U_{F_2,G_1},A)\leq(U_{F_1,G_2},A)$,

(3) If $(F,A)\leq(G,A)$, then $(1_{\triangle},A)\leq(U_{F,G},A)$,

(4) For every $(U_{F,G},A)\in S(X\times X,A)$ and $(H,A)\in S(X,A)$, we have $(U_{H,G},A)\circ(U_{F,H},A)\leq(U_{F,G},A)$.

(5) $(U_{F_1,G_1}, A) \odot (U_{F_2,G_2}, A) \le (U_{F_1\odot F_2, G_1\odot G_2}, A)$

(6) $(U_{F_1,G_1}, A) \odot (U_{F_2,G_2}, A) \le (U_{F_1\oplus F_2, G_1\oplus G_2}, A)$

(7) $(U^{-1}_{F,G}, A) = (U_{G^*,F^*}, A)$

(8) $(U^{-1}_{F_1,\odot F_2, G_1\odot G_2}, A) = (U_{G_1^*\oplus G_2^*, F_1^*\oplus F_2^*}, A)$

(9) $(U^{-1}_{F_1\oplus F_2, G_1\oplus G_2}, A) = (U_{G_1^*\odot G_2^*, F_1^*\odot F_2^*}, A)$

Proof. (1)

$$\begin{aligned} &1_{X\times X}(a)(x,y) = 1 = U_{0_X,0_X}(a)(x,y) = 0_X(a)(x) \to 0_X(a)(y) \\ &= 1_X(a)(x) \to 1_X(a)(y) = U_{1_X,1_X}(a)(x,y). \end{aligned}$$

(2) Let $(F_1, A) \le (F_2, A)$ and $(G_1, A) \le (G_2, A)$.

$$\begin{aligned} &U_{F_2,G_1}(a)(x,y) = F_2(a)(x) \to G_1(a)(y) \\ &\le F_1(a)(x) \to G_2(a)(y) = U_{F_1,G_2}(a)(x,y). \end{aligned}$$

(3) Let $(F, A) \le (G, A)$. We have $(1_\triangle, A) \le (U_{F,G}, A)$ from

$$U_{F,G}(a)(x,x) = F(a)(x) \to G(a)(x) = 1.$$

(4)

$$\begin{aligned} &U_{H,G}(a)(x,z) \circ U_{F,H}(a)(x,z) \\ &= \bigvee_{y\in X}((H(a)(y) \to G(a)(z)) \odot (F(a)(x) \to H(a)(y))) \\ &\le F(a)(x) \to G(a)(z) = U_{F,G}(a)(x,z). \end{aligned}$$

(5) By Lemma 2(13), we have

$$\begin{aligned} &U_{F_1,G_1}(a)(x,y) \odot U_{F_2,G_2}(a)(x,y) \\ &= (F_1(a)(x) \to G_1(a)(y)) \odot (F_2(a)(x) \to G_2(a)(y)) \\ &\le (F_1(a)(x) \odot F_2(a)(x) \to G_1(a)(y) \odot G_2(a)(y) \\ &= U_{F_1\odot F_2, G_1\odot G_2}(a)(x,y). \end{aligned}$$

(6) By Lemma 2(14), we have

$$\begin{aligned} &U_{F_1,G_1}(a)(x,y) \odot U_{F_2,G_2}(a)(x,y) \\ &= (F_1(a)(x) \to G_1(a)(y)) \odot (F_2(a)(x) \to G_2(a)(y)) \\ &\le (F_1(a)(x) \oplus F_2(a)(x) \to G_1(a)(y) \oplus G_2(a)(y) \\ &= U_{F_1\oplus F_2, G_1\oplus G_2}(a)(x,y). \end{aligned}$$

(7) By Lemma 2(17), we have

$$U^{-1}_{F,G}(a)(x,y) = U_{F,G}(a)(y,x) = F(a)(y) \to G(a)(x)$$
$$= G(a)^*(x) \to F(a)^*(y) = U_{G^*,F^*}(a)(x,y).$$

(8)

$$U^{-1}_{F_1\odot F_2, G_1\odot G_2} = U_{(G_1\odot G_2)^*,(F_1\odot F_2)^*} = U_{G_1^*\oplus G_2^*, F_1^*\oplus F_2^*}.$$

(9)

$$U^{-1}_{F_1\oplus F_2, G_1\oplus G_2} = U_{(G_1\oplus G_2)^*,(F_1\oplus F_2)^*} = U_{G_1^*\odot G_2^*, F_1^*\odot F_2^*}.$$

Lemma 12. Let $(X, A, \mathcal{U})$ be a soft L-fuzzy quasi uniform space. For each $(U,A) \in S(X\times X, A)$ and $(F,A) \in S(X,A)$, the images $(U,A)[(F,A)], (U,A)[[(F,A)]]$ of (F,A) with respect to (U,A) are defined by, for all $x \in X, a \in A$,

$$(U,A)[(F,A)](a)(x) = \bigvee_{y\in X} (F(a)(y) \odot U(a)(y,x)),$$

$$(U,A)[[(F,A)]](a)(x) = \bigvee_{y\in X} (F(a)(y) \odot U(a)(x,y)).$$

For each $(U,A),(V,A),(U_1,A),(U_2,A) \in S(X\times X, A)$ and $(F,A),(G,A),(F_1,A),(F_2,A),(F_i,A) \in S(X,A)$, we have

(1) $(F,A) \le (U,A)[(F,A)]$ and $(F,A) \le (U,A)[[(F,A)]]$ for each $\mathcal{U}((U,A)) > 0$,

(2) $(U,A) \le (U,A)\circ(U,A)$, for each $\mathcal{U}((U,A)) > 0$,

(3) $((V,A)\circ(U,A))[(F,A)] = (V,A)[(U,A)[(F,A)]]$ $((V,A)\circ(U,A))[[(F,A)]] = (V,A)[[(U,A)[[(F,A)]]]]$,

(4) $(U,A)[\bigvee_i (F_i,A)] = \bigvee_i (U,A)[(F_i,A)]$ and $(U,A)[[\bigvee_i (F_i,A)]] = \bigvee_i (U,A)[[(F_i,A)]]$

(5) $((U_1,A)\odot(U_2,A))[(F_1,A)\odot(F_2,A)] \le (U_1,A)[(F_1,A)]\odot(U_2,A)[(F_2,A)]$,

(6) $((U_1,A)\odot(U_2,A))[[(F_1,A)\odot(F_2,A)]] \le (U_1,A)[[(F_1,A)]]\odot(U_2,A)[[(F_2,A)]]$,

(7) $((U_1,A)\odot((U_2,A),A))[(F_1,A)\oplus(F_2,A)] \le (U_1,A)[(F_1,A)]\oplus((U_2,A),A)[(F_2,A)]$,

(8) $((U_1,A)\odot((U_2,A),A))[[(F_1,A)\oplus(F_2,A)]]$
$\le (U_1,A)[[(F_1,A)]]\oplus((U_2,A),A)[[(F_2,A)]]$.

(9) $(U_{F,G},A)=\bigvee\{(W,A)\in S(X\times X,A)|(W,A)[(F,A)]\le(G,A)\}$.

(10) $(U^{-1}_{F,G},A)=\bigvee\{(W,A)\in S(X\times X,A)|(W,A)[[(F,A)]]\le(G,A)\}$.

(11) $(U_{F,G},A)[(F,A)]\le(G,A)$ and $(U_{F,G},A)[[(G,A)^*]]\le(F,A)^*$. Moreover, $(U_{F,F},A)[(F,A)]=(F,A)$ and $(U_{F,F},A)[(F,A)^*]=(F,A)^*$.

Proof. (1)-(8) follows from [9].

(9) Since

$$(W,A)[(F,A)](a)(y)=\bigvee_{x\in X}((W,A)(a)(x,y)\odot(F,A)(a)(x))\le(G,A)(y),$$

then $(W,A)(a)(x,y)\le F(a)(x)\to G(a)(y)=U_{F,G}(a)(x,y)$. So, $U_{F,G}(a)(x,y)\odot(F,A)(a)(x)=(F(a)(x)\to G(a)(y))\odot(F,A)(a)(x)\le G(a)(y)$. Hence

$$(U_{F,G},A)=\bigvee\{(W,A)\in S(X\times X,A)|(W,A)[(F,A)]\le(G,A)\}.$$

(10) Since $(W,A)[[(F,A)]](a)(y)=\bigvee_{x\in X}((W,A)(a)(y,x)\odot(F,A)(a)(x))\le(G,A)(y)$, then $(W,A)(a)(y,x)\le F(a)(x)\to G(a)(y)=U_{F,G}(a)(x,y)=U^{-1}_{F,G}(a)(y,x)$. So, $U^{-1}_{F,G}(a)(y,x)\odot(F,A)(a)(x)=(F(a)(x)\to G(a)(y))\odot(F,A)(a)(x)\le G(a)(y)$. Hence

$$(U^{-1}_{F,G},A)=\bigvee\{(W,A)\in S(X\times X,A)|(W,A)[[(F,A)]]\le(G,A)\}.$$

(11) Since

$$(U_{F,G},A)[(F,A)](a)(y)=\bigvee_{x\in X}((F(a)(x)\to G(a)(y))\odot F(a)(x))\le G(a)(y),$$

then $(U_{F,G},A)[(F,A)]\le G$. Since

$$\begin{array}{l}(U_{F,G},A)[[(G^*,A)]](a)(x)=\bigvee_{y\in X}((F(a)(x)\to G(a)(y))\odot G^*(a)(y))\\ =\bigvee_{y\in X}((G^*(a)(y)\to F^*(a)(x))\odot G^*(a)(y))\le F^*(a)(x),\end{array}$$

then $(U_{F,G},A)[[(G,A)^*]]\le(F,A)^*$. Moreover,

$$\begin{array}{l}(U_{F,F},A)[(F,A)](a)(y)\ge U_{F,F}(y,y)\odot F(a)(y)\\ (F(a)(y)\to F(a)(y))\odot F(a)(y)=F(a)(y).\end{array}$$

$$(U_{F,F}, A)[[(F, A)^*]](a)(y) \geq U_{F,F}(y, y) \odot F^*(a)(y)$$
$$(F^*(a)(y) \to F^*(a)(y)) \odot F^*(a)(y) = F^*(a)(y).$$

Theorem 13. [9] *Let $(X, A, \mathcal{U})$ be a soft L-fuzzy quasi-uniform space. Define mappings $\xi^r_{\mathcal{U}}, \xi^l_{\mathcal{U}} : S(X, A) \times S(X, A) \to L$ by*

$$\xi^r_{\mathcal{U}}((F, A), (G, A)) = \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[(F, A)] \leq (G, A)\},$$
$$\xi^l_{\mathcal{U}}((F, A), (G, A)) = \bigvee\{\mathcal{U}((U, A)) \mid (U, A)[[(F, A)]] \leq (G, A)\}.$$

Then $\xi^r_{\mathcal{U}}$ and $\xi^l_{\mathcal{U}}$ are soft L-fuzzy bitopogenous stuctures.

In the following theorem, we obtain a soft L-fuzzy pre-uniform structure from a soft L-fuzzy topogenous order.

Theorem 14. Let ξ be a soft L-fuzzy topogenous order on (X, A). Define $\mathcal{U}_\xi : S(X \times X, A) \to L$ by

$$\mathcal{U}_\xi((U, A)) = \bigvee\{\odot^n_{i=1}\xi((F_i, A), (G_i, A)) \mid \odot^n_{i=1} (U_{F_i,G_i}, A) \leq (U, A)\},$$

where $\bigvee$ is taken over every finite family $\{(U_{F_i,G_i}, A) \mid i = 1, 2, 3, ..., n\}$. Then

(1) $\mathcal{U}_\xi((U_{\odot^n_{i=1}F_i,\odot^n_{i=1}G_i}, A)) = \xi(\odot^n_{i=1}(F_i, A), \odot^n_{i=1}(G_i, A))$.

(2) $\mathcal{U}_\xi$ is a soft L-fuzzy pre-uniformity on X.

(3) If ξ is a soft L-fuzzy topogenous structure on (X, A), then $\mathcal{U}_\xi$ is a soft L-fuzzy quasi-uniformity on X.

(4) $\xi^r_{\mathcal{U}_\xi} = \xi$ and $\xi^l_{\mathcal{U}_\xi} = \xi^s$.

Proof. (1) Since $\odot^n_{i=1}(U_{F_i,G_i}, A) \leq (U_{\odot^n_{i=1}F_i,\odot^n_{i=1}G_i}, A)$ from Lemma 11 (5) and, by (T),

$$\odot^n_{i=1}\xi((F_i, A), (G_i, A)) \leq \xi(\odot^n_{i=1}(F_i, A), \odot^n_{i=1}(G_i, A)),$$

$$\mathcal{U}_\xi((U_{\odot^n_{i=1}F_i,\odot^n_{i=1}G_i}, A)) = \xi(\odot^n_{i=1}(F_i, A), \odot^n_{i=1}(G_i, A)).$$

(2) (SU1) Since $\xi((0_X, A), (0_X, A)) = \xi((1_X, A), (1_X, A)) = 1$, there exists $(1_{X\times X}, A) = (U_{0_X,0_X}, A) = (U_{1_X,1_X}, A) \in S(X \times X, A)$. It follows

$$\mathcal{U}_\xi((1_{X\times X}, A)) = 1.$$

(SU2) It is trivial from the definition of $\mathcal{U}_\xi$.

(SU3) For every $(U,A),(V,A)\in S(X\times X,A)$, each two families $\{U_{F_i,G_i}\mid \odot_{i=1}^n U_{F_i,G_i}\le U\}$ and $\{U_{H_j,K_j}\mid \odot_{j=1}^k U_{H_j,K_j}\le V\}$, we have

$$\begin{array}{l}\mathcal{U}_\xi((U,A))\odot\mathcal{U}_\xi((V,A))\\ =\big(\bigvee\{\odot_{i=1}^n\xi((F_i,A),(G_i,A))\mid \odot_{i=1}^n(U_{F_i,G_i},A)\le(U,A)\}\big)\\ \odot\big(\bigvee\{\odot_{j=1}^k\xi((H_i,A),(K_i,A))\mid \odot_{j=1}^k(U_{H_i,K_i},A)\le(V,A)\}\big)\\ \le\bigvee\{\odot_{i=1}^n\xi((F_i,A),(G_i,A))\odot(\odot_{j=1}^k\xi((H_i,A),(K_i,A))\mid\\ \odot_{i=1}^n(U_{F_i,G_i},A)\odot\odot_{j=1}^k(U_{H_i,K_i},A)\le(U,A)\odot(V,A)\}\\ \le\mathcal{U}_\xi((U,A)\odot(V,A)).\end{array}$$

(SU4) If $\mathcal{U}((U,A))\ne 0$, there exists a family $\{(U_{F_i,G_i},A)\mid\odot_{i=1}^m(U_{F_i,G_i},A)\le(U,A)\}$ such that

$$\odot_{i=1}^m\xi((F_i,A),(G_i,A))\ne 0.$$

Since $\xi((F_i,A),(G_i,A))\ne 0$ for $i=1,...m$, then $(F_i,A)\le(G_i,A)$ for $i=1,...m$, i.e. $(1_\triangle,A)\le(U_{F_i,G_i},A)$. Thus

$(1_\triangle,A)\le\odot_{i=1}^m(U_{F_i,G_i},A)\le(U,A)$. Then $\mathcal{U}_\xi$ is a soft L-fuzzy pre-uniformity on X.

(3) Let ξ be a soft L-fuzzy topogenous structure on (X,A).

(Q) Suppose there exists $(U,A)\in S(X\times X,A)$ such that

$$\begin{array}{l}t=\bigvee\{\mathcal{U}_\xi((V,A))\odot\mathcal{U}_\xi((W,A))\mid(V,A)\circ(W,A)\le(U,A)\}\\ \not\ge\mathcal{U}_\xi((U,A)).\end{array}$$

From the definition of $\mathcal{U}_\xi((U,A))$, there exists a family

$$\{(U_{F_i,G_i},A)\mid\odot_{i=1}^m(U_{F_i,G_i},A)\le(U,A)\}$$

such that

$$t\not\ge\odot_{i=1}^m\xi((F_i,A),(G_i,A)).$$

Since $\xi\circ\xi\ge\xi$,

$$\begin{array}{rl}t&\not\ge\odot_{i=1}^m(\xi\circ\xi)((F_i,A),(G_i,A))\\ &=\odot_{i=1}^m\{\bigvee_{(H,A)\in S(X,A)}(\xi((H,A),(G_i,A))\odot\xi((F_i,A),(H,A))\}.\end{array}$$

Since L is a complete residuated lattice, there exists $(H_i,A)\in S(X,A)$ such that

$$t\not\ge\odot_{i=1}^m(\xi((H_i,A),(G_i,A))\odot\xi((F_i,A),(H_i,A))).\qquad\text{(I)}$$

On the other hand, put $(V_i,A)=(U_{H_i,G_i},A),(W_i,A)=(U_{F_i,H_i},A)$. From Lemma 11(4), it satisfies

$$(V_i,A)\circ(W_i,A)=(U_{H_i,G_i},A)\circ(U_{F_i,H_i},A)\le(U_{F_i,G_i},A),$$

$$\mathcal{U}_\xi((V_i,A)) \geq \xi((H_i,A),(G_i,A)),$$

$$\mathcal{U}_\xi((W_i,A)) \geq \xi((F_i,A),(H_i,A)).$$

Let $(V,A) = \odot_{i=1}^m (V_i,A)$ and $(W,A) = \odot_{i=1}^m (W_i,A)$ be given. Since $(V_i,A) \circ (W_i,A) \leq (U_{F_i,G_i},A)$ for each $i = 1,...,m$, we have

$$\begin{aligned}&(\odot_{i=1}^m (V_i,A)) \circ (\odot_{i=1}^m (W_i,A)) = \odot_{i=1}^m ((V_i,A) \circ (W_i,A))\\&\leq \odot_{i=1}^m (U_{F_i,G_i},A) \leq (U,A).\end{aligned}$$

Then we have $(V,A) \circ (W,A) \leq (U,A)$ and $\mathcal{U}_\xi((V,A)) \geq \odot_{i=1}^m \mathcal{U}_\xi((V_i,A))$ and $\mathcal{U}_\xi((W,A)) \geq \odot_{i=1}^m \mathcal{U}_\xi((W_i,A))$. Thus,

$$\begin{aligned}&t = \bigvee\{\mathcal{U}_\xi((V,A)) \odot \mathcal{U}_\xi((W,A)) \mid (V,A) \circ (W,A) \leq (U,A)\}\\&\geq \mathcal{U}_\xi((V,A)) \odot \mathcal{U}_\xi((W,A))\\&\geq \odot_{i=1}^m \mathcal{U}_\xi((V_i,A)) \odot \odot_{i=1}^m \mathcal{U}_\xi((W_i,A))\\&\geq \odot_{i=1}^m (\xi((H_i,A),(G_i,A)) \odot \xi((F_i,A),(H_i,A)).\end{aligned}$$

It is a contradiction for the equation (I).

Then $\mathcal{U}_\xi$ is a soft L-fuzzy quasi uniformity on X.

(4)

$$\begin{aligned}&\xi^r_{\mathcal{U}_\xi}((F,A),(G,A)) = \bigvee\{\mathcal{U}_\xi((U,A)) \mid (U,A)[(F,A)] \leq (G,A)\}\\&= \mathcal{U}_\xi(U_{F,G}) = \xi((F,A),(G,A)).\end{aligned}$$

$$\begin{aligned}&\xi^l_{\mathcal{U}_\xi}((F,A),(G,A)) = \bigvee\{\mathcal{U}_\xi((U,A)) \mid (U,A)[[(F,A)]] \leq (G,A)\}\\&= \mathcal{U}_\xi(U^{-1}_{F,G}) = \xi^s((F,A),(G,A)).\end{aligned}$$

Theorem 15. Let ξ be a soft L-fuzzy cotopogenous order on (X,A). Define $\mathcal{U}_\xi : S(X \times X, A) \to L$ by

$$\begin{aligned}\mathcal{U}_\xi((U,A)) \ &= \bigvee\{\odot_{i=1}^n \xi((F_i,A),(G_i,A)) \mid\\&\odot_{i=1}^n (U_{F_i,G_i},A) \leq (U,A)\},\end{aligned}$$

where $\bigvee$ is taken over every finite family $\{(U_{F_i,G_i},A) \mid i = 1,2,3,....,n\}$. Then

(1) $\mathcal{U}_\xi((U_{\oplus_{i=1}^n F_i, \oplus_{i=1}^n G_i},A)) = \xi(\oplus_{i=1}^n (F_i,A), \oplus_{i=1}^n (G_i,A))$.

(2) $\mathcal{U}_\xi$ is a soft L-fuzzy pre-uniformity on X.

(3) If ξ is a soft L-fuzzy cotopogenous structure on (X,A), then $\mathcal{U}_\xi$ is a soft L-fuzzy quasi-uniformity on X.

(4) $\mathcal{U}_{\xi^s}((U,A)) = \mathcal{U}_\xi((U,A)^{-1})$ for all $(U,A) \in S(X \times X, L)$.

(5) $\xi^r_{\mathcal{U}_\xi} = \xi$ and $\xi^l_{\mathcal{U}_\xi} = \xi^s$.

Proof. (1) Since $\odot_{i=1}^n(U_{F_i,G_i},A) \leq (U_{\oplus_{i=1}^n F_i,\oplus_{i=1}^n G_i},A)$ from Lemma 11 (6) and, by (CT),
$\odot_{i=1}^n \xi((F_i,A),(G_i,A)) \leq \xi(\oplus_{i=1}^n(F_i,A),\oplus_{i=1}^n(G_i,A))$, then

$$\mathcal{U}_\xi((U_{\oplus_{i=1}^n F_i,\oplus_{i=1}^n G_i},A)) = \xi(\oplus_{i=1}^n(F_i,A),\oplus_{i=1}^n(G_i,A)).$$

(4)

$$\begin{aligned}
&\mathcal{U}_{\xi^s}((U,A)) = \bigvee\{\odot_{i=1}^n \xi^s((F_i,A),(G_i,A)) \mid \odot_{i=1}^n (U_{F_i,G_i},A) \leq (U,A)\}\\
&= \bigvee\{\odot_{i=1}^n \xi((G_i^*,A),(F_i^*,A)) \mid \odot_{i=1}^n (U_{F_i,G_i}^{-1},A) \leq (U,A)^{-1}\}\\
&= \bigvee\{\odot_{i=1}^n \xi((G_i^*,A),(F_i^*,A)) \mid \odot_{i=1}^n (U_{G_i^*,F_i^*},A) \leq (U,A)^{-1}\}\\
&= \mathcal{U}_\xi((U,A)^{-1}).
\end{aligned}$$

(2), (3) and (5) are similarly proved as Theorem 14.

Example 16. Let $U = \{h_i \mid i = \{1,...,6\}\}$ with h_i=house and $E = \{e,b,w,c,i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Define a binary operation $\odot$ on $[0,1]$ by

$$x \odot y = \max\{0, x+y-1\}, \quad x \to y = \min\{1-x+y, 1\},$$

$$x \oplus y = \min\{1, x+y\}, \quad x^* = 1-x.$$

Then $([0,1],\wedge,\to,0,1)$ is a complete residuated lattice (ref.[4.5]). Let $A = \{b,c,i\} \subset E$ and $X = \{h^1,h^4,h^5,h^6\}$. Put (H,A) be a fuzzy soft set as follow:

(H,A)	h^1	h^4	h^5	h^6
b	0.5	0.6	0.2	0.6
c	0.1	0.5	0.5	0.6
i	0.4	0.6	0.6	0.5

$(H,A)\odot(H,A)$	h^1	h^4	h^5	h^6
b	0.0	0.2	0.0	0.2
c	0.0	0.0	0.0	0.2
i	0.0	0.2	0.2	0.0

(H^*,A)	h^1	h^4	h^5	h^6
b	0.5	0.4	0.8	0.4
c	0.9	0.5	0.5	0.4
i	0.6	0.4	0.4	0.5

$(H^*,A)\oplus(H^*,A)$	h^1	h^4	h^5	h^6
b	1.0	0.8	1.0	0.8
c	1.0	1.0	1.0	0.8
i	1.0	0.8	0.8	1.0

(1) Define a soft L-fuzzy topogenous order $\xi : S(X,A)\times S(X,A)\to L$ as follows

$$\xi((F,A),(G,A)) = \begin{cases} 1, & \text{if } (F,A)=(\overline{0},A) \text{ or } (G,A)=(\overline{1},A) \\ 0.6, & \text{if } (F,A)\le (H,A)\le (G,A), \\ & (F,A)\not\le (H,A)\odot(H,A) \\ 0.3, & \text{if } (\overline{0},A)\ne (F,A)\le (H,A)\odot(H,A) \\ & \le (G,A), (H,A)\not\le (G,A), \\ 0, & \text{otherwise,} \end{cases}$$

But it is not a soft L-fuzzy topogenous structure because

$$\begin{aligned}&\bigvee_{(F,A)\in S(X,A)}(\xi((H,A)\odot(H,A),(F,A))\\ &\odot\xi((F,A),(H,A)\odot(H,A)))=0\\ &\not\ge \xi((H,A)\odot(H,A),(H,A)\odot(H,A))=0.3.\end{aligned}$$

We obtain $(U_{H,H},A),(U_{H\odot H,H\odot H},A)\in S(X\times X,A)$ such that, for $a\in A$, $U_{H,H}(a)\in L^{X\times X}$ with $U_{H,H}(a)(x,y)=H(a)(x)\to H(a)(y)$,

$$U_{H,H}(b)=\begin{pmatrix} 1 & 1 & 0.7 & 1\\ 0.9 & 1 & 0.6 & 1\\ 1 & 1 & 1 & 1\\ 0.9 & 1 & 0.6 & 1 \end{pmatrix}, U_{H,H}(c)=\begin{pmatrix} 1 & 1 & 1 & 1\\ 0.6 & 1 & 1 & 1\\ 0.6 & 1 & 1 & 1\\ 0.5 & 0.9 & 0.9 & 1 \end{pmatrix}$$

$$U_{H,H}(i)=\begin{pmatrix} 1 & 1 & 1 & 1\\ 0.8 & 1 & 1 & 0.9\\ 0.8 & 1 & 1 & 0.9\\ 0.9 & 1 & 1 & 1 \end{pmatrix}, U_{H\odot H,H\odot H}(b)=\begin{pmatrix} 1 & 1 & 1 & 1\\ 0.8 & 1 & 0.8 & 1\\ 1 & 1 & 1 & 1\\ 0.8 & 1 & 0.8 & 1 \end{pmatrix}$$

$$U_{H\odot H,H\odot H}(c)=\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 0.8 & 0.8 & 0.8 & 1 \end{pmatrix}, U_{H\odot H,H\odot H}(i)=\begin{pmatrix} 1 & 1 & 1 & 1\\ 0.8 & 1 & 1 & 0.8\\ 0.8 & 1 & 1 & 0.8\\ 1 & 1 & 1 & 1 \end{pmatrix}$$

By Theorem 14, we obtain a soft L-fuzzy pre-uniformity $\mathcal{U}_\xi : S(X\times X,A)\to$

L as follows

$$\mathcal{U}_\xi((U,A)) = \begin{cases} 1, & \text{if } (U,A) = (1_{X\times X}, A), \\ 0.6, & \text{if } (U_{H,H}, A) \leq (U,A) \neq (1_{X\times X}, A), \\ 0.3, & \text{if } (U_{H\odot H, H\odot H}, A) \leq (U,A) \\ & (U,A) \not\geq (U_{H,H}, A), \\ 0.2, & \text{if } (U_{H,H}, A) \odot (U_{H,H}, A) \leq (U,A), \\ & (U,A) \not\geq (U_{H\odot H, H\odot H}, A), \\ 0, & \text{otherwise.} \end{cases}$$

Since $H(b) \circ H(b) = H(b)$, we have

$$\begin{aligned} &\mathcal{U}_\xi \circ \mathcal{U}_\xi((U_{H,H}, A) \odot (U_{H,H}, A)) = 0 \\ &\neq 0.2 = \mathcal{U}_\xi((U_{H,H}, A) \odot (U_{H,H}, A)). \end{aligned}$$

Hence $\mathcal{U}_\xi$ is not a soft L-fuzzy quasi-uniformity.

(2) By Remark 6 and (1), we obtain a soft L-fuzzy cotopogenous order $\xi^s : S(X,A) \times S(X,A) \to L$ as follows

$$\xi^s((F,A),(G,A)) = \begin{cases} 1, & \text{if } (F,A) = (\overline{0}, A) \text{ or } (G,A) = (\overline{1}, A) \\ 0.6, & \text{if } (F,A) \leq (H,A)^* \leq (G,A), \\ & (G,A) \not\geq (H,A)^* \oplus (H,A)^* \\ 0.3, & \text{if } \ (F,A) \leq (H,A)^* \oplus (H,A)^* \\ & \leq (G,A) \neq (\overline{1}, A), (F,A) \not\leq (H,A)^*, \\ 0, & \text{otherwise,} \end{cases}$$

But it is not a soft L-fuzzy cotopogenous structure because

$$\begin{aligned} &\bigvee_{(F,A)\in S(X,A)} (\xi^s((H,A)^* \oplus (H,A)^*, (F,A)) \\ &\odot \xi^s((F,A), (H,A)^* \oplus (H,A)^*)) = 0 \\ &\not\geq \xi^s((H,A)^* \oplus (H,A)^*, (H,A)^* \oplus (H,A)^*) \end{aligned}$$

We obtain $(U_{H,H}, A), (U_{H\odot H, H\odot H}, A) \in S(X\times X, A)$ such that, for $a \in A$, $U_{H,H}(a) \in L^{X\times X}$ with $U_{H,H}(a)(x,y) = H(a)(x) \to H(a)(y)$,

$$U_{H^*,H^*}(b) = \begin{pmatrix} 1 & 0.9 & 1 & 0.9 \\ 1 & 1 & 1 & 1 \\ 0.7 & 0.6 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{pmatrix}, U_{H^*,H^*}(c) = \begin{pmatrix} 1 & 0.6 & 0.6 & 0.5 \\ 1 & 1 & 1 & 0.9 \\ 1 & 1 & 1 & 0.9 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$U_{H^*,H^*}(i) = \begin{pmatrix} 1 & 0.8 & 0.8 & 0.9 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0.9 & 0.9 & 1 \end{pmatrix}, U_{H^*\oplus H^*, H^*\oplus H^*}(b) = \begin{pmatrix} 1 & 0.8 & 1 & 0.8 \\ 1 & 1 & 1 & 1 \\ 1 & 0.8 & 1 & 0.8 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$U_{H^*\oplus H^*,H^*\oplus H^*}(c) = \begin{pmatrix} 1 & 1 & 1 & 0.8 \\ 1 & 1 & 1 & 0.8 \\ 1 & 1 & 1 & 0.8 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$U_{H^*\oplus H^*,H^*\oplus H^*}(i) = \begin{pmatrix} 1 & 0.8 & 0.8 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0.8 & 0.8 & 1 \end{pmatrix}$$

By Theorem 15, we obtain a soft L-fuzzy pre-uniformity $\mathcal{U}_{\xi^s} : S(X \times X, A) \to L$ as follows

$$\mathcal{U}_{\xi^s}((U,A)) = \begin{cases} 1, & \text{if } (U,A) = (1_{X\times X}, A), \\ 0.6, & \text{if } (U_{H^*,H^*}, A) \leq (U,A) \neq (1_{X\times X}, A), \\ 0.3, & \text{if } (U_{H^*\oplus H^*,H^*\oplus H^*}, A) \leq (U,A) \\ & (U,A) \not\geq (U_{H^*,H^*}, A), \\ 0.2, & \text{if } (U_{H^*,H^*}, A) \odot (U_{H^*,H^*}, A) \leq (U,A), \\ & (U,A) \not\geq (U_{H^*\oplus H^*,H^*\oplus H^*}, A), \\ 0, & \text{otherwise.} \end{cases}$$

Since $H^*(b) \circ H^*(b) = H^*(b)$, we have

$$\mathcal{U}_{\xi^s} \circ \mathcal{U}_{\xi^s}((U_{H^*,H^*}, A) \odot (U_{H^*,H^*}, A)) = 0$$
$$\neq 0.2 = \mathcal{U}_{\xi^s}((U_{H^*,H^*}, A) \odot (U_{H^*,H^*}, A)).$$

Hence $\mathcal{U}_{\xi^s}$ is not a soft L-fuzzy quasi-uniformity.

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