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ON A PROBLEM OF MINIMAL NON-FC-GROUPS

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Abstract: In this paper Problem 17.13 by A.O. Asar in The Kourovka Notebook is studied which is 'Let G be a totally imprimitive $p-group$ of finitary permutations on an infinite set. Suppose that the support of any cycle in the cyclic decomposition of every element of G is a block for G. Does G necessarily contain a $minimal\ non-FC-subgroup$?' and an example of a group G satisfying these conditions but not having a $minimal\ non-FC-subgroup$ is given.

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1. Introduction

Let Ω be an infinite set and Ω_1,Ω_2,... be subsets of Ω such that $\bigcup_{i\in I}\Omega_i$, $\Omega_i\bigcap(\bigcup_{i\neq j}\Omega_j) = \emptyset$ and $|\Omega_i| = 3$ for all i. Consider the group G which is generated by $3-cycles$ of Ω_i's. Clearly G is a finitary permutation $3-group$. Let us denote the $3-cycle$ of Ω_i by σ_i. Namely if $\Omega_i = \{\alpha_{i1}, \alpha_{i2}, \alpha_{i3}\}$ then $\sigma_i = (\alpha_{i1}\alpha_{i2}\alpha_{i3})$ or $\sigma_i = (\alpha_{i1}\alpha_{i3}\alpha_{i2})$,it is not important which one you decide to be σ_i. For any pair of permutations $\sigma, \tau \in G$, σ and τ are product of finitely many distinct $3-cycles$. So $G \leq FSym(\Omega)$.

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2. Preliminaries

Some definitions need to be given for this study. Firstly a *support* of a permutation x is defined as $supp(x) = \{\alpha \in \Omega | \alpha^x \neq \alpha\}$. A group G is said to be *finitary permutation group* if every element of G has finite support. A subset Δ of Ω is called a *block* for a group G if $\Delta^x = \Delta$ or $\Delta^x \cap \Delta = \emptyset$ for all $x \in G$. Thus Ω and $\{\alpha\}$, for every $\alpha \in \Omega$, are blocks for G. These blocks are called *proper blocks*. If G has no blocks except proper blocks G is called *primitive*, otherwise G is called *imprimitive group*. For an imprimitive group G, if there exists an infinite strictly ascending sequence of finite blocks, G is said to be *totally imprimitive group*.[2] A group G is called an $FC-group$ if every element of G has finitely many conjugates in G and a group G is called $minimal\ non-FC-group$ if G is not an $FC-group$ but which each proper subgroup is an $FC-group$.[3].

3. On Solution

Theorem 1. *Support of any cycle in the cyclic decomposition of every element of G which is constructed above, is a block for G.*

Proof. Let any pair of permutations $\sigma, \tau \in G$ are written in cyclic decomposition as follows: $\sigma = \sigma_1 ... \sigma_n$ and $\tau = \tau_1 ... \tau_m$. Of course σ_i and τ_j are $3-cycles$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$. To show that $supp(\sigma_i)$ is a block for G we need to analyze that in two possible cases: Case 1: $supp(\sigma_i) \bigcap supp(\tau_j) = \emptyset$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In this case since any element of $supp(\sigma_i)$ does not appear in any cycle of τ, $(supp(\sigma_i))^\tau = supp(\sigma_i)$ for every $1 \leq i \leq n$. Case 2: At least one cycle contains same elements both in σ and τ (at least one cycle in σ and τ has same support). It is enough to prove it for one cycle. Let σ_i and τ_j have same support in σ and τ . Hence every element of $supp(\sigma_i)$ is moved by τ, especially by τ_j. But $supp(\sigma_i)$ is fixed setwise by τ. So $(supp(\sigma_i))^\tau = supp(\sigma_i)$. Finally support of any cycle in the cyclic decomposition of every element of G , is a block for G. □

Also we now see that for all cases, support of any cycle in the cyclic decomposition of every element of G, is fixed setwise.

Theorem 2. *Any union of finitely many Ω_i is block for G.*

Proof. We concluded that every permutation in G fixes the sets Ω_i, for all i. Since $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $(\Omega_i \cup \Omega_j)^\sigma = \Omega_i^\sigma \cup \Omega_j^\sigma = \Omega_i \cup \Omega_j$, for all $\sigma \in G$.

So $\Omega_i \cup \Omega_j$ is a block for G. □

Corollary 3. *By adding different Ω_i to previous term we obtain an infinite strictly ascending sequence of finite blocks $\Omega_1 \subset \Omega_1 \cup \Omega_2 \subset ...$ of G. So G is a totally imprimitive group.*

Proof. One can easily see that it is a conclusion of Proposition 3.2 . □

Theorem 4. *The group G, which is defined above is an $FC - group$.*

Proof. Let $\sigma, \tau \in G$, $\sigma = \sigma_1...\sigma_n$ and $\tau = \tau_1...\tau_m$ in cyclic decomposition. We will prove it in similar way with proof of Proposition 3.1. Case 1: $supp(\sigma_i) \cap supp(\tau_j) = \emptyset$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. In this case $\sigma^\tau = \sigma$ for all $\sigma, \tau \in G$. Case 2: At least one cycle contains same elements both in σ and τ (at least one cycle in σ and τ has same support). We will prove it for one cycle. Let σ_i and τ_j have same support in σ and τ. For simplicity we can write $\sigma = \sigma_i\sigma_1...\sigma_n$ and $\tau = \tau_1...\tau_m\tau_j$. Then we have $\sigma^\tau = \tau\sigma\tau^{-1} = \tau_1...\tau_m\tau_j\sigma_i\sigma_1...\sigma_n\tau_j^{-1}\tau_m^{-1}...\tau_1^{-1}$. Since τ_k and σ_t are distinct $3 - cycles$ for $k \neq j$ and $t \neq i$ we obtain $\sigma^\tau = \tau_j\sigma_i\sigma_1...\sigma_n\tau_j^{-1} = \tau_j\sigma_i\tau_j^{-1}\sigma_1...\sigma_n$. Now σ_i and τ_j are $3 - cycles$ we have two subcases: Case 2.1: $\sigma_i = \tau_j$. In this case $\sigma^\tau = \tau_j\sigma_i\tau_j^{-1}\sigma_1...\sigma_n = \sigma$, Case 2.2: $\sigma_i^{-1} = \tau_j$. In this case $\sigma^\tau = \tau_j\sigma_i\tau_j^{-1}\sigma_1...\sigma_n = \sigma$. Anyone can easily see that $\sigma^\tau = \sigma$ for any pair of permutations $\sigma, \tau \in G$ which have one than more cycle contain same elements both in σ and τ by induction. So G is an $FC - group$. □

Corollary 5. *G does not have a minimal non-FC-Subgroup*

Proof. Every subgroup of an $FC - group$ is an $FC - group$ by [1]. Hence G does not have a $minimal\ non - FC - subgroup$. □

Someone can ask question what if $|\Omega_i| = p$, for some prime p. In this case G would not be a $p - group$. For example if $p = 5$, though two $5 - cycles$ $\sigma = (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)$ and $\tau = (\alpha_1\alpha_2\alpha_4\alpha_3\alpha_5)$ are belong to G, $\sigma\tau = (\alpha_1\alpha_3)(\alpha_2\alpha_5)$ is not a $5 - element$ so G is not a $5 - group$.

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