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ON THE TAIL BEHAVIOR OF FUNCTIONS OF RANDOM VARIABLES

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Abstract: It is shown that the tail behavior of the function of nonnegative random variables can be characterized using deterministic functions satisfying certain properties. Also, the upper and lower bounds for the tail of product of random variables are given. Applications of these results are given to some of the well-known models in economics and risk theory.

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1. Introduction

The tail behavior of functions of random variables (rvs) is an important area of research. For theoretical development on this topic, see [11, 8, 2] and references therein. In applied probability, some of the branches that rely on the analysis of the stochastic model, described by given function(s) of rvs, it is important to estimate the behavior for the given function(s) of rvs. For example, in reliability theory, the tail behavior of failure distribution plays an important role (see [1, 7]). In risk modelling, the behavior of the distribution of ruin, for risk model is important (see [9]). In this paper, we focus on some aspects of the tail behavior and generalize the existing results for various functions of rvs, such as sum, maximum and product under dependent and independent setup. Also, the result for moment and exponential indices follows as a special case of our results, provided they exist.

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We next define the necessary terminology required for the discussion. Let X be the rv with cumulative distribution function (cdf) $F_X(x)$. Then $\bar{F}_X(x) = 1 - F_X(x)$ known as the *tail function* of X. The moment and exponential index, (see [4] and [5]), for a rv X are defined as

$$\begin{aligned} \mathbb{I}(X) &= \liminf_{x\to\infty} \frac{R_X(x)}{\ln(x)} = \sup\left\{s \geq 0 : \mathbb{E}\left[\left(X^+\right)^s\right] < \infty\right\}, \\ \mathcal{E}(X) &= \liminf_{x\to\infty} \frac{R_X(x)}{x} = \sup\left\{s \geq 0 : \mathbb{E}\left(e^{sX}\right) < \infty\right\}, \end{aligned}$$

respectively, where $x^+ = \max(0, x)$ and $R_X(x) = -\ln\left(\bar{F}_X(x)\right)$, the *hazard function* of X. A function $h : [0,\infty) \to [0,\infty)$ such that h is increasing and $h(x) \to \infty$ as $x \to \infty$ known as a *scale function.* If h is continuous then we can generalize the definitions of moment and exponential index (see Theorem 2.1 of [6]) as follows.

$$\mathbb{I}_h(X) = \liminf_{x\to\infty} \frac{R_X(x)}{h(x)} = \sup\left\{s \geq 0 : \mathbb{E}\left(e^{sh(X)}\right) < \infty\right\}, \tag{1}$$

is called the *h-order* of X and h is said to be *natural scale function* if $\mathbb{I}_h(X) = 1$. Using this definition, the tail of the rv X can be compared as, for $\epsilon > 0$,

$$\bar{F}_X(x) \leq e^{-(1-\epsilon)h(x)}, \tag{2}$$

and for the rvs X and Y, if $\liminf_{x\to\infty} R_X(x)/R_Y(x) = c$ and $c > 0$. Then the tail comparison (for details, see [6]) is given by, for any small $\epsilon > 0$, there exists x_N such that for all $x > x_N$,

$$\bar{F}_X(x) \leq \left[\bar{F}_Y(x)\right]^{c-\epsilon}. \tag{3}$$

Next, we introduce a result of the existence of h satisfying the required properties for discussing the tail behavior of a nonnegative rv.

Lemma 1.1. *Let X be nonnegative rv. Then there exists a monotone concave function $h : [0,\infty) \to [0,\infty)$ satisfying $h(x) = o(x)$ as $x \to \infty$ and $\mathbb{E}\left(e^{h(X)}\right) < \infty$.*

Proof. It is clear that X is either heavy-tailed or light-tailed rv. For heavy-tailed rv X, Theorem 2.9 of [11] gives the required result. For light-tailed rvs, we can take $h(x) = x^\alpha$ for any $0 < \alpha < 1$ is a monotone concave function satisfying $h(x) = o(x)$ and $\mathbb{E}e^{h(X)} < \infty$.

In this paper, we consider nonnegative continuous rvs with right unbounded support, that is, for a rv X, $\mathbb{P}(X > c) > 0$ for all $c > 0$. The structure of the paper is as follows. In Section 2, we first prove the theorem to characterize the tail behavior of functions of (dependent or independent) rvs, such as sum, maximum and product. Next, we derive a method to find the natural scale function for differentiable functions of rvs and use this method to prove the theorem which gives the bounds for the tail of rv XY. Finally, in Section 3, we apply the results of Section 2 to Cobb-Douglas production model and discrete time risk model.

2. The Moment Index for Sum, Maximum and Product of Random Variables

In this section, we obtain some of the results about the tail behavior of functions of rvs based on the h-order defined in (1). We exemplify the approach of Lemma 1.1 to various functions of rvs such as sum, maximum and product in the following results.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be nonnegative rvs. Then there exists a monotone concave function h and $0 < c_1 \leq c_2 \leq 1$ such that*

$$\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) = c_1 \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\} \leq \mathbb{I}_h\left(\max_{1\leq i\leq n}\{X_i\}\right) = c_2 \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\}.$$

Theorem 2.2. *Let $X_1, \ldots, X_n$ be nonnegative rvs. Then there exist functions h and $\hat{h}$ such that*

(a) $$\min_{1\leq i\leq n}\left\{\mathbb{I}_{\hat{h}}(X_i)\right\} \leq \mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \min_{1\leq i\leq n}\left\{\mathbb{I}_h(X_i)\right\}.$$

(b) $$\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) = m \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\},$$ *for some $m \in (0, 1]$.*

(c) $$\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) = r \min_{1\leq i\leq n}\left\{\mathbb{I}_{\hat{h}}(X_i)\right\}$$ *for some $r \in [1, \infty)$.*

Corollaries 2.1. *Let $X_1, \ldots, X_n$ be independent nonnegative rvs. Then there exists a monotone concave function h such that*

(a) $$\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) = \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\}.$$

(b) $\mathbb{I}_h\left(\max_{1\leq i\leq n} X_i\right) = \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\}.$

Remarks 2.1. 1. Observe that, if h satisfies $h\left(\sum_{i=1}^n x_i\right) \leq \sum_{i=1}^n h(x_i)$, the condition of concavity can also be relaxed (see Theorem 4 of [6]).

2. If h satisfies $h\left(\prod_{i=1}^n x_i\right) \leq \sum_{i=1}^n h(x_i)$, then we have $\mathbb{I}_h\left(\prod_{i=1}^n X_i\right) = \min_{1\leq i\leq n}\{\mathbb{I}_h(X_i)\}$.

3. Observe that, for $X = 0$ and $h(0) = 0$, applying (1), we get $\mathbb{I}_h(0) = \sup\{s \geq 0 : \mathbb{E}\left(e^{sh(0)}\right) < \infty\} = \sup\{s \geq 0 : \mathbb{E}(1) < \infty\} = \infty$. Also, if for $s > 0$, either $\mathbb{E}\left(e^{sh(X_1)}\right) = \infty\Big(\text{i.e., } \mathbb{I}_h(X_1) = 0\Big)$ or $\mathbb{E}\left(e^{sh(X_2)}\right) = \infty\Big(\text{i.e., } \mathbb{I}_h(X_2) = 0\Big)$. Then $\mathbb{I}_h(X_1 + X_2) = 0$.

4. Observe also that, if $h(x) = x$ or $h(x) = \ln(x)$ (although $\ln(x)$ is not a scale function, as $h : [0,\infty) \to (-\infty,\infty)$), our results for the case of exponential and moment indices follows immediately provided they exist. However, our results give flexibility for the choice of h_i's.

2.1. Natural Scale Function

Recall from (1) that, a scale function h is called natural scale function for a rv X if $\mathbb{I}_h(X) = 1$. Next, we describe a method to find the natural scale function for functions of rvs via transformation technique.

Method. Let $X_1, \ldots, X_n$ be continuous rvs with support $\mathcal{S}$. In particular, assume $Y_1 = g(X_1, \ldots, X_n)$ is a differentiable function of n rvs. Then, the problem is to find the natural scale function for Y_1. Consider an integral

$$\int \cdots \int_R f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)dx_1 \cdots dx_n,$$

where $R \subset \mathbb{R}^n$. Now, take the transformation of the form $y_1 = g(x_1, \ldots, x_n)$ and $y_i = x_i$ for $i = 2, 3, \ldots, n$ with support $\mathcal{T}$, together with inverse functions $x_1 = w(y_1, \ldots, y_n)$ and $x_i = y_i$ for $i = 2, 3, \ldots, n$. Then, using transformation technique (see [10], pp-124), it is well-known that the joint pdf of rvs $Y_1 = g(X_1, \ldots, X_n)$, $Y_2 = X_2, \ldots, Y_n = X_n$ is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = |J| f_{X_1,\ldots,X_n}(w(y_1, \ldots, y_n), y_2, \ldots, y_n),$$

where $|J|$ is the determinant of the Jacobian matrix J. Also, the marginal distribution of Y_1 is

$$f_{Y_1}(y_1) = \underbrace{\int \cdots \int}_{(n-1)\ with\ supp \in \mathcal{T}} |J| f_{X_1,\ldots,X_n}(w(y_1,\ldots,y_n), y_2,\ldots,y_n)\, dy_2 \ldots dy_n$$

and the tail function is given by $\bar{F}_{Y_1}(y_1) = \int_{y_1}^{\infty} f_{Y_1}(t)dt$. Hence, the natural scale function is $h(y_1) = -\ln\left(\bar{F}_{Y_1}(y_1)\right)$.

We can compare the tail of the random variable $Y_1 = g(X_1,\ldots,X_n)$ with the help of natural scale function h. That is, $\bar{F}_{Y_1}(y_1) = \bar{F}\left(g(x_1,\ldots,x_n)\right) \leq e^{-(1-\epsilon)h(y_1)}$. In particular, we have shown that for any differentiable function of $g(X_1,\ldots,X_n)$, we can find a natural scale function h, such that, for any $\epsilon > 0$, $\bar{F}_{Y_1}(y_1)$ (the tail function of g) is dominated by $e^{-(1-\epsilon)h(y_1)}$.

Next, we present the upper and lower bounds for the product of two iid rvs.

Theorem 2.3. *Let X and Y be nonnegative iid rvs. Then, for small $\epsilon > 0$, we have the following inequalities, for $x > 1$,*

(a) *If $h(\cdot) = 2R_X(\cdot)$, then $\bar{F}_{XY}(x), \bar{F}_X(x) \leq e^{-(\frac{1}{2}-\epsilon)h(x)}$.*

(b) *If $h(\cdot) = R_{XY}(\cdot)$, then $\bar{F}_{XY}(x), \bar{F}_X(x) \leq e^{-(1-\epsilon)h(x)}$.*

(c) $\left[\bar{F}_X(x)\right]^{1/(1-\epsilon)} \leq \bar{F}_{XY}(x) \leq \left[\bar{F}_X(x)\right]^{1-\epsilon}$.

Remark 2.1. In the proof of Theorem 2.3, the natural scale function h defined in Case 1 and Case 2 are different. In Case 1, it is twice the natural scale function of X and in Case 2, it is the natural scale function of XY.

3. Applications

In this section, we give applications of our results to some well-known models in economics and risk theory.

3.1. Cobb-Douglas Production Model

In economics, the Cobb-Douglas production model describes the relationship between the output and input variables. This has been widely used since its introduction by Knut Wicksell (1851-1926).

The first significant application of this model is given in [3], where they studied the growth of American economy during the period 1899-1922 using this model, and they were able to present a simplified view of the economy in which the production output is determined by the amount of labor involved and capital invested.

It is also of importance in economics, to study the long-term behavior of production function in order to formulate certain policies. We next describe the mathematical formulation of the Cobb-Douglas model.

Let $T(X,Y) = bX^{\alpha}Y^{\beta}$, where T =total production (the monetary value of all goods produced in a year):
X =labor involved (the total number of person-hours worked in a year),
Y =capital invested (the monetary worth of all machinery, equipment, and buildings),
b =total factor productivity,
$\alpha > 0$, $\beta > 0$ are the output elasticities of labor and capital, respectively. These values are constants determined by available technology.

We next give the implications of our results to Cobb-Douglas model. Now, consider the following conditions on α and β, for a detailed analysis.

Case (i). $\alpha + \beta < 1$, there are decreasing return to scale (i.e., output decreases proportional to change in inputs).

Case (ii). $\alpha + \beta = 1$, there are constant return to scale (i.e., output is proportional to change in inputs).

Case (iii). $\alpha + \beta > 1$, there are increasing return to scale (i.e., output increases proportional to change in inputs).

Remark 3.1. Suppose, for any value of b, α and β positive, $T(X,Y) = bX^{\alpha}Y^{\beta}$, where X and Y are any nonnegative rvs. Then using the method given in 2.1, we can find the dominated function for T.

We demonstrate this phenomenon through following examples, for various conditions of α and β. First, consider for Pareto distribution with parameters a and k, and the condition $\alpha \neq \beta$ in Cobb-Douglas production model.

Example 3.1. Let X and Y are iid Pareto distributed rvs with common pdf

$$f_X(x) = \frac{ak^a}{x^{a+1}}$$

where $k \leq x < \infty$ and $a, k > 0$. Now using technique given in 2.1 with $y_1 = g(x,y) = x^{\alpha}y^{\beta}$ and $w(u,v) = \left(u^{1/\alpha}\Big/v^{\alpha/\beta}\right)$. Therefore, $|J| = \left(u^{(1/\alpha)-1}\Big/\alpha v^{\beta/\alpha}\right)$. Hence, it can be easily seen that $f(u,v) = \left(a^2k^{2a}\Big/\alpha u^{(a/\alpha)+1}v^{1+a-(\beta a/\alpha)}\right)$.

Since $k \le x, y < \infty$ implies that $k^{\alpha+\beta} \le k^{\alpha} v^{\beta} \le u < \infty$. Therefore, the marginal distribution of $g(X,Y)$ is $f_{g(X,Y)}(u) = \frac{ak^a}{(\beta-\alpha)} \left\{ \left(1 \Big/ k^{-a\alpha/\beta} u^{(a/\beta)+1}\right) - \left(1 \Big/ u^{(a/\alpha)+1} k^{-a\beta/\alpha}\right) \right\}$, where $k^{\alpha+\beta} \le u < \infty$. The tail function of $g(X,Y)$ is $\bar{F}_{g(X,Y)}(u) = \frac{k^a}{(\beta-\alpha)} \left\{ \left(\beta \Big/ k^{-a\alpha/\beta} u^{a/\beta}\right) - \left(\alpha \Big/ u^{a/\alpha} k^{-a\beta/\alpha}\right) \right\}$.

Let

$$c = \left(k^a \Big/ (\beta - \alpha)\right),$$

then

$$\bar{F}_{g(X,Y)}(u) = c \left\{ \left(\beta \Big/ k^{-a\alpha/\beta} u^{a/\beta}\right) - \left(\alpha \Big/ u^{a/\alpha} k^{-a\beta/\alpha}\right) \right\}$$

and the natural scale function of $g(X,Y)$ is $h(u) = -\ln \left[c \left\{ \left(\beta \Big/ k^{-a\alpha/\beta} u^{a/\beta}\right) - \left(\alpha \Big/ u^{a/\alpha} k^{-a\beta/\alpha}\right) \right\} \right]$. From (2), for $\epsilon > 0$

$$\bar{F}_{g(X,Y)}(u) \le e^{-(1-\epsilon)h(u)} = c^{1-\epsilon} \left(\frac{\beta}{k^{-a\alpha/\beta} u^{a/\beta}} - \frac{\alpha}{u^{a/\alpha} k^{-a\beta/\alpha}} \right)^{1-\epsilon}.$$

Hence the tail of the production function dominated by the above function for $\alpha, \beta > 0$ and $\alpha \ne \beta$. That is, all three cases (Increasing return to scale, constant return to scale, decreasing return to scale) whenever $\alpha \ne \beta$ tail of the production function dominated by the above function.

Now, consider the case when $\alpha = \beta = 1$ and Theorem 2.3 for the Pareto distribution.

Example 3.2. Suppose X and Y are iid Pareto distributed rv with pdf given by

$$f(x) = \frac{ak^a}{x^{a+1}}$$

where $a > 0$ *and* $k \le x < \infty$. Now using technique given in 2.1 with $y_1 = g(x,y) = xy$ and $w(u,v) = u/v$. Therefore, $|J| = 1/v$.

Hence, it is easy to see that $f(u,v) = \left(a^2 k^{2a} / u^{a+1} v\right)$. Since $k \le x < \infty$ implies that $k^2 \le kv \le u < \infty$. Also, the marginal pdf of XY is $f_{XY}(u) = \left(a^2 k^{2a} \ln(u/k^2) \Big/ u^{a+1}\right)$, where $k^2 \le u < \infty$. The tail function XY is $\bar{F}_{XY}(u) = \left(k^{2a}\left(1 + a\ln(u/k^2)\right) \Big/ u^a\right)$. Hence, the natural scale function of XY is $h_{XY}(u) = a\ln(u) - \ln\left(1 + a\ln\left(u/k^2\right)\right) - 2a\ln(k)$.

It is clear that

$$\mathbb{I}_{h_{XY}}(X) = \liminf_{x\to\infty} \frac{R_X(x)}{h_{XY}(x)} = \liminf_{x\to\infty} \frac{a\ln(x) - a\ln(k)}{a\ln(x) - \ln(1 + a\ln(\frac{x}{k^2})) - 2a\ln(k)} = 1.$$

Similarly, $\mathbb{I}_{h_{XY}}(Y) = 1$, hence for $c = 1$, it is satisfied Theorem 2.1.

Compare with production function of Cobb-Douglas model, we have $\alpha = \beta = b = 1$. That is, $T(X,Y) = XY$, i.e., for $x > 1$, we have the case in which increasing return to scale, From Theorem 2.3, the tail function of XY and X are dominated by the function,

$$\exp\{-(1-\epsilon)h_{XY}(x)\} = \left(\frac{k^{2a}(1 + a\ln(x/k^2))}{x^a}\right)^{1-\epsilon}$$

and

$$\left[\bar{F}_X(x)\right]^{1/(1-\epsilon)} \leq \bar{F}_{XY}(x) \leq \left[\bar{F}_X(x)\right]^{1-\epsilon}.$$

3.2. Discrete Time Risk Model

Let X_i be the net payout of the insurer at year i, and Y_i be the discount factor (from year i to $i-1$) related to the return on the investment, $i = 1, 2, \dots$. Then the discounted value of the total risk amount accumulated till the end of year n can be modeled by a discrete time stochastic process

$$W_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j.$$

The basic assumptions for this model (see [9]) are as follows.

A_1. Let A_n be the net income within year n. Assume $\{A_n : n = 1, 2, \dots\}$ constitute a sequence of iid rvs with support $(-\infty, \infty)$.

A_2. Let r_n be the rate of interest on the reserve invested is risky assets. Assume $\{r_n : n = 1, 2, \dots\}$ is a sequence of iid rvs with support $(-1, \infty)$.

A_3. Also, assume $\{A_n : n = 1, 2, \dots\}$ and $\{r_n : n = 1, 2, \dots\}$ are mutually independent.

Define $B_n = 1 + r_n$, also known as the *inflation coefficient* from year $n-1$ to year n and let $Y_n = B_n^{-1}$ be the *discount factor* from year n to year $n-1$, $n = 1, 2, \dots$. The rvs $X = -A$ and Y as the insurance risk and financial risk, respectively. Clearly, $\mathbb{P}(0 < Y < \infty) = 1$.

Let the initial capital of the insurer be $x \geq 0$. The surplus of the insurer accumulated till the end of year n can be characterized by S_n which satisfies the following recurrence equation

$$S_0 = x, \quad S_n = B_n S_{n-1} + A_n,$$

where $B_n = 1 + r_n$, $n = 1, 2, \dots$. The recurrence relation can be written as

$$S_0 = x, \quad S_n = x \prod_{j=1}^{n} B_j + \sum_{i=1}^{n} A_i \prod_{j=i+1}^{n} B_j, \qquad n = 1, 2, \dots, \tag{4}$$

where $\prod_{j=n+1}^{n} = 1$.

Next, the time of ruin for the risk model (4) is defined as $\tau(x) = \inf\{n = 1, 2, \dots : S_n < 0 | S_0 = x\}$. Hence, the finite time ruin probability, $\psi(x, n)$, and of ultimate ruin probability, $\psi(x)$, can be defined as $\psi(x, n) = \mathbb{P}(\tau(x) \le n)$, respectively, $\psi(x) = \psi(x, \infty) = \mathbb{P}(\tau(x) < \infty)$. Clearly, the probability that the ruin occurs exactly at year n, can be defined as $\mathbb{P}(\tau(x) = n) = \psi(x, n) - \psi(x, n-1)$, $n = 1, 2, \dots$.

A more significant calculation might be $\mathbb{P}(\tau_y(x) \le n)$ or $\mathbb{P}(\tau_y(x) < \infty)$ for $x > 0$ and $n = 1, 2, \dots$, where $\tau_y(x)$ is a stopping time, defined by $\tau_y(x) = \inf\{n = 1, 2, \dots : S_n \le y | S_0 = x\}$ for any regulatory or trigger boundary $y \ge 0$. This stopping time $\tau_y(x)$ may be interpreted as the first time at which there is a need to raise the capital in order to maintain solvency. We can rewrite the discounted value of the surplus S_n in as

$$\tilde{S}_0 = x, \quad \tilde{S}_n = S_n \prod_{j=1}^{n} Y_j = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j = x - W_n.$$

Hence, for each $n = 0, 1, 2, \dots$, $\psi(x, n) = \mathbb{P}(U_n > x)$, where

$$U_n = \max\{0, \max_{1 \le k \le n} W_k\} \quad \textit{with} \quad U_0 = 0.$$

Now, define $V_0 = 0$, $V_n = Y_n \max\{0, X_n + V_{n-1}\}$, $n = 1, 2, \dots$. Then Theorem 2.1 of [9], it is clear that $U_n \overset{\mathcal{L}}{=} V_n$.

Hence, the following result shows that the relation $\psi(x, n) = \mathbb{P}(V_n > x)$ holds for each $n = 1, 2, \dots$ under the assumptions A_1, A_2 and A_3.

We next apply our results to discuss the tail behavior of U_n. Using Theorem 2.1, Theorem 2.2 and Corollary 2.1, there exists a monotone concave function h such that

$$\mathbb{I}_h(U_n) = \mathbb{I}_h\left(\max\{0, \max_{1 \le k \le n} W_k\}\right) = \mathbb{I}_h\left(\max_{1 \le k \le n} W_k\right) = c_n \min_{1 \le k \le n} \{\mathbb{I}_h(W_k)\} \tag{5}$$

for some constant $c_n \in [\frac{1}{n}, 1]$.

Suppose constant corresponding to $\left(\sum_{k=1}^{i} X_k \prod_{j=1}^{k} Y_j\right) + \left(X_{i+1} \prod_{j=1}^{i+1} Y_j\right)$ is c_i, where $c_i \in [\frac{1}{2}, 1]$ for $i = 1, 2, \dots, n-1$. Now use Theorem 2.1 in

$W_2, W_2, \ldots, W_n$, we get

$$\mathbb{I}_h(W_j) = \min_{0\leq k\leq j-1} \left\{ \prod_{i=k}^{j-1} c_i \mathbb{I}_h \left(X_{k+1} \prod_{l=1}^{k+1} Y_l \right) \right\} \tag{6}$$

with assumption that $c_0 = 1$, combining (5) and (6), we get

$$\mathbb{I}_h(U_n) = c_n \min_{0\leq k\leq n-1} \left\{ \prod_{i=k}^{n-1} c_i \mathbb{I}_h \left(X_{k+1} \prod_{l=1}^{k+1} Y_l \right) \right\}, \tag{7}$$

where $0 < c_i \leq 1$, for $i = 1, 2, \ldots, n-1$. Now, suppose constant corresponding to $X_i(\prod_{j=1}^{i} Y_j)$ is k_i, $i = 1, 2, \ldots, n$. Then, we can write

$$\mathbb{I}_h \left(X_j \prod_{j=1}^{i} Y_j \right) = \min \left\{ k_j \mathbb{I}_h(X_j), k_j \mathbb{I}_h \left(\prod_{i=1}^{j} Y_j \right) \right\} \tag{8}$$

Combining (7) and (8), we get

$$\mathbb{I}_h(U_n) = c_n \min_{0\leq m\leq n-1} k_{m+1} \prod_{i=m}^{n-1} c_i \left\{ \mathbb{I}_h(X_{m+1}), \mathbb{I}_h \left(\prod_{l=1}^{m+1} Y_l \right) \right\}. \tag{9}$$

Now, again take constants corresponding to $(\prod_{j=1}^{i} Y_j)Y_{i+1}$ is d_i, $i = 1, 2, \ldots, n-1$, where $d_i \in (0, 1]$ are some constants.

$$\mathbb{I}_h \left(\prod_{i=1}^{n} Y_i \right) = \min_{0\leq m\leq n-1} \left\{ \prod_{i=m}^{n-1} d_i \mathbb{I}_h(Y_{m+1}) \right\}, \tag{10}$$

with assumption $d_0 = 1$. Combining (9) and (10)

$$\begin{aligned} \mathbb{I}_h(U_n) = c_n \min \Bigg\{ & \min_{0\leq m\leq n-1} \left\{ k_{m+1} \prod_{i=m}^{n-1} c_i \mathbb{I}_h(X_{m+1}) \right\}, \\ & \min_{0\leq l\leq n-1} \left\{ \min_{l\leq m\leq n-1} \left\{ k_{m+1} \prod_{i=m}^{n-1} c_i \prod_{j=l}^{m} d_j \mathbb{I}_h(Y_{l+1}) \right\} \right\} \Bigg\}. \end{aligned} \tag{11}$$

The expression given in (11) is a general representation for h-order of U_n. Further to simplify representation for (11), we consider the following cases.

Case 1. If we can arrange c_i and d_i such that $d_1 \leq c_{i_1}$ for $i_1 \in \{1, 2, \ldots, n-1\}$,$d_2 \leq c_{i_2}$ for $i_2 \neq i_1$, ..., $d_{n-1} \leq c_{i_{n-1}}$ for $i_{n-1} \neq i_1 \neq i_2 \neq \cdots \neq i_{n-2}$ and $k_n = \min_{1\leq p\leq n}\{k_p\}$. Then

$$\mathbb{I}_h(U_n) = c_n \min_{0\leq m\leq n-1} \left\{ k_{m+1} \prod_{i=m}^{n-1} c_i \mathbb{I}_h(X_{m+1}), k_n c_{n-1} \prod_{i=m}^{n-1} d_i \mathbb{I}_h(Y_{m+1}) \right\}.$$

Take

$$\begin{aligned}
\mathbb{C} &= diag \left\{ k_1 \prod_{i=0}^{n-1} c_i, k_2 \prod_{i=1}^{n-1} c_i, k_3 \prod_{i=2}^{n-1} c_i, \ldots, k_n c_{n-1} \right\}. \\
\mathbb{D} &= diag \left\{ k_n c_{n-1} \prod_{i=0}^{n-1} d_i, k_n c_{n-1} \prod_{i=1}^{n-1} d_i, k_n c_{n-1} \prod_{i=2}^{n-1} d_i, \ldots, k_n c_{n-1} d_{n-1} \right\}. \\
\mathbf{X} &= (X_1, X_2, \ldots, X_n) \quad \text{and} \quad \mathbf{Y} = (Y_1, Y_2, \ldots, Y_n).
\end{aligned}$$

Then

$$\mathbb{I}_h(U_n) = c_n \min \{\mathbb{I}_h(\mathbf{X})\mathbb{C}, \mathbb{I}_h(\mathbf{Y})\mathbb{D}\}.$$

Case 2. If we can arrange c_i and d_i such that $c_1 \leq d_{i_1}$ for $i_1 \in \{1, 2, \ldots, n-1\}$, $c_2 \leq d_{i_2}$ for $i_2 \neq i_1$, ..., $c_{n-1} \leq d_{i_{n-1}}$ for $i_{n-1} \neq i_1 \neq i_2 \neq \cdots \neq i_{n-2}$. Then

$$\mathbb{I}_h(U_n) =$$
$$c_n \min_{0\leq m\leq n-1} \left\{ k_{m+1} \prod_{i=m}^{n-1} c_i \mathbb{I}_h(X_{m+1}), k_2 d_1 \prod_{i=1}^{n-1} c_i \mathbb{I}_h(Y_1), k_{m+1} d_m \prod_{i=m}^{n-1} c_i \mathbb{I}_h(Y_{m+1}) \right\}.$$

Take

$$\begin{aligned}
\mathbb{C} &= diag \left\{ k_1 \prod_{i=1}^{n-1} c_i, k_2 \prod_{i=1}^{n-1} c_i, k_3 \prod_{i=2}^{n-1} c_i, \ldots, k_n c_{n-1} \right\}. \\
\mathbb{D} &= diag \left\{ k_2 d_1 \prod_{i=1}^{n-1} c_i, k_2 d_1 \prod_{i=1}^{n-1} c_i, k_3 d_2 \prod_{i=2}^{n-1} c_i, \ldots, k_n c_{n-1} d_{n-1} \right\}.
\end{aligned}$$

Then

$$\mathbb{I}_h(U_n) = c_n \min \{\mathbb{I}_h(\mathbf{X})\mathbb{C}, \mathbb{I}_h(\mathbf{Y})\mathbb{D}\}.$$

Remark 3.2. If $X_1, X_2, \dots, X_n$ are iid rvs and $Y_1, Y_2, \dots, Y_n$ are also iid rvs. Then for Case 1, we have

$$\mathbb{I}_h(U_n) = c_n \min\left\{k_n c_{n-1}\prod_{i=1}^{n-1} d_i \mathbb{I}_h(Y_1), \min\{k_1, k_2\}\prod_{i=1}^{n-1} c_i \mathbb{I}_h(X_1)\right\},$$

and for Case 2,

$$\mathbb{I}_h(U_n) = c_n \min\left\{k_2 d_1 \prod_{i=1}^{n-1} c_i \mathbb{I}_h(Y_1), \min\{k_1, k_2\}\prod_{i=1}^{n-1} c_i \mathbb{I}_h(X_1)\right\}$$

which are very easy to calculate.

Remark 3.3. Observe that, from Theorem 2.1, Theorem 2.2 and Corollary 2.1, if the natural scale function of X_i and Y_i are h_{X_i} and h_{Y_i} respectively, then the scale function for U_n is given by

$$h(x) = \sum_{i=1}^{n}\Big(h_{X_i}(x) + (n-i+1)h_{Y_i}(x)\Big).$$

Next we consider the tail behavior of V_n, defined as

$$V_n = Y_n \max\{0, X_n + V_{n-1}\}.$$

Now let the constants between $(Y_i)(\max\{0, X_i + V_{i-1}\})$ and $(X_j) + (V_{j-1})$ are c_i and d_i respectively, where $c_i \in (0,1]$ and $d_j \in [\frac{1}{2}, 1]$ for $i = 1, 2, \dots, n$ and $j = 2, 3, \dots, n$. Hence,

$$\begin{aligned}
\mathbb{I}_h(V_n) &= \min\{c_n \mathbb{I}_h(Y_n), c_n \mathbb{I}_h(0, X_n + V_{n-1})\}\\
&= \min\{c_n \mathbb{I}_h(Y_n), c_n \mathbb{I}_h(X_n + V_{n-1})\}\\
&= \min\{c_n \mathbb{I}_h(Y_n), c_n d_n \mathbb{I}_h(X_n), c_n d_n \mathbb{I}_h(V_{n-1})\}\\
&\;\vdots\\
&= \min_{1\le m\le n}\left\{\prod_{i=m}^{n} c_i d_i \mathbb{I}_h(X_m), \prod_{i=m}^{n} c_i d_{i+1} \mathbb{I}_h(Y_m)\right\}. \qquad (12)
\end{aligned}$$

We assume in above expression that $d_{n+1} = 1 = d_1$.

Now take

$$\mathbb{C} = diag\left\{\prod_{i=1}^{n} c_i d_{i+1}, \dots, c_n d_n\right\}$$

and

$$\mathbb{D} = diag\left\{\prod_{i=1}^{n} c_i d_{i+1}, \ldots, c_n\right\}.$$

Hence, we get

$$\mathbb{I}_h(V_n) = \min\left\{\mathbb{I}_h(\mathbf{X})\mathbb{C}, \mathbb{I}_h(\mathbf{Y})\mathbb{D}\right\}.$$

Remark 3.4. If $X_1, X_2, \ldots, X_n$ are iid rvs and $Y_1, Y_2, \ldots, Y_n$ are also iid rvs. Then $\mathbb{I}_h(V_n) = \prod_{i=1}^{n} c_i d_{i+1} \min\left\{\mathbb{I}_h(X_1), \mathbb{I}_h(Y_1)\right\}$, which is easy to calculate. We know that $V_n = Y_n \max\left\{0, X_n + V_{n-1}\right\}$. If the scale function of X_i and Y_i are h_{X_i} and h_{Y_i} respectively. Then, using Theorem 2.1, Theorem 2.2 and Corollary 2.1, the scale function h for V_n is given by

$$h(x) = \sum_{i=1}^{n}\Big(h_{X_i} + h_{Y_i}\Big).$$

Remark 3.5. As mentioned earlier, U_n and V_n have same distribution and it can be seen from (11) and (13), the representation of V_n is preferable over the representation of U_n due to the ease of computation and simplicity in the construction of h.

4. Proofs

Proof of Theorem 2.1. Since $X_1, \ldots, X_n$ are nonnegative rvs. Using Lemma 1.1, there exist monotone concave functions $h_1, \ldots, h_n$ such that $\mathbb{E}\left(e^{h_i(X)}\right) < \infty$ and $h_i(x) = o(x)$ as $x \to \infty$, for $i = 1, \ldots, n$. Define

$$h(x) = \sum_{i=1}^{n} h_i(x) - \sum_{i=1}^{n} h_i(0),$$

therefore, h is a monotone concave function and $h(0) = 0$. Hence, by Remark 2 of [6], h is continuous and increasing function with $h(x_1 + x_2) \le h(x_1) + h(x_2)$ for all $x_1, x_2 > 0$. We know that $\max_{1\le i\le n} x_i \ge x_i$ for $i = 1, \ldots, n$, this implies for $s > 0$, we have

$$\mathbb{E}\left(e^{sh\left(\max_{1\le i\le n} X_i\right)}\right) \ge \mathbb{E}\left(e^{sh(X_i)}\right).$$

Therefore, $\mathbb{I}_h(\max_{1\le i\le n} X_i) \le \mathbb{I}_h(X_i)$. Hence,

$$\mathbb{I}_h\left(\max_{1\le i\le n} X_i\right) \le \min_{1\le i\le n}\left\{\mathbb{I}_h(X_i)\right\}. \tag{13}$$

Similarly, it is easy to see that

$$\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) \leq \mathbb{I}_h\left(\max_{1\leq i\leq n} X_i\right) \tag{14}$$

Hence combine (14) and (15), we get required result.

Proof of Theorem 2.2. Using Lemma 1.1, there exist monotone concave functions $h_1,\ldots,h_n$ such that $\mathbb{E}\left(e^{h_i(X)}\right) < \infty$ and $h_i(x) = o(x)$ as $x \to \infty$, for $i = 1,\ldots,n$. Let $h(x) = \sum_{i=1}^{n} h_i(x) - \sum_{i=1}^{n} h_i(0)$, therefore, h is monotone concave with $h(0) = 0$. For $i = 1,\ldots,n$, define $\hat{h}(x_i) = x^{n-1}h(x_i)$, where $x = \max_{1\leq i\leq n}\{x_i + 1\}$. It is clear that $x_i x^{n-1} \geq \prod_{i=1}^{n} x_i$, This implies

$$h(x_i) \geq h\left(\prod_{i=1}^{n} x_i/x^{n-1}\right) \geq \frac{1}{x^{n-1}} h\left(\prod_{i=1}^{n} x_i\right).$$

That is,

$$h\left(\prod_{i=1}^{n} x_i\right) \leq x^{n-1}h(x_i) = \hat{h}(x_i).$$

Therefore,

$$\mathbb{E}\left(e^{sh\left(\prod_{i=1}^{n} X_i\right)}\right) \leq \mathbb{E}\left(e^{s\hat{h}(X_i)}\right).$$

Hence,

$$\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \geq \min_{1\leq i\leq n}\left\{\mathbb{I}_{\hat{h}}(X_i)\right\}.$$

Now, consider

$$\begin{aligned}&\mathbb{E}\left(e^{sh\left(\prod_{i=1}^{n} X_i\right)}\right)\\ &\quad\geq \mathbb{E}\left(e^{sh\left(\prod_{i=1}^{n} X_i\right)}\mathbf{1}\left(X_2 \geq 1,\ldots,X_n \geq 1\right)\right) \geq \mathbb{E}\left(e^{sh(X_1)}\right)\prod_{i=2}^{n}\mathbb{P}(X_i \geq 1),\end{aligned}$$

we get $\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \mathbb{I}_h(X_1)$, similarly, for $i = 2,\ldots,n$, $\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \mathbb{I}_h(X_i)$. Hence, $\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \min_{1\leq i\leq n}\left\{\mathbb{I}_h(X_i)\right\}$. Hence,

$$\min_{1\leq i\leq n}\left\{\mathbb{I}_{\hat{h}}(X_i)\right\} \leq \mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \min_{1\leq i\leq n}\left\{\mathbb{I}_h(X_i)\right\}.$$

This proves (a). It is easy to see that $\mathbb{I}_{\hat{h}}(X_i) \leq \mathbb{I}_h(X_i)$, for $i = 1, \ldots, n$. Choose $c \geq 1$ such that $c\mathbb{I}_{\hat{h}}(X_i) \geq \mathbb{I}_h(X_i)$, therefore,

$$\frac{1}{c} \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\} \leq \min_{1\leq i\leq n} \left\{\mathbb{I}_{\hat{h}}(X_i)\right\} \leq \mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) \leq \min_{1\leq i\leq n} \mathbb{I}_h(X_i) \leq c \min_{1\leq i\leq n} \left\{\mathbb{I}_{\hat{h}}(X_i)\right\}.$$

Hence, there exist $m \in (0, 1]$ and $r \in [1, \infty)$ such that

$$\mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) = m \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\} \quad \text{and} \quad \mathbb{I}_h\left(\prod_{i=1}^{n} X_i\right) = r \min_{1\leq i\leq n} \left\{\mathbb{I}_{\hat{h}}(X_i)\right\}.$$

This proves (b) and (c).

Proof of Corollary 2.1. It follows from the proof of Theorem 2.1,

$$\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) \leq \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\}.$$

To prove the equality, it is known that $h\left(\sum_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} h(x_i)$ and X_i are independent rvs. Then $\mathbb{E}\left(e^{sh\left(\sum_{i=1}^{n} X_i\right)}\right) \leq \prod_{i=1}^{n} \mathbb{E}\left(e^{sh(X_i)}\right)$. Therefore, $\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) \geq \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\}$. Hence,

$$\mathbb{I}_h\left(\sum_{i=1}^{n} X_i\right) = \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\}.$$

This proves (a).

Next, from (14), we know that

$$\mathbb{I}_h\left(\max_{1\leq i\leq n} X_i\right) \leq \min_{1\leq i\leq n} \{\mathbb{I}_h(X_i)\}.$$

Now, it is clear that

$$\max_{1\leq i\leq n} x_i \leq \sum_{i=1}^{n} x_i,$$

this implies

$$\mathbb{E}\left(e^{sh(\max_{1\leq i\leq n} X_i)}\right) \leq \mathbb{E}\left(e^{sh\left(\sum_{i=1}^{n} X_i\right)}\right) \leq \prod_{i=1}^{n} \mathbb{E}\left(e^{sh(X_i)}\right).$$

Therefore,

$$\mathbb{I}_h\left(\max_{1\le i\le n} X_i\right) \ge \min_{1\le i\le n}\left\{\mathbb{I}_h(X_i)\right\}.$$

Hence,

$$\mathbb{I}_h\left(\max_{1\le i\le n} X_i\right) = \min_{1\le i\le n}\left\{\mathbb{I}_h(X_i)\right\}.$$

This proves (b).

Proof of Theorem 2.3. Recall from Theorem 2.2 and Remarks 2.1, we have $\mathbb{I}_h(XY) = \min(\mathbb{I}_h(X), \mathbb{I}_h(Y))$. For iid rvs $\mathbb{I}_h(X) = \mathbb{I}_h(Y)$. Hence, $\mathbb{I}_h(XY) = \mathbb{I}_h(X)$. Then we can find the tail behavior of the rvs in the following ways.

Case 1. Suppose h_1 is a natural scale function of X, therefore $h = 2h_1$ and $\mathbb{I}_h(X) = \frac{1}{2}\mathbb{I}_{h_1}(X) = 1/2$. Then $\mathbb{I}_h(XY) = \mathbb{I}_h(X) = 1/2$, for small $\epsilon > 0$, using the definition of lim inf,

$$\bar{F}_{XY}(x), \bar{F}_X(x) \le e^{-(\frac{1}{2}-\epsilon)h(x)}.$$

This proves (a).

Since, h_1 is the natural scale function X, therefore, $h(x) = 2h_1(x) = 2R_X(x)$ and $\mathbb{I}_h(XY) = 1/2$, hence,

$$\liminf_{x\to\infty} \frac{R_{XY}(x)}{2R_X(x)} = 1/2,$$

that is,

$$\liminf_{x\to\infty} \frac{R_{XY}(x)}{R_X(x)} = 1.$$

From (3),

$$\bar{F}_{XY}(x) \le \left[\bar{F}_X(x)\right]^{1-\epsilon}. \tag{15}$$

Case 2. Using the method described in 2.1, we can find the natural scale function h for XY then $\mathbb{I}_h(XY) = \mathbb{I}_h(X) = 1$. Using the definition of lim inf, for $\epsilon > 0$,

$$\bar{F}_{XY}(x), \bar{F}_X(x) \le e^{-(1-\epsilon)h(x)}.$$

This proves (b).

Next, note that h is the natural scale function XY, therefore, $h(x) = R_{XY}(x)$ and $\mathbb{I}_h(XY) = \mathbb{I}_h(X) = 1$, that is, $\mathbb{I}_h(X) = 1$, hence,

$$\liminf_{x\to\infty} \frac{R_X(x)}{R_{XY}(x)} = 1.$$

Therefore, from (3),

$$\bar{F}_X(x) \leq \left[\bar{F}_{XY}(x)\right]^{1-\epsilon}. \tag{16}$$

Combining (16) and (17), it is clear that

$$\left[\bar{F}_X(x)\right]^{1/(1-\epsilon)} \leq \bar{F}_{XY}(x) \leq \left[\bar{F}_X(x)\right]^{1-\epsilon}.$$

This proves (c).

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