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# A CRITERION TO UNIFORM STABILITY FOR FUNCTIONAL PERTURBED DIFFERENTIAL EQUATIONS

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**Abstract:** In this paper we consider a class of non autonomous ODEs with a functional perturbation. For the unperturbed equation a Lyapunov function bounded by two quadratic forms is known. The Lipschitzean rate of the vector field along with some additional requirements to the derivatives of the Lyapunov function guarantee existence of uniform stable solutions. A sufficient condition that guarantees uniform stability of the zero-solution to the equation under consideration is discussed.

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**Key Words:** functional differential equations, stability, Lyapunov function, functional perturbation

## 1. Introduction

In this paper we consider the class of Functional Differential Equations (FDE) having the general form:

$$\left| \begin{aligned} &\dot{x} = F(t, x, \lambda_t(x)), \quad t \geq t_0, \\ &x(t) = \varphi(t), \quad t \in [-\alpha, t_0] \ \text{ initial data} \end{aligned} \right. \tag{1}$$

with an initial function

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$$\varphi : [-\alpha, t_0] \to \mathbb{R}^n,\ \alpha > 0, \quad \varphi \in C, \quad |\varphi| \le c_\varphi, \ \ t_0 > 0,$$

where $c_\varphi$ =const, $\lambda_t(x)$ is a linear functional,

$$\lambda_t :\ C([t-\alpha, t], \mathbb{R}^n) \to \mathbb{R}^n, \ \ t \ge t_0,$$

$F$ is continuous function w.r.t. its arguments, and $\lambda_t(x)$ is such that in the case of delay FDE, for instance, then $\lambda_t(x) = x(t+\theta)$,

$$\theta \in [-\alpha, t_0].$$

There are cases when the functional can be of of integral type. There are classes of mathematical models of integro-differential equations describing different phenomena in Biology, Physics and Engineering sciences , [4], that have linear functionals in their vector fields.

The FDEs with delay have the form

$$\dot{x} = f(t, x(t), x(t-\sigma)), \tag{2}$$

where $f$ is most generally a nonlinear function or operator w.r.t. $t$ (very often it is the time), and $\lambda_t$ in this case in the right-hand side of (2) is a functional which yield the delay, and particularly can be linear, that is, $\lambda_t(x) = x(t-\sigma)$. Here note that, in the case when the function $f$ is nonlinear, then it is required $f$ to be completely continuous and to satisfy additional smoothness conditions in order to ensure the existence and uniqueness of the solution $x(t, \alpha, \varphi)$. For the stability and boundedness of linear delay DE we refer the reader to [10, 11].

In this paper we assume that the standard requirements for continuity and Lipschitzean of the vector field in the right-hand side of (1) are satisfied. Thus the existence and uniqueness of the problem under consideration are guaranteed.

The main goal of our investigation is by the second method of Lyapunov and some requirements imposed on the vector field of (1) to prove uniform stability (by Lyapunov) of the zero-solution to the problem under consideration. The method that we use here is similar to those used in [12] (see, e.g., [11, 16]).

Let $V : \mathbb{R} \times C \to \mathbb{R}$ be continuous function, and $x(t, \alpha, \varphi)$ the solution of (1), then define

$$\dot{V}(t, \varphi) = \overline{\lim_{h \to 0+}} \Big\{ \frac{1}{h} \Big[ V(t+h, \lambda_{t+h}(x)) - V(t, \varphi) \Big] \Big\}.$$

The stated function $\dot{V}$ is the upper right-hand derivative along the solution $x$ of (1). Here note that most authors prefer to use the notation $\dot{V}_{(1)}(t, \varphi)$.

In the next section we give some preliminaries, notations and known facts to the FDEs as well as some requirements imposed on the functions in the right-hand side of (1).

In the third section we prove five auxiliary assertions as well as the main statement, that is, a criterion tor existence of uniformly stable solution of (1).

## 2. Preliminaries and Notations

Consider a function $F : \mathbb{R} \times C \times C \to \mathbb{R}^n$, which is continuous, and another function $\varphi : [-\alpha, t_0] \to \mathbb{R}^n$, also continuous and bounded, $|\varphi| \leq C_0 = \text{const}$.

Given a FDE in a general form (1)

Here consider a case of linear $\lambda_t$.

Assume that the nonlinear function $F$ in (1) has the form

$$F(t, x, \lambda_t(x)) = f(t, x) + \varepsilon R(\lambda_t(x)), \tag{3}$$

where $f$ is a smooth function, $\varepsilon$ small parameter, and $R$ is a differentiable in the sense of Gâteaux.

Consider the unperturbed ODE

$$\dot{\tilde{x}} = f(t, \tilde{x}), \quad f(t, \tilde{x}) = A(t)\tilde{x} + \tilde{f}(t, \tilde{x}), \tag{4}$$

where $A(t)$ is a smooth linear $n \times n$ matrix operator, and $\tilde{f}(t, \tilde{x})$ is the nonlinear summand.

Given an initial value problem (1), under some initial condition, that is an initial function $\varphi(s), \quad s \in [-\alpha, t_0]$, continuous on its closed domain. In some special cases the domain may have equal to zero measure. Therefore, $|\varphi(s)| \leq C_\varphi$, where $C_\varphi$ is a positive constant. Further, assume without infringing the generality of the problem under consideration that $\varphi(s) \geq 0$ for $s \in [-\alpha, t_0]$.

Assume the following hypothesis hold:

**H1**. The linear functional $\lambda_t$ is bounded, that is, $\|\lambda_t\| < c_1$ ($c_1 = \text{const} > 0$). The linear operator $A(t)$ is also bounded

$$\|A(t)\| < M \quad \forall t \in [0, \infty), \tag{5}$$

where $M > 0$ is a constant.

Assume that both $f$ and $R$ are Lipschitzean with the same constant $L$, i.e.,

$$\begin{aligned} &(a) \qquad f \in Lip(L; \mathbb{R} \times \mathbb{R}^n), \\ &(b) \qquad R \in Lip(L; \mathbb{R}^n), \end{aligned} \tag{6}$$

where $Lip(L;\Omega)$ is the space of all Lipschitzean functions on the set $\Omega$ having a Lipschitzean constant $L$. Suppose that $t_0 = 0$, and $R(0) = f(t,0) = 0$ $(t \in \mathbb{R})$. Hence the function $\tilde{f}$ is also Lipschitzean with a constant $M+L$, and $\tilde{f}(t,0) = 0$.

Next, suppose a second hypothesis hold:

**H2**. The system (4) has stable equilibrium state $x = 0$ which is guaranteed by the existence of Lyapunov function $V(t,x)$ for the unperturbed system (4) such that

$$\frac{\partial V}{\partial t} + \nabla V \cdot f(t,x) \le 0 \tag{7}$$

There exist a pair of positively definite functions $\tilde{a}(\|x\|)$ and $\tilde{b}(\|x\|)$ such that

$$\tilde{a}(\|x\|) \le V(t,x) \le \tilde{b}(\|x\|). \tag{8}$$

Note that the above condition is similar to one considered in [12].

Introduce the quantity $\Phi(t,x) \equiv \langle \nabla V,\ R(\lambda_t(x)) \rangle$, which is differentiable in the sense of Gâteaux. Here by $\langle \cdot, \cdot \rangle$ we have denoted the scalar product.

Note that the origin $O$ is an equilibrium point for the system (4), hence $f(t,0) = 0 = \tilde{f}(t,0) = 0$.

Define the variational system

$$\dot{z} = A(t)z \tag{9}$$

Here remind the Gâteaux derivative and also gradient.

Let consider $G(x)$ which is a nonlinear function defined in a linear and dense set $D(G) \subset E$, where $E$ is Banach space, i.e. $D(G)$ is normed and dense in $E$.

**Definition 1.** Suppose that there exists

$$\frac{d}{dt} G(x + \tau h)|_{\tau=0} = \lim_{\tau \to 0} \frac{G(x+\tau h) - G(x)}{\tau} = VG(x)(h)$$

for every $h \in D(G)$. Then $VG(x)(h)$ call the variation or differential of Gâteaux at the point $x$ of the function $G$.

We note that, $VG(x)(h)$ is a homogeneous operator of $h$, i.e. $VG(x)(\alpha h) = \alpha VG(x)(h)$, but it is not always additive. There are examples when

$$VG(x)(h_1 + h_2) \ne VG(x)(h_1) + VG(x)(h_2).$$

In this paper we assume that $VG(x)(h)$ is a linear operator, that is, homogeneous and additive. Thus we may assume that $DG(x)(h)$ has the form

$$DG(x)(h) = P(x)h,$$

where $P(x)$ call the derivative of $G$ at $x$ and denote $P(x) \equiv G'(x)$.

In the case $G'(x)$ is a bounded operator, then it may be continuously extended to an operator acting on every vector $h \in E_x$.

The same definition hold true in the case if $G$ is a nonlinear functional or nonlinear operator, then denote

$$DG(x,h) = G'(x)h,$$

where $G'(x)$ is the Gâteaux derivative of $G$ at the point $x$, and any vector $h \in D(G)$.

In this case $G'(x)$ is a linear functional at a fixed $x$ with domain containing all vectors $h \in D(G)$. The continuous extension of $G'$ for any $h \in E_x$, call the gradient of $G$ at $x$, and denote by either

$$\text{grad } G(x) \equiv \nabla G(x) \equiv \frac{\partial G}{\partial x}.$$

Let $\tilde{G}(x) = \text{grad } G(x)$. Obviously, $\tilde{G}(x)$ is a linear and continuous functional acting on any vector $h \in E_x$. The functional $\tilde{G}$ is an element of the conjugated space $E^*$.

If the gradient exists on some set $A \subset D(G)$, then it maps $A$ into $E^*$, i.e. grad $G(x)$ is an operator on $A$ into $E^*$.

Thus we may define

$$\Big\langle \text{grad } G(x), h \Big\rangle = \frac{d}{dt} G(x + \tau h)\big|_{\tau=0} = \lim_{\tau \to 0} \frac{G(x+\tau h) - G(x)}{\tau}$$

where $\langle z, x \rangle$ is the value of a linear functional $z \in E^*$ on the vector $x \in E$.

The existence, uniqueness and stability for (14) is considered by many authors (see e.g. [4] - [11]).

The applicability of the considered class ODEs is indisputably. So we refer the reader to K. Gopalsamy, [10], and J. Hale [11] for the applications.

## 3. Main Result

In this paragraph we discuss the stability of zero solution of (1) with $F$ in the form (3) provided that the conditions **H1, H2** hold true. Here we define the norm of $\lambda_t$, that is $\|\lambda_t\|$, but it is unclear whether $\|\lambda_s - \lambda_\tau\|$ has maximal value for $s$, $\tau$ in some interval.

Consider the space of linear functionals $\{\lambda_s\}_{s\in\Omega}$, where $s$ is some real parameter in the set $\Omega \subset \mathbb{R}$. Define the set

$$S_k \equiv \{m_k(s,\tau) = \|\lambda_s - \lambda_\tau\| : \ 0 \le s \le \tau, \ k\alpha \le \tau < (k+1)\alpha\}, \tag{10}$$

where $k = 1, 2, \ldots$. Here the pairs of real parameters $(s,\tau)$ sweep the intervals given in (10), apparently, depending on $k = 1, 2, \ldots$.

For $S_k$ there exists a binary relation

$$m_k(s,\tau) \prec m_k(r,\tau)$$

between certain pairs $m_k(s,\tau), m_k(r,\tau) \in S_k$ with the properties:

$$\begin{aligned} &(i) \quad m_k(s,\tau) \prec m_k(s,\tau); \\ &(ii) \quad \text{if } \ m_k(a,\tau) \prec m_k(b,\tau) \ \text{ and } \ m_k(b,\tau) \prec m_k(a,\tau), \\ &\qquad\qquad \text{then } \ m_k(a,\tau) = m_k(b,\tau); \\ &(iii) \quad \text{if } \ m_k(a,\tau) \prec m_k(b,\tau) \ \text{ and } \ m_k(b,\tau) \prec m_k(c,\tau), \\ &\qquad\qquad \text{then } \ m_k(a,\tau) \prec m_k(c,\tau) \\ &\qquad\qquad \text{(transitivity)}. \end{aligned} \tag{11}$$

Then in this case, $S_k$ is partially ordered (semi-ordered) by the relation "$\prec$".

Apparently, for every $m_k(a,\tau)$ and $m_k(b,\tau)$ there is $m_k(c,\tau)$ such that $m_k(a,\tau) \prec m_k(c,\tau)$, and $m_k(b,\tau) \prec m_k(c,\tau)$. Then $m_k(c,\tau)$ is an upper bound for $m_k(a,\tau)$ and $m_k(b,\tau)$.

Taking into account all these notes we state the Zorn's lemma for our case.

**Lemma 1.** *(Zorn's lemma)*
*$S_k$ is partially ordered set with the property that every linearly ordered subset of $S_k$ has an upper bound in $S_k$. Then $S_k$ contains at least one maximal element, which denote*

$$M_k \equiv \max_{s\in[0,\tau]} \|\lambda_s - \lambda_\tau\|$$

*($k\alpha \le \tau < (k+1)\alpha$). It is true for $k = 1, 2, \ldots$.*

**Lemma 2.** *Consider the perturbed problem (1) with right-hand side in the form (3). The functions $f$ and $R$ satisfy the Lipschitz condition (6) with the same constant $L$ in the domain $\|x\| < C_x$ ($C_x$ = const $> 0$). Then the* *solution $x(t)$ of (1) satisfies the estimate $\|x\| \le \|x_0\| e^{L\theta_k(t)}$, where $x_0 = x(0)$, and*

$$\theta_k(t) \equiv [1 + \varepsilon(M_k + c_1)]t, \quad M_k \equiv \max_{[0,\tau]} \|\lambda_s - \lambda_\tau\|,$$
$$k\alpha \le t \le \tau \le (k+1)\alpha \quad (k = 1, 2, \ldots).$$

*Proof.*

1) Consider the interval $0 \le t \le \tau < \alpha$.

After integrating (1), obtain

$$\|x\| \le \|x_0\| + \int_0^t \{\|f(s, x(s))\| + \|\varepsilon R(\lambda_s x)\|\} ds \le$$

$$\le \|x_0\| + \int_0^t L\{\|x(s)\| + \varepsilon c_1 \|x(s)\|\} ds = \|x_0\| + L\tilde{c} \int_0^t \|x(s)\| ds,$$

where $\tilde{c} = \varepsilon c_1 + 1$. Thus, making use of the Grünwall lemma obtain

$$\|x\| \le \|x_0\| e^{\tilde{c}Lt}.$$

2) Consider the interval $\alpha \le t \le \tau < 2\alpha$.

Integrate the same system in 1),

$$\|x\| \le \|x_0\| + L \int_0^t \{\|x(s)\| ds + \varepsilon L \int_0^t \|\lambda_s x\| ds \le$$

$$\le \|x_0\| + L \int_0^t \|x(s)\| ds + \varepsilon L \int_0^t \|\lambda_s x - \lambda_\tau x\| ds + \varepsilon L \int_0^t \|\lambda_\tau x\| ds \le$$

$$\le \|x_0\| + L \int_0^t \|x(s)\| ds + \varepsilon L \max_{[0,\tau]} \|\lambda_s - \lambda_\tau\| \int_0^t \|x(s)\| ds + \varepsilon L c_1 \int_0^t \|x(s)\| ds \le$$

$$\le \|x_0\| + L(1 + \varepsilon M_1) \int_0^t \|x(s)\| ds + \varepsilon L c_1 \int_0^t \|x(s)\| ds,$$

where have set $M_1 \equiv \max_{[0,\tau]} \|\lambda_s - \lambda_\tau\|$. Here note that the existence of the quantity $M_1$ follows from the Zorn's Lemma 1. Thus, get the estimate

$$\|x(s)\| \le \|x_0\| + L\theta_1(1) \int_0^t \|x(s)\| ds,$$

hence by Grünwall lemma obtain

$$\|x(s)\| \leq \|x_0\| e^{L\theta_2(t)}.$$

Following the same method obtain the general estimate in the interval
k) $k\alpha \leq t \leq \tau < (k+1)\alpha$ $(k = 1, 2, \ldots)$.

$$\|x(s)\| \leq \|x_0\| e^{L\theta_k(t)}, \quad \theta_k(t) \equiv [1 + \varepsilon(M_k + c_1)]t.$$

□

**Lemma 3.** *Consider the problem (4). The function $f$ satisfy the Lipschitz condition with the constant $L$ in the domain $\|x\| < C_x$ ($C_x = const$). Then the solution $\tilde{x}(t)$ of (4) satisfies the estimate $\|\tilde{x}\| \leq \|x_0\| e^{L\tilde{c}t}$, where $0 \leq t \leq l \leq k\alpha$ $(k = 1, 2, \ldots)$, $x_0 = x(0)$, and $\tilde{c} = 1 + c_1\varepsilon$.*

*Proof.*

$$\|\tilde{x}\| \leq \|x_0\| + \int_0^t \{\|A(s)\tilde{x}\| + \|\tilde{f}(s, \tilde{x}(s))\|\} ds \leq$$

$$\leq \|x_0\| + \int_0^t 2M\|x(s)\| ds.$$

Thus it follows by the same argument that

$$\|\tilde{x}\| \leq \|x_0\| e^{2Mt}.$$

□

**Lemma 4.** *For the problem (9) the following inequality hold:*

$$\|z\| \leq \|x_0\| e^{Mt}.$$

The proof is analogical to those in Lemma 1 and 2.

**Lemma 5.** *Let $x(t)$ and $\tilde{x}(t)$ be solutions of (1) and (4), respectively. Assume the requirements of Lemma 2 are satisfied. Then the following estimate hold:*

$$\|x(t) - \tilde{x}(t)\| \leq \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \big(e^{L\theta_k(\tau)} - 1\big) e^{Lt} \quad (k = 1, 2, \ldots)$$

*where, $\theta_k(\tau)$ is the same in Lemma 1.*

*Proof.* Integrate the equation (1) with right-hand side in the form (3), then get

$$x = x_0 + \int_0^t \{f(s, x(s)) + \varepsilon R(\lambda_s(x))\} ds, \quad 0 \le t \le \tau,$$

and do the same on the unperturbed equation (4)

$$\tilde{x} = x_0 + \int_0^t f(s, \tilde{x}(s)) ds.$$

1) Consider the solutions in the interval $0 \le t \le \tau < \alpha$.

Thus obtain

$$\|x - \tilde{x}\| \le \int_0^t \|f(s, x(s)) - f(s, \tilde{x}(s)) + \varepsilon R(\lambda_s(x))\| ds \le$$
$$\le \int_0^t \big(L\|x(s) - \tilde{x}(s)\| + \varepsilon L\|\lambda_s(x)\|\big) ds \le$$
$$\le L \int_0^t \|x(s) - \tilde{x}(s)\| ds + \varepsilon \int_0^t L c_1 \|x(s)\| ds.$$

Hence from Lemma 1 obtain

$$\|x - \tilde{x}\| \le L \int_0^t \|x - \tilde{x}\| ds + \varepsilon c_1 \|x_0\| \frac{1}{\tilde{c}} (e^{\tilde{c}L\tau} - 1).$$

Finally, again from Grünwall lemma it follows

$$\|x(t) - \tilde{x}(t)\| \le \varepsilon c_1 \|x_0\| \frac{1}{\tilde{c}} (e^{L\tilde{c}\tau} - 1) e^{Lt}.$$

2) Consider the problem under consideration in the interval $\alpha \le t \le \tau < 2\alpha$.

Using the same argument we get

$$\|x-\tilde{x}\| \le \int_0^t \|f(s,x(s)) - f(s,\tilde{x}(s)) + \varepsilon R(\lambda_s(x))\| ds \le$$
$$\le \int_0^t \big(L\|x(s)-\tilde{x}(s)\| + \varepsilon L\|\lambda_s(x)\|\big) ds \le$$
$$\le L\int_0^t \|x(s)-\tilde{x}(s)\| ds + \varepsilon L c_1 \int_0^t \|x_0\| e^{L\theta_1(s)} ds \le$$
$$\le L\int_0^t \|x(s)-\tilde{x}(s)\| ds + \frac{\varepsilon c_1\|x_0\|}{\theta_1(1)}\Big(e^{L\theta_1(\tau)}-1\Big),$$

hence by Grünwall lemma obtain

$$\|x-\tilde{x}\| \le \frac{\varepsilon c_1\|x_0\|}{\theta_1(1)}\Big(e^{L\theta_1(\tau)}-1\Big)e^{Lt}.$$

Further, we have the $k$-th interval $k=1,2,\ldots$:
k) $k\alpha \le t \le \tau < (k+1)\alpha$.

Thus, get

$$\|x-\tilde{x}\| \le \frac{\varepsilon c_1\|x_0\|}{\theta_k(1)}\Big(e^{L\theta_k(\tau)}-1\Big)e^{Lt} \quad (k=1,2,\ldots).$$

□

Note. By the above estimate in the proof of Lemma 5 we observe that after "switching on" the disturbance $R$ the difference between disturbed and undisturbed solutions gets too great through increasing of $t$. So one may control this difference particularly by the change of $L$, and $\varepsilon$.

**Theorem 6.** *Let the following conditions be satisfied:*
*1. The hypotheses H1, H2 for system* (1)*;*
*2. In the set* $\omega \equiv \{(t,x): t\in[-\alpha,\infty),\ \|x\|<C_x\}$, $(\alpha>0,\ C_x>0)$

$$|\Phi| \le M\|x\|^d, \quad \left|\frac{\partial\Phi}{\partial x}\right| \le M\|x\|^{d-1}$$

*for some* $d\ge 1$*; here* $\frac{\partial\Phi}{\partial x}$ *is the linear continuous functional of Gâteaux.*

*3.* $\exists \tau > 0, \quad \exists \delta > 0$, *such that* $\forall \alpha \geq 0$ *and* $\|x_0\| < \varepsilon_0 (< H)$,

$$x_0 \equiv \varphi(t_0),$$

*and*

$$\int_0^\tau \Phi(z(t)dt \leq -\delta \|x_0\|^q, \quad 0 < q < d,$$

*where* $z(t)$ *is the solution of the variational system* (9) *with initial data* $z(0) = x_0$, $t_0 = 0$.

*Then the zero solution of* (1) *is uniformly stable in Lyapunov sense.*

**Remark 1.** $\dfrac{\partial \Phi}{\partial x}$ is the gradient, by definition,

$$\left\langle \frac{\partial \Phi}{\partial x}(t,x), h \right\rangle = \frac{d}{d\mu} \Phi(t, x + \mu h) \Big|_{\mu=0} = \lim_{\mu \to 0} \frac{\Phi(t, x+\mu h) - \Phi(t,x)}{\mu}$$

for any $h \in D(\Phi)$.

*Proof.* Assume that we have already found the triplet of solutions $x(t)$, $\tilde{x}(t)$, $z(t)$ to the systems (1), (4), (9), respectively. After integrating (9)

$$\|z\| \leq \|x_0\| + \int_0^t \|A(s)\|.\|z(s)\|ds \leq \|x_0\| + \int_0^t M\|z(s)\|ds, \tag{12}$$

and using (5) (from hypothesis $H1$) for $0 \leq t \leq \tau, \quad (\tau < \alpha)$ and Grünwall lemma, we get

$$\|z\| \leq \|x_0\| e^{M\tau}. \tag{13}$$

Next, integrate (1) after $\varepsilon$-perturbing, i.e. $F$ has the form (3):

$$x = x_0 + \int_0^t F(s, x(s), \lambda_s(x))ds = x_0 + \int_0^t f(s, x(s))ds + \varepsilon \int_0^t R(\lambda_s(x))ds,$$

and

$$\tilde{x} = x_0 + \int_0^t A(s)\tilde{x}ds + \int_0^t \tilde{f}(s, \tilde{x}(s))ds.$$

Integrating (1) and (4), and taking into account Lemma 5 obtain

$$\|x(t) - \tilde{x}(t)\| \leq \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \left( e^{L\theta_k(\tau)} - 1 \right) e^{Lt}. \tag{14}$$

Next estimate $\|x(t) - z(t)\|$:

$$\|x(t) - z(t)\| \leq \int_0^t \|A(s)\|\|x(s) - z(s)\|ds + \int_0^t \|\tilde{f}(s, x(s)) + \varepsilon R(\lambda_s(x))\|ds \leq$$

$$\leq M \int_0^t \|x(s) - z(s)\|ds + \int_0^t [(M + L)\|x(s)\| + \varepsilon L c_1 \|x\|]ds \leq$$

$$\leq M \int_0^t \|x(s) - z(s)\|ds + [M + L(1 + \varepsilon c_1)] \int_0^t \|x(s)\|ds \leq$$

$$\leq M \int_0^t \|x(s) - z(s)\|ds + [M + L(1 + \varepsilon c_1)]\|x_0\| \int_0^t e^{L\theta_k(s)} ds =$$

$$= M \int_0^t \|x(s) - z(s)\|ds + \frac{[M + L\tilde{c}]\|x_0\|}{L\theta_k(1)} \left(e^{L\theta_k(t)} - 1\right). \tag{15}$$

Therefore,

$$\|x(t) - z(t)\| \leq \frac{[M + L\tilde{c}]\|x_0\|}{L\theta_k(1)} \left(e^{L\theta_k(\tau)} - 1\right) e^{Mt}. \tag{16}$$

Now introduce an auxiliary function:

$$\tilde{\Phi}(\kappa) \equiv \Phi(t, x + \kappa(\tilde{x} - x)), \tag{17}$$

where $\tilde{x}$, $x$ and $t$ being parameters, and $0 \leq \kappa \leq 1$. Then one has that

$$|\Phi(t, x) - \Phi(t, \tilde{x})| = |\tilde{\Phi}(0) - \tilde{\Phi}(1)| \leq \max_{0 \leq \kappa \leq 1} \left| \frac{d\tilde{\Phi}}{d\kappa} \right| \leq$$

$$\leq \max_{0 \leq \kappa \leq 1} \|\nabla \Phi(t, x + \kappa(\tilde{x} - x))\| \|\tilde{x} - x\| \leq$$

$$\leq M\|x + \kappa(\tilde{x} - x)\|^{d-1} \|\tilde{x} - x\| \leq$$

$$\leq M(2\|x\| + \|\tilde{x}\|)^{d-1} \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \left(e^{L\theta_k(\tau)} - 1\right) e^{Lt} \leq$$

$$\leq M \left(2\|x_0\| e^{L\theta_k(t)} + \|x_0\| e^{2Mt}\right)^{d-1} \frac{\varepsilon c_1 \|x_0\|}{\theta_k(1)} \left(e^{L\theta_k(\tau)} - 1\right) e^{Lt}.$$

Hence, obtain the estimate:

$$|\Phi(t, x) - \Phi(t, \tilde{x})| \leq \frac{\varepsilon M c_1 \|x_0\|^d}{\theta_k(1)} B_1(t), \tag{18}$$

where have denoted

$$B_1(t) \equiv \left(2e^{L\theta_k(t)} + e^{2Mt}\right)^{d-1} \left(e^{L\theta_k(\tau)} - 1\right)e^{Lt}.$$

Also by analogy we get

$$|\Phi(t,x) - \Phi(t,z)| = |\tilde{\Phi}(0) - \tilde{\Phi}(1)| \le \max\left|\frac{d\tilde{\Phi}}{d\kappa}\right| \le$$
$$\le \max \|\nabla\Phi(t, x + \kappa(z-x))\| \; \|z - x\| \le M\|x + \kappa(z-x)\|^{d-1}\|z-x\| \le$$
$$\le M(2\|x\| + \|z\|)^{d-1}\frac{(M+L\tilde{c})\|x_0\|}{L\theta_k(1)}\left(e^{L\theta_k(\tau)} - 1\right)e^{Mt} \le$$
$$\le M\left(2\|x_0\|e^{L\theta_k(t)} + \|x_0\|e^{Mt}\right)^{d-1}\frac{(M+L\tilde{c})\|x_0\|^d}{L\theta_k(1)}\left(e^{L\theta_k(\tau)} - 1\right)e^{Mt}.$$

After letting above

$$B_2(t) \equiv \left(2e^{L\theta_k(t)} + e^{Mt}\right)^{d-1} \left(e^{L\theta_k(\tau)} - 1\right)e^{Mt},$$

obtain

$$|\Phi(t,x) - \Phi(t,z)| \le \frac{(M+L\tilde{c})\|x_0\|^d}{L\theta_k(1)}B_2(t). \tag{19}$$

Now take an arbitrary $\tilde{\varepsilon} > 0$ and fix it. Consider a trajectory $x(t)$ of (1), which starts at $t_0 = 0$, for some $x_0$. Then get that

$$V(0, x_0) \le \tilde{a}\left(\frac{\tilde{\varepsilon}}{2}\right), \tag{20}$$

where $\tilde{a}(\cdot) > 0$, given in (8). Here note that $x_0$ can be also arbitrary.

Consider the last inequality (20) and taking into account the condition (8) obtain

$$\|\tilde{x}(t)\| \le \frac{\tilde{\varepsilon}}{2}, \quad t \ge 0. \tag{21}$$

Therefore, the trajectory $x(t), \quad (t \in [0,\alpha])$ does not abandon $\tilde{\varepsilon}$-neighborhood of $\tilde{x}(t)$ for sufficiently small $\tilde{\varepsilon}$ as far as the estimate (14) hold true, i.e.

$$\|x(t) - \tilde{x}(t)\| \le \frac{\varepsilon c_1\|x_0\|}{\theta_k(1)}\left(e^{L\theta_k(\tau)} - 1\right)e^{Lt}.$$

Next, analyze the function $V(t, x(t))$. One has that

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big[f(t,x) + \varepsilon R(\lambda(x))\Big] \equiv V_t + \langle \nabla V,\ f + \varepsilon \Phi(t,x)\rangle. \tag{22}$$

From H2 $\Phi(t,x) = \langle \nabla V,\ R(\lambda_t(x))\rangle$

After integrating (22), and making use of (7) obtain

$$V(t,x(t)) - V(0,x_0) \le \varepsilon \int_0^t \big(\Phi(s,x(s)) - \Phi(s,z(s)) + \Phi(s,z(s))\big)ds.$$

Then set $t = \tau$ and from (19) and 3) get

$$V(t,x(t)) - V(0,x_0) \le \varepsilon \int_0^t |\Phi(s,x(s)) - \Phi(s,z(s))|ds + \varepsilon \int_0^t |\Phi(s,z(s))|ds \le$$
$$\le \varepsilon(M + L\tilde{c})\frac{\tau\|x_0\|^d B_2(\tau)}{L\theta_k(1)} - \varepsilon\delta\|x_0\|^q \le$$
$$\le \varepsilon\|x_0\|^q\left(\frac{(M + L\tilde{c})\|x_0\|^{d-q}}{L\theta_k(1)}\tau B_2(\tau) - \delta\right),$$

hence

$$V(\tau,x(\tau)) \le V(0,x_0) + \varepsilon\|x_0\|^q\left(\frac{(M + L\tilde{c})\|x_0\|^{d-q}}{L\theta_k(1)}\tau B_2(\tau) - \delta\right). \tag{23}$$

Now, if assume that

$$\frac{(M + L\tilde{c})\|x_0\|^{d-q}}{L\theta_k(\tau)}B_2(\tau) < \frac{\delta}{2},$$

then we have

$$V(\tau,x(\tau)) \le V(0,x_0) - \frac{\varepsilon\tau\|x_0\|^q\delta}{2}. \tag{24}$$

Note that from

$$V(0,x_0) \le \tilde{a}\left(\frac{\tilde{\varepsilon}}{2}\right)$$

it follows

$$V(\tau,x(\tau)) \le \tilde{a}\left(\frac{\tilde{\varepsilon}}{2}\right) - \frac{\varepsilon\tau\|x_0\|^q\delta}{2} < \tilde{a}\left(\frac{\tilde{\varepsilon}}{2}\right),$$

whence conclude that the same trajectory $x(t)$ has returned in the domain defined by (21).

Thus, conclude that

$$\|x(t)\| < \tilde{\varepsilon}, \quad \forall\, t > 0,$$

and then the zero-solution $x = 0$ for (1) is Lyapunov stable in accordance with the definition.

The above stated analyses shows that all estimates are uniform w.r.t. $t_o > 0$ and $\|x_0\| \leq \tilde{\varepsilon}$. Therefore, $\|x(t)\| < \tilde{\varepsilon}$, $\forall t > 0$, i.e., the zero solution $x \equiv 0$ of (1) is Lyapunov stable.

□

**Conclusion.**

We note that our investigation is based on the theory represented in [12].

The result shows that if the Lyapunov function $V(t, x)$ for the unperturbed system is known, and the hypotheses **H1, H2** hold, then the zero solution of the perturbed system is uniformly stable. However, to find $V$ remains a difficult task (see, e.g., [1, 4, 11, 16]). The reader may prove in addition the existence of uniform asymptotic stability of the zero-solution under the same requirements using for this purpose almost the same methods.

Note here that by the same method one could show stability for impulsive FDEs, also it is applicable for problems with inclusions, [2, 5, 6, 8] as well as fuzzy FDEs, [9]. Similar methods can be used for FDEs with "maxima" and delay in the cases considered in [3, 13, 14].

Other applications of the methods used in the present paper are the cases of evolutionary DEs (for instance parabolic PDEs) with "maxima" and/or delay. The problem for stability and asymptotic stability can be resolved also with aid of similar estimates. In parabolic case one may reduce the problem to an FDE and the above stated estimates can be applied as well (see, e.g., [7], [15]).

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