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FIXED POINT THEOREMS IN THE STUDY OF OPERATOR EQUATIONS IN ORDERED BANACH SPACES

Mohammed Said El Khannoussi[1 §], Abderrahim Zertiti[2]
[1,2]Department of Mathematics
Faculty of Sciences
University Abdelmalek Essaadi
B.P. 2121, Tetouan, MOROCCO

Abstract: In this paper, we shall study completely continuous maps which leave a cone invariant in an ordered Banach space. A generalisation of some results of Krasnosel'skii and Amann, where the map are supposed to be monotone, will be given.

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1. Introduction

In this paper we are interested in giving some abstract fixed point theorems for not necessarily increasing, asymptotically linear and completely continuous maps leaving invariant a cone in ordered Banach spaces. Our results generalize some of those giving by Krasnosel'skii and Amann in [8] and [1], respectively, where the maps are supposed to have the restrictive hypothesis of monotonicity. The proof will be based on the fixed point index for completely continuous maps. Finally, in order to prove the importance of our abstract results, we shall give two important applications.

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§Correspondence author

Let $(E, \|\cdot\|_E)$ be a real Banach space and P be a nonempty closed convex set in E.

P is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0 \Longrightarrow \lambda x \in P$

(ii) $x \in P, -x \in P \Longrightarrow x = \theta$, where θ denotes the zero element in E.

The cone P defines a linear ordering in E by

$$x \leq y \quad \text{iff} \quad y - x \in P.$$

The cone P is said to be normal if there exists a constant $N > 0$ such that

$$\theta \leq x \leq y \Longrightarrow \|x\| \leq N\|y\|, \quad x, y \in P.$$

For every $L : E \to E$ a bounded linear operator, define $r(L)$, the spectral radius of L by

$$r(L) = \lim_{n \to +\infty} \|L^n\|^{\frac{1}{n}}.$$

For every bounded open subset U of P (from now on, the topological notions of subsets of P refer to the relative topology of P as a topological subspace of E) and every completely continuous map $f : \bar{U} \to P$ which has no fixed points on ∂U, there exists an integer, $i_p(f, U)$, called the fixed point index of F on U with respect to P, satisfying the usual properties of the Leray-Schauder degree.

If $r > 0$, we denote

$$P_r = \{x \in P : \|x\|_E < r\}, \quad S_r = \{x \in P : \|x\|_E = r\}.$$

Let X be a nonempty set and let Y be an ordered set. Following Amann [1], a map $g : X \to Y$ is said to be a majorant of the map $f : X \to Y$ if $f(x) \leq g(x)$ for all $x \in X$. minorant are defined by reversing the above inequality sign.

2. Main Results

In the proof of our main result we will use the following theorem of M.G.Krein and M.A.Rutman [10, Theorem 6.2].

Theorem 1. *Let A be a completely continuous linear operator satisfying the following two conditions:*

(α) $A(P) \subset P$,

(β) *there exists an element* $u \in P, \|u\| = 1$, *a scalar* $c > 0$, *and a natural number* p *such that* $A^p u \geq cu$.

Then A *has nonzero eigenvalues; among those of maximal modulus there is a positive one not less than* $c^{\frac{1}{p}}$, *to which corresponds a characteristic vector* $v \in P$ *of the operator* A:

$$Av = \rho v \quad (\rho \geq c^{\frac{1}{p}}, v \in P, v \neq 0).$$

This theorem may also be found in Krasnoselskii's book [8, p.67]. It should be noted that the requirement $\|u\| = 1$ in the above theorem is not essential.

After these preparations we are ready for the statement of our main results

Theorem 2. *Let* (E, P) *be an (OBS) with normal cone, and let* $f : P \to P$ *be a completely continuous map. Suppose that* f *has a completely continuous, asymptotically linear majorant* $g : P \to P$, *such that* $g'(\infty)$ *does not possess a positive eigenvector to an eigenvalue greater than or equal to 1. Then f has a fixed point.*

Proof. We claim that there exists $R > 0$ such that if $x \in P$ and $t \in [0, 1]$ satisfy $x = tf(x)$, then $\|x\| \leq R$.

In fact, if the claim is true, then $i(f, P_R) = 1$ which implies the existence of a fixed point of f in P_R.

Suppose that is not true, then we can find sequences $\lambda_n \in [0, 1]$ and $x_n \in P$ such that $\|x_n\| \longrightarrow \infty$ as $n \to \infty$ and

$$x_n = \lambda_n f(x_n) \leq f(x_n) \leq g(x_n),$$

So that

$$\begin{aligned} \frac{x_n}{\|x_n\|} &\leq \frac{g(x_n)}{\|x_n\|} \\ &= \frac{g(x_n) - g'(\infty)x_n}{\|x_n\|} + \frac{g'(\infty)x_n}{\|x_n\|}. \end{aligned}$$

By letting $\delta_n = \frac{x_n}{\|x_n\|}$ we get

$$\frac{g(x_n) - g'(\infty)x_n}{\|x_n\|} + g'(\infty)\delta_n - \delta_n \in P. \tag{1}$$

By using the fact that $g'(\infty)$ is a positive linear map (see Theorem 7.3 in [1]) we obtain

$$g'(\infty)\Big(\frac{g(x_n) - g'(\infty)x_n}{\|x_n\|}\Big) + g'(\infty)(g'(\infty)\delta_n) - g'(\infty)\delta_n \in P. \tag{2}$$

On the other hand, since $g'(\infty) \setminus P$ is completely continuous (see Theorem 7.3 in [1]), we may as well assume that $g'(\infty)\delta_n \to y \in P$. Then by letting $n \to \infty$ in (2) we get

$$g'(\infty)y - y \in P.$$

Next, we prove that $y \in P \setminus \{0\}$. Indeed, suppose that $y = 0$.

From (1) we have

$$0 \leq \delta_n \leq \frac{gx_n - g'(\infty)x_n}{\|x_n\|} + g'(\infty)\delta_n.$$

Since P is normal we get

$$1 \leq N\left\| \frac{gx_n - g'(\infty)x_n}{\|x_n\|} + g'(\infty)\delta_n \right\|$$

where N is the normal constant of P.

But from the limits

$$\frac{g(x_n) - g'(\infty)x_n}{\|x_n\|} + g'(\infty)\delta_n \to 0 \quad \text{as} \quad n \to \infty,$$

we obtain a contradiction.

Consequently, $y \in P \setminus \{0\}$ and verifies the inequality $g'(\infty)y - y \in P$, which implies that $g'(\infty)y \geq y$. Hence by using Theorem (1), $g'(\infty)$ has a positive eigenvector to an eigenvalue greater than or equal to 1, which contradicts the hypothesis of the theorem. This completes the proof of the theorem. □

Remark 1. Note that the last theorem is a generalisation of Theorems 7.4, 13.7 given by Amann in [1] and of Theorems 4.9′,4.8 given by Krasnosel'skii in [8], where the maps and the norm are supposed to be monotone.

As a consequence of the previous result we give the following statement:

Theorem 3. *Let (E, P) be an (OBS) whose positive cone is normal and let $f : P \to P$ be a completely continuous, right differentiable map. Suppose that there exists a positive compact endomorphism T of E which does not possess a positive eigenvector to an eigenvalue greater than or equal to 1 and such that*

$$f'_+(x) \leq T$$

for all $x \in P$. Then f has exactly one fixed point $\bar{x} \in P$.

Proof. It follows from $(T-f)'_+(x) = T - f'_+(x) \geq 0$ that $T-f$ is increasing. Hence f cannot have two comparable distinct fixed points. In order to be convinced of this, suppose that there exist fixed points x_1 and x_2 of f such that $x_1 < x_2$, then $x_2 - x_1 = f(x_2) - f(x_1) > 0$, and this implies the inequality

$$x_2 - x_1 \leq T(x_2 - x_1).$$

It follows from the last inequality and from Theorem (1) that T possesses a positive eigenvector to an eigenvalue greater than or equal to 1, which contradicts the hypothesis of the theorem.

Furthermore, since $T - f$ is increasing, it is obvious that $T(x) - f(x) \geq -f(0)$, for all $x \in P$, from which it follows that the map $g = f(0) + T$ is a completely continuous majorant of f. Clearly, g is asymptotically linear, and $g'(\infty) = T$. Hence the assertion follows from Theorem (2). □

Remark 2. The above theorem is a generalization of Theorem 8.2 of Amann in [1], where the map $f \setminus P$ are supposed to be increasing.

Remark 3. Theorem (2) does not require the monotonicity of f, but has no the conclusion

$$\bar{x} = \lim_{k \to +\infty} f^k(0).$$

of Theorem 8.2 of Amann in [1]. In order to solve this problem, we suppose the following additional condition.

Theorem 4. *Let (E, P) be an (OBS) whose positive cone is normal and let $f : E \to E, (f(P) \subset P)$ be a completely continuous map which have a Frechet derivative at every point x of the space E. Suppose that there exists a positive compact endomorphism T of E satisfying the following condition*

$$-T \leq f'(x) \leq T. \tag{3}$$

for all $x \in E$. Then if $r(T) < 1$, f has exactly one fixed point $\bar{x}$, and $\bar{x}$ is the limit of the sequence

$$\bar{x} = \lim_{k \to +\infty} f^k(0)$$

that is, the fixed point $\bar{x}$ can be computed iteratively by means of the iteration scheme

$$x_0 = 0,$$
$$x_{k+1} = f(x_k) \quad k = 0, 1, 2, \ldots.$$

Proof. The existence of a fixed point $\bar{x}$ of f is a consequence of Theorem (3). By virtue of (3) we have the following inequality

$$-T\bar{x} \leq \bar{x} - f(0) \leq T\bar{x},$$

from which it follows that

$$\bar{x} \geq \frac{1}{2}(\bar{x} + f(0) - T\bar{x}), \quad f(0) \geq \frac{1}{2}(\bar{x} + f(0) - T\bar{x}).$$

By using inequalities (3), we get

$$-T\Big(\frac{\bar{x} - f(0) + T\bar{x}}{2}\Big) \leq \bar{x} - f\Big(\frac{\bar{x} + f(0) - T\bar{x}}{2}\Big) \leq T\Big(\frac{\bar{x} - f(0) + T\bar{x}}{2}\Big) \quad (4)$$

and

$$-T\Big(\frac{f(0) - \bar{x} + T\bar{x}}{2}\Big) \leq f^2(0) - f\Big(\frac{\bar{x} + f(0) - T\bar{x}}{2}\Big) \leq T\Big(\frac{f(0) - \bar{x} + T\bar{x}}{2}\Big). \quad (5)$$

By subtracting (5) for (4), then we have

$$-T^2\bar{x} \leq \bar{x} - f^2(0) \leq T^2\bar{x}.$$

By repeating this argument n times, we obtain the inequality

$$-T^n\bar{x} \leq \bar{x} - f^n(0) \leq T^n\bar{x}.$$

On the other hand, it follows from $r(T) < 1$ that $T^n\bar{x} \to 0$ as $n \to \infty$. From this, and from the normality of the cone P, the equality

$$\bar{x} = \lim_{k \to +\infty} f^k(0).$$

holds. This completes the proof of the theorem. □

Remark 4. Note that Krasnosel'skii and Zabreiko have shown a similar result by the more restrictive assumption that the cone P is reproducing and $\|T\| < 1$ and the less restrictive assumption that f does not have the property of compactness and positiveness (see [9], see also Theorem 3.1.14 in [7]).

It should be remarked above that if $f : P \to P$ is an increasing, right differentiable map, where the derivative $f'_+(x)$ satisfies the condition

$$0 \leq f'_+(x) \leq T$$

(as in Theorem 8.2 of Amann in [1]) and $r(T) < 1$ where T is a positive compact endomorphism of E. Then it follows from the inequality $f(x) \leq Tx + f(0)$ for every $x \in P$ that $f(v_0) \leq v_0$ where $v_0 = (I - T)^{-1} f(0) = \sum_{n=0}^{\infty} T^n f(0)$. In order to be convinced of this, it suffices to observe that

$$f(\sum_{n=0}^{\infty} T^n f(0)) \leq T(\sum_{n=0}^{\infty} T^n f(0)) + f(0) \tag{6}$$

$$= \sum_{n=0}^{\infty} T^n f(0). \tag{7}$$

From which it follows that f leaves the interval $[0, v_0]$ invariant. Hence by using Theorem 4.1 of Krasnosel'skii in [10], it suffices that anyone of the following conditions be satisfied for the existence on $[0, v_0]$ of at least one fixed point for the map f.

(a) The cone P is strongly minihedral;

(b) The cone P is regular, the map f is continuous;

(c) The cone P is normal, the map f is completely continuous;

(d) The cone P is normal, the space E is weakly complete, the unit sphere in E is weakly compact, the map f is weakly continuous.

Also, it not hard to see that with the fulfillment of the condition (b) or condition (c) or condition (d) the fixed point $\bar{x}$ of f can be obtained as the limit of the sequence

$$x_n = f(x_{n-1}), \quad n = 1, 2, ...$$

where $x_0 = 0$, that is $\bar{x}$ can be computed iteratively.

3. First Application

In this section, we apply the abstract results of the preceding section to the non-linear integral equations of the form

$$x(t) = \int_{\Omega} f(t, s, x(s)) \, ds, \tag{8}$$

where, Ω denote a bonded closed of a finite-dimensional space.

In order that the Uryson operator

$$Fx(t) = \int_{\Omega} f(t, s, x(s)) \, ds, \tag{9}$$

act and be completely continuous in the space C of continuous real-valued functions on Ω, the continuity of the function $f(t,s,y)$ $(t,s\in\Omega,-\infty<y<\infty)$ with respect to the collection of variables is sufficient. The proof is obvious (see [8]).

Denote by P the normal cone of non-negative functions of the space C, where if $x\in C$

$$\|x\| = \max_{t\in\Omega} |x(t)|.$$

From now on, it will be assumed that the condition

$$f(t,s,y)\geq 0, \quad f(t,s,0)\neq 0 \quad (t,s\in\Omega, y\geq 0)$$

is satisfied. Then the Uryson operator (9) will leave the cone P invariant.

In order that the theorems (which were proved in the previous section) can be considered as statement about equation with an integral operator (9), it is necessary to prove the existence of $F'(x_0)h(t)$ the derivative of F at every point x_0 of the space C. In this connection, we suppose that the function $f(t,s,y)$ is continuous together with its derivative $f'_y(t,s,y)$ with respect to all the variables $t,s\in\Omega,\quad -\infty<y<\infty$. Therefore, by virtue of the formula of finite increments, (see [8] p.234) $F'(x_0)h(t)$ is defined by the formula

$$F'(x_0)h(t) = \int_\Omega f'_y(t,s,x_0(s))h(s)\ ds, \quad \forall x_0\in C.$$

As a consequence of this and Theorem (4) we have the following theorem.

Theorem 5. *Suppose that f satisfies the previous conditions and*

(H1) there exists a positive function $a\in C(\Omega\times\Omega)$ such that

$$-a(t,s)\leq f'_y(t,s,y)\leq a(t,s), \quad \forall(t,s\in\Omega,-\infty<y<\infty)$$

Then if

$$r(T)<1, \tag{10}$$

where $T: C\longrightarrow C$ is the linear operator defined by

$$Tx(t) = \int_\Omega a(t,s)x(s)\ ds, \quad \forall x\in C,$$

equation (8) has a solution in $P\backslash\{0\}$, and we get the solution from the following iterative sequence defined by $x(t)=\lim_{n\to+\infty} u_n(t)$, with

$$u_0(t)=0$$

$$u_n(t) = \int_\Omega f(t, s, u_{n-1}(s))\, ds,$$

Now we present an example of theorem 5

Example 6. Let $E = C[0, 1]$ and $P = \{x \in C[0, 1] : x(t) \geq 0\}$. Then P is a normal cone in E. Consider the following integral equation

$$x(t) = \int_0^1 d(t, s)h(y)ds$$

where $h : \mathbb{R} \to \mathbb{R}^+, \quad h(0) \neq 0$ is a C^1 function verifying

$$| h'(x) | \leq 1$$

and $d : [0, 1] \to \mathbb{R}$ a continuous, positive function.

If

$$f(t, s, y) = d(t, s)h(y) \quad \forall (t, s, y) \in [0, 1] \times [0, 1] \times \mathbb{R}$$

then hypothesis (H1) of Theorem 5 is satisfied with $a(t, s) = d(t, s)$ Consequently if

$$\max_{0 \leq t \leq 1} \int_0^1 d(t, s)ds < 1 \tag{11}$$

equation (8) has a solution in $P \setminus \{0\}$.

Here, we use the fact that

$$r(T) < \|T\| < \max_{0 \leq t \leq 1} \int_0^1 d(t, s)ds.$$

Note that in the particular case where $d(t) \equiv d \in \mathbb{R}^+$ conditions (11) are satisfied if we take

$$d < 1.$$

4. Second Application

In this section we shall study the existence of positive solutions of integral equations of the form

$$x(t) = \int_0^{\tau(t)} f(t, s, x(t - s - l))\, ds, \tag{12}$$

which formulate a model to explain the evolution of certain infectious diseases and it may also be considered as a growth equation for single species populations when the birth rate varies seasonally. It include, on a particular case, different equations suggested by other authors (see [3], [4], [6], [12], [13] and [11]). We are interested in producing sufficient conditions for the existence of positive periodic solution to (12) in the special case where $f(t,s,y) \leq g(t,s,y), \forall (t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,+\infty[$ under the following assumptions (H) on functions f and g:

$f, g : \mathbb{R} \times \mathbb{R} \times [0,+\infty[\longrightarrow \mathbb{R}$ are continuous functions with:

(F1) $f(t,s,0) = 0$ for all $(t,s) \in \mathbb{R} \times \mathbb{R}$,

(F2) $f(t,s,y) \geq 0, g(t,s,y) \geq 0, f(t,s,y) \leq g(t,s,y), \forall (t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[$ and there exists a positive number w ($w > 0$) such that $f(t+w,s,y) = f(t,s,y)$ and $g(t+w,s,y) = g(t,s,y), \forall (t,s,y) \in \mathbb{R} \times \mathbb{R} \times [0,+\infty[$,

(F3) l is a nonnegative constant and $\tau : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous and λ-periodic function ($\lambda > 0$) such that $\frac{\omega}{\lambda} = \frac{p}{q}, p, q \in \mathbb{N}$.

Denote by P the cone of nonnegative functions in the real Banach space E, of all real and continuous $q\omega-$ periodic functions defined on $\mathbb{R}$, where if $x \in E$

$$\|x\| = \max_{0 \leq t \leq q\omega} |x(t)|.$$

We are interested in the existence of solution of (12) in $P \setminus \{0\}$. Define the operator $F, G : E \longrightarrow E$ by

$$Fx(t) = \int_0^{\tau(t)} f(t,s,x(t-s-l))\, ds.$$

$$Gx(t) = \int_0^{\tau(t)} g(t,s,x(t-s-l))\, ds.$$

Then equation (12) has a continuous, nonnegative, and nontrivial $q\omega$-periodic solution iff there exists $x \in P \setminus \{0\}$ verifying

$$x = Fx.$$

Now, we present and prove our main results.

Theorem 7. *Suppose that f and g satisfy assumptions (H) and:*

(H1) there exists a continuous function $a : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\lim_{y \to 0^+} \frac{f(t,s,y)}{y} = a(t,s), \quad \textit{uniformly in } (t,s) \in \mathbb{R} \times \mathbb{R},$$

(H2) there exists a continuous function $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y \to +\infty} \frac{g(t,s,y)}{y} = b(t,s), \quad \textit{uniformly in } (t,s) \in \mathbb{R} \times \mathbb{R},$$

(H3) $\overset{\circ}{A_t} = \emptyset \quad \forall t \in \mathbb{R}$, *where* $A_t = \{s \in \mathbb{R} : a(t, t-s) = 0\}$.

Then if

$$r(L(\tau, a)) > 1, \quad \textit{and} \quad r(L(\tau, b)) < 1, \tag{13}$$

equation (12) *has a solution in* $P \setminus \{0\}$, *where* $r(L(\tau, a))$ *means the spectral radius of the linear operator* $L(\tau, a) : E \longrightarrow E$ *defined by*

$$L(\tau, a)x(t) = \int_0^{\tau(t)} a(t,s)x(t-s-l)\, ds, \quad \forall x \in E,$$

(analogously for $r(L(\tau, b))$ *and* $L(\tau, b)$*).*

Proof. We must observe that (E, P) is an ordered Banach space with $\overset{\circ}{P} \neq \emptyset$ and P is normal. Also it is not difficult to see that $F, G : P \to P$ are completely continuous. Moreover:

(a) It is easy to prove (see[3, Theorem 2.1]) the existence of $G'(+\infty)$, the derivative of G along P at infinity. In fact

$$G'(+\infty)(x)(t) = L(\tau, b)x(t), \quad \forall x \in E$$

(b) Also, it's not difficult to prove the existence of $F'_+(0)$ the right derivative of F along P at 0, and

$$F'_+(0)(x)(t) = L(\tau, a)x(t), \quad \forall x \in E,$$

(c) It is easily seen (see[3, Theorem 2.1]) that $F'_+(0)$ is strongly positive.

Now, from (b),(c) and Lemma 13.1 in Amann [1] and by virtue of (13) there exists a number $\sigma > 0$ such that $i_P(F, P_\sigma) = 0$. On the other hand from the proof of Theorem (2) and from (a) and by virtue of (13) there exists a number $R > \sigma > 0$ such that $i_P(F, P_R) = 1$. Hence by the additivity property $i(F, P_R \setminus \bar{P_\sigma}) = 1$. Consequently, the solution property of the fixed point index implies the existence of at least one fixed point x with $\sigma < \|x\|_E < R$.

Now we present an example of Theorem 7 which cannot be studied from the results of [3], [4] and [11].

Example 8. Let $h : [0, +\infty] \to \mathbb{R}^+$ be a continuous function verifying

$$h(0) = 0 \quad h'(0) = \alpha > 0, \quad \lim_{y \to +\infty} \frac{h(y)}{y} = \beta > 0$$

and take $d : \mathbb{R} \to \mathbb{R}$ a continuous, positive and ω-periodic function ($\omega > 0$) and $l = 0$.

If

$$f(t, s, y) = d(t - s)h(y)(1 + \sin^2 y), \quad \forall (t, s, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty]$$

and

$$g(t, s, y) = 2d(t - s)h(y), \quad \forall (t, s, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty]$$

hypotheses (H1)-(H3) of Theorem 7 are satisfied with $a(t, s) = \alpha d(t - s)$, and $b(t, s) = 2\beta d(t - s)$.

Consequently if

$$1 < r(L(\tau, a)), \quad r(L(\tau, b)) < 1, \tag{14}$$

equation (12) has a solution in $P \setminus \{0\}$.

Note that in the particular case where $d(t) \equiv d \in \mathbb{R}^+$ conditions (14) are satisfied if we take

$$\frac{1}{\alpha d} < \min_{t \in \mathbb{R}} \tau(t) \le \max_{t \in \mathbb{R}} \tau(t) < \frac{1}{2\beta d}.$$

Here we use that fact that (see [11] and [3]).

$$\min_{t \in \mathbb{R}} \int_0^{\tau_1(t)} \alpha(t, s) ds \le r(L(\tau_1, \alpha)) \le \max_{t \in \mathbb{R}} \int_0^{\tau_1(t)} \alpha(t, s) ds.$$

for every continuous function $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is ω-periodic in t.

Note that this example cannot be studied by Theorem 2.1 in [3] because the condition [H2] is not satisfied ($\nexists \lim_{y \to +\infty} \frac{f(t,s,y)}{y}$).

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