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KANNAN–TYPE FIXED POINT THEOREM IN CONE PENTAGONAL METRIC SPACES

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Abstract: In this paper, we prove Kannan - type fixed point theorem for two self mappings in non-normal cone pentagonal metric spaces. Our results extend and improve the recent results announced by many authors.

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1. Introduction

A mapping $T : X \to X$ on a metric space (X, d) is called Kannan contraction if there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X. \tag{1.1}$$

Kannan [10] proved that if X is complete, then every Kannan contraction has a fixed point.

The study of existence and uniqueness of fixed point of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Kannan contraction principle in various generalized metric spaces (e.g., see [2, 7, 8, 11]).

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Long-Guang and Xian [7] introduced the concept of a cone metric space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1, 3, 5, 6, 12]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Garg and Agarwal [6] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Motivated and inspired by the results of [6, 11], it is our purpose in this paper to continue the study of common fixed points for a two self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [2, 8, 11], and many others.

2. Preliminaries

The following definitions and Lemmas, introduced in [1, 2, 3, 6, 7], are needed in the sequel.

Definition 2.1. Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}, \quad a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.2. Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [7]).

Definition 2.3. Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \le \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.4. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [3]).

Definition 2.5. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \le d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ [Pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2.6. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [6]).

Let (X, d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone pentagonal metric space.

Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Also, T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 2.7. *Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

Lemma 2.8. *Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for $a, c, u, v, w \in E$, we have*

(1) If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$.

(2) If $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.

(3) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(4) If $c \in int(P)$ and $a_n \to 0$, then $\exists n_0 \in \mathbb{N} : \forall n > n_0$, $a_n \ll c$.

Lemma 2.9. *Let (X, d) be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:*

1. *$x_n = x_m$ for all $n, m > N$;*
2. *x_n, x are distinct points in X for all $n > N$;*
3. *x_n, y are distinct points in X for all $n > N$;*
4. *$x_n \to x$ and $x_n \to y$ as $n \to \infty$.*

Then $x = y$.

3. Main Results

In this section, we prove Kannan - type contraction principle in cone pentagonal metric spaces for two self mappings. We give an example to illustrate the result.

Theorem 3.1. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $f, g : X \to X$ satisfy the contractive condition:*

$$d(fx, fy) \leq \lambda\big(d(gx, fx) + d(gy, fy)\big), \tag{3.1}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \subseteq g(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then the mappings f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_n = gx_{n+1} \text{ for all } n = 0, 1, 2, \cdots .$$

Now, we define a sequence $\{y_n\}$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n = 0, 1, 2, \cdots$. If $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then $y_k = fx_{k+1} = gx_{k+1}$. That is, f and g have a point of coincidence y_k in X. We assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Then, from (3.1), we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\leq \lambda\big(d(gx_n, fx_n) + d(gx_{n+1}, fx_{n+1})\big) \\ &= \lambda\big(d(y_{n-1}, y_n) + d(y_n, y_{n+1})\big). \end{aligned}$$

So that,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(y_{n-1}, y_n) \\ &\leq r d(y_{n-1}, y_n), \text{ where } r = \frac{\lambda}{1-\lambda} \in [0, 1) \\ &\leq r^2 d(y_{n-2}, y_{n-1}) \\ &\vdots \\ &\leq r^n\big(d(y_0, y_1)\big), \quad \forall n \geq 1. \end{aligned} \tag{3.2}$$

Also from (3.1) and (3.2), we obtain

$$\begin{aligned} d(y_n, y_{n+2}) &= d(fx_n, fx_{n+2}) \\ &\leq \lambda\big(d(gx_n, fx_n) + d(gx_{n+2}, fx_{n+2})\big) \\ &\leq \lambda\big(d(y_{n-1}, y_n) + d(y_{n+1}, y_{n+2})\big) \\ &\leq \lambda\big(r^{n-1} d(y_0, y_1) + r^{n+1} d(y_0, y_1)\big) \\ &\leq \lambda r^{n-1}(1 + r^2) d(y_0, y_1). \end{aligned}$$

That is,

$$d(y_n, y_{n+2}) \leq \alpha r^{n-1} d(y_0, y_1), \quad \forall n \geq 1, \tag{3.3}$$

where $\alpha = \lambda(1+r^2) > 0$.
For the sequence $\{y_n\}$, we consider $d(y_n, y_{n+p})$ in two cases as follows:
If p is odd say $p = 2m+1$, where $m \geq 1$, then by pentagonal property and (3.2), we have

$$\begin{aligned} d(y_n, y_{n+2m+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+2m+1}) \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots \\ &\quad + d(y_{n+2m-1}, y_{n+2m}) + d(y_{n+2m}, y_{n+2m+1}) \\ &\leq r^n d(y_0, y_1) + r^{n+1} d(y_0, y_1) + r^{n+2} d(y_0, y_1) + \cdots \\ &\quad + r^{n+2m-1} d(y_0, y_1) + r^{n+2m} d(y_0, y_1) \\ &\leq \frac{r^n}{1-r} d(y_0, y_1), \quad \forall n \geq 1. \end{aligned}$$

If p is even say $p = 2m$, where $m \geq 2$, then by pentagonal property, (3.2) and (3.3), we have

$$\begin{aligned} d(y_n, y_{n+2m}) &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+2m}) \\ &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + \cdots \\ &\quad + d(y_{n+2m-2}, y_{n+2m-1}) + d(y_{n+2m-1}, y_{n+2m}) \\ &\leq \alpha r^{n-1} d(y_0, y_1) + r^{n+2} d(y_0, y_1) + r^{n+3} d(y_0, y_1) + \cdots \\ &\quad + r^{n+2m-2} d(y_0, y_1) + r^{n+2m-1} d(y_0, y_1) \\ &\leq \alpha r^{n-1} d(y_0, y_1) + \frac{r^n}{1-r} d(y_0, y_1). \end{aligned}$$

Since $r \in [0,1)$, we get, as $n \to \infty$, $\frac{r^n}{1-r} \to 0$ and $\alpha r^{n-1} \to 0$. Hence, for every $c \in E$ with $c \gg 0$, $\exists n_0 \in \mathbb{N}$ such that

$$d(y_n, y_{n+p}) \ll c, \text{ for all } n \geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in (X,d). Since $g(X)$ is a complete subspace of X, there exists a points $u, v \in g(X)$ such that $\lim_{n\to\infty} y_n = v = gu$.

Now, we show that $gu = fu$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(v, y_n) \ll \frac{c(1-\lambda)}{3}$, $\forall n \geq M_1$, $d(y_n, y_{n+1}) \ll \frac{c(1-\lambda)}{3}$, $\forall n \geq M_2$ and $d(y_{n+1}, y_{n+2}) \ll \frac{c(1-\lambda)}{3(1+\lambda)}$, $\forall n \geq M_3$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$\begin{aligned} d(gu, fu) &\leq d(gu, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, fu) \\ &\leq d(v, y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(fx_{n+2}, fu) \end{aligned}$$

$$\begin{aligned}
&\leq d(v,y_n)+d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})\\
&\quad+\lambda\big(d(gu,fu)+d(gx_{n+2},fx_{n+2})\big)\\
&\leq d(v,y_n)+d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})\\
&\quad+\lambda\big(d(gu,fu)+d(y_{n+1},y_{n+2})\big)\\
d(gu,fu)&\leq \frac{1}{1-\lambda}\big(d(v,y_n)+d(y_n,y_{n+1})+(1+\lambda)d(y_{n+1},y_{n+2})\big)\\
&\ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3}=c,\ \text{for all } n\geq M,
\end{aligned}$$

where $M := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(gu, fu) \ll \frac{c}{m}$, $\forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gu, fu) \to -d(gu, fu)$ as $m \to \infty$. Since P is closed, $-d(gu, fu) \in P$. Hence $d(gu, fu) \in P \cap -P$. By definition of cone we get that $d(gu, fu) = 0$, and so $gu = fu = v$. Hence, v is a point of coincidence of f and g. Similarly, if $f(X)$ is a complete subspace of X the result holds.

Next, we show that v is unique. For suppose v' be another point of coincidence of f and g, that is $gu' = fu' = v'$, for some $u' \in X$, then

$$d(v,v')=d(fu,fu')\leq\lambda\big(d(gu,fu)+d(gu',fu')\big)\leq\lambda\big(d(v,v)+d(v',v')\big).$$

Hence, $v = v'$. Since (f, g) is weakly compatible, by Lemma 2.7, v is the unique common fixed point of f and g. This completes the proof of the theorem. □

To illustrate Theorem 3.1, we give the following example.

Example 3.2. Let $X = \{a, b, c, d, e\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $\rho : X \times X \to E$ as follows:

$$\begin{aligned}
\rho(x,x)&=0,\forall x\in X;\\
\rho(a,b)&=\rho(b,a)=(4,16);\\
\rho(a,c)&=\rho(c,a)=\rho(c,d)=\rho(d,c)=\rho(b,c)=\rho(c,b)\\
&=\rho(b,d)=\rho(d,b)=\rho(a,d)=\rho(d,a)=(1,4);\\
\rho(a,e)&=\rho(e,a)=\rho(b,e)=\rho(e,b)=\rho(c,e)\\
&=\rho(e,c)=\rho(d,e)=\rho(e,d)=(5,20).
\end{aligned}$$

Then (X, ρ) is a complete cone pentagonal metric space, but (X, ρ) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$(4,16)=\rho(a,b)>\rho(a,c)+\rho(c,d)+\rho(d,b)$$

$$= (1,4) + (1,4) + (1,4)$$
$$= (3,12), \text{ as } (4,16) - (3,12) = (1,4) \in P.$$

Define a mapping $f, g : X \to X$ as follows:

$$f(x) = \begin{cases} d, & \text{if } x \neq e; \\ b, & \text{if } x = e. \end{cases}$$

$$g(x) = \begin{cases} c, & \text{if } x = a; \\ a, & \text{if } x = b; \\ b, & \text{if } x = c; \\ d, & \text{if } x = d; \\ e, & \text{if } x = e. \end{cases}$$

Clearly $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also f and g are weakly compatible mappings. Thus, the conditions of Theorem 3.1 holds for all $x, y \in X$, where $\lambda = \frac{1}{5}$ and $d \in X$ is the unique common fixed point of the mappings f and g.

Corollary 3.3. (see [2]) *Let (X, d) be a complete cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big(d(x, Sx) + d(y, Sy)\big), \tag{3.4}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. *S has a unique fixed point in X.*
2. *For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.*

Proof. Take $g = I$ and P be a normal cone in Theorem 3.1. This completes the proof. □

Corollary 3.4. (see [11]) *Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big(d(gx, Sx) + d(gy, Sy)\big),$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \subseteq g(X)$ and $S(X)$ or $g(X)$ is a complete subspace of X, then the mappings S and g have a unique coincidence point in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. This follows from Remark 2.6 and Theorem 3.1, where P is a normal cone. □

Corollary 3.5. (see [8]) *Let (X, d) be a complete cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big(d(x, Sx) + d(y, Sy)\big), \tag{3.5}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. *S has a unique fixed point in X.*

2. *For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.*

Proof. Take $g = I$ and P be a normal cone in Theorem 3.1 and Remark 2.6. This completes the proof. □

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