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SOME FIXED POINT RESULTS FOR T-WEAK CONTRACTIONS IN GENERALIZED CONE b-METRIC SPACE

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Abstract: In this paper, we prove some fixed point theorems of T-weak contractions in generalized cone b-metric spaces established by R. George, et al. in [15], without assuming normality of cone. Our results improve and generalize many recent known results in cone metric, cone b-metric and cone rectangular metric spaces.

AMS Subject Classification: 47H10, 54H25
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1. Introduction and Preliminaries

The well known fixed point theorem is Banach contraction principle [3] in metric space, which has been used, generalized and improved in many types. The well known generalizations of metric space are Rectangular metric space [4], b-metric space [5], cone metric space [6], cone cone b-metric space [7] and rectangular metric space [2]. Recently, R. George, et al. [15] introduced the concept of

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generalized cone b-metric space which generalizes the concepts of metric space, b-metric space, cone metric space, cone b-metric space and cone rectangular metric space. They have also proved an analogue of the Banach contraction principle and Kannan type fixed point theorem in generalized cone b-metric space.

In this paper we have proved some common fixed point results for T-weak contractions in generalized cone b-metric spaces. Our results extend and improve the results of Huang and Zhang [6], Vetro [16], Azam et al. [2], Malhotra et al. [10, 11, 12], Jleli et al. [8] and Reny George et al. [15] on generalized cone b-metric space.

We need the following definitions and results.

Definition 1.1. (see [6]) A subset P of a real Banach space E is called a *cone*, if it has following properties:

1. P is non empty, closed and $P \neq \{\theta\}$;
2. $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;
3. $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$, if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in int(P)$, where $int(P)$ denotes the interior of P and θ is a zero vector in E. A cone P is called *solid* if $int(P) \neq \emptyset$.

A cone P is called *normal* if there is a number $k > 1$ such that for all $x, y \in X$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. The least positive number k satisfying this condition is called the *normal constant* of P.

Remark 1.2 (see [9]) Let E be an ordered Banach space with a positive cone P and $a, b, v \in P$. The following properties hold:

1. If $a \preceq b$ and $b \ll c$, then $a \ll c$;
2. If $\theta \preceq u \ll c$ for each $c \in int(P)$, then $u = \theta$;
3. If $a \preceq b + c$ for each $c \in int(P)$, then $a \preceq b$;
4. If $\theta \preceq x \preceq y$, and $a \geq 0$, then $\theta \preceq ax \preceq ay$;
5. If $\theta \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $\theta \preceq x \preceq y$;

6. If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $d(x_n, x) \ll c$, where x_n and x are a sequence and a given point in X, respectively;

7. If $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$;

8. If $c \in int(P)$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll \quad c$.

Definition 1.3. (see [6]) Let X be a non empty set, E be a real Banach space, P be a solid cone in E and $\preceq$ be a partial ordering with respect to P.

Suppose that the map $d : X \times X \to E$ satisfies:

1. $\theta \prec d(x, y)$, for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$, if and only if $x = y$;

2. $d(x, y) = d(y, x)$, for all $x, y \in X$;

3. $d(x, y)\, d(x, z) + d(z, y)$, for all $x, y, z \in X$ (triangular inequality).

Then d is called a *cone metrics* on X and (X, d) is called a *cone metric space.*

Definition 1.4. (see [2]) Let X be a non empty set, E be a real Banach space, P be a solid cone in E, and $\preceq$ be a partial ordering with respect to P. Let the map $d : X \times X \to E$ be such that for all $x, y \in X$ and for all distinct points $w, z \in X$ (distinct from x and y):

1. $\theta \prec d(x, y)$, for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$, if and only if $x = y$;

2. $d(x, y) = d(y, x)$, for all $x, y \in X$;

3. $d(x, y)\, d(x, w) + d(w, z) + d(z, y)$ (rectangular inequality).

Then d is called a *cone rectangular metric* on X and (X, d) is called a *cone rectangular metric space.*

Definition 1.5. (see [15]) Let X be a nonempty set, E be a real Banach space, P be a solid cone in E, $\preceq$ be a partial ordering with respect to P and $s \geq 1$ be a real number. Let the map $d : X \times X \to E$ be such that for all $x, y \in X$ and for all distinct points $w, z \in X$ (distinct from x and y):

1. $\theta \prec d(x, y)$, for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

2. $d(x,y) = d(y,x)$, for all $x, y \in X$;

3. $d(x,y)\, s\,[d(x,w) + d(w,z) + d(z,y)]$ (b-rectangular inequality).

Then d is called a *generalized cone b-metric* on X and (X,d) is called a *generalized cone b-metric space.*

Definition 1.6. (see [5]) Let (X,d) be a generalized cone b-metric space with $s \geq 1$. The sequence $\{xn\}$ in X is said to be:

1. a *convergent sequence* if for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(xn, x) \ll c$ for some $x \in X$. We say that the sequence $\{xn\}$ converges to x and we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$.

2. a *Cauchy sequence* if for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(xn, xm) \ll c$.

3. The generalized cone b-metric space (X,d) is said to be *complete* if every Cauchy sequence is convergent in X.

Definition 1.7. (see [12, 14]) A self map $S : X \to X$ on a metric space (X,d) is called *Reich contraction*, if for all $x, y \in X$, there exists $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ such that,

$$d(Sx, Sy) \preceq \lambda_1 d(x,y) + \lambda_2 d(x, Sx) + \lambda_3 d(y, Sy).$$

Definition 1.8. (see [10, 16]) A self map $S : X \to X$ on a metric space (X,d) is called T-*weak contraction* if for all $x, y \in X$, there exists $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ such that,

$$d(Sx, Sy) \preceq \lambda_1 d(Tx, Ty) + \lambda_2 d(Tx, TSx) + \lambda_3 d(Ty, TSy).$$

2. Main Results

Theorem 2.1. *Let (X,d) be a generalized cone b-metric space with $s > 1$, P be a solid cone and let the maps $S, T : X \to X$ such that $S(X) \subset T(X)$ and $T(X)$ is complete. Suppose that the following condition satisfies:*

$$d(Sx, Sy) \preceq \lambda_1 d(Tx, Ty) + \lambda_2 d(Tx, TSx) + \lambda_3 d(Ty, TSy) \tag{2.1}$$

for all $x, y \in X$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1/s$, then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible then S and T have a unique common fixed point in X.

Proof. Let x_0 be any arbitrary point of X. Since, $S(X) \subset T(X)$ starting with x_0 we define the sequence $\{yn\}$ such that,

$$y_n = Sx_n = Tx_{n+1}, \quad \forall\, n = 0, 1, 2, \ldots\,. \tag{2.2}$$

If $y_n = y_{n+1}$, then $y_n = Sx_{n+1} = Tx_{n+1}$, that is y_n is a point of coincidence of S and T with coincidence point x_{n+1}.

Assume that, $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Then from (2.1), it follows that

$$\begin{aligned} d(y_n, y_{n+1}) =& d(Sx_n, Sx_{n+1}) \\ \preceq& \lambda_1 d(Tx_n, Tx_{n+1}) + \lambda_2 d(Tx_n, Sx_n) + \lambda_3 d(Tx_{n+1}, Sx_{n+1}) \\ \preceq& \lambda_1 d(y_{n-1}, y_n) + \lambda_2 d(y_{n-1}, y_n) + \lambda_3 d(y_n, y_{n+1}) \\ =& (\lambda_1 + \lambda_2) d(y_{n-1}, y_n) + \lambda_3 d(y_n, y_{n+1}) \end{aligned}$$

which implies that

$$d(y_n, y_{n+1}) \lambda d(y_{n-1}, y_n), \forall n \in \mathbb{N},$$

where

$$\lambda = \frac{\lambda_1 + \lambda_2}{1 - \lambda_3}.$$

It is easy to see that, $0 < \lambda < 1/s$.

By repeating this process, we obtain

$$d(y_n, y_{n+1}) \preceq \lambda d(y_{n-1}, y_n) \preceq \lambda^2 d(y_{n-2}, y_{n-1}) \preceq \cdots \preceq \lambda^n d(y_0, y_1), \quad \forall n \in \mathbb{N}, \tag{2.3}$$

where $\lambda < 1/s$.

Using b-rectangular inequality, (2.1), (2.3) and the fact that $\lambda_1 < 1/s$, we get,

$$\begin{aligned} d(y_n, y_{n+2}) =& d(Sx_n, Sx_{n+2}) \\ \preceq& \lambda_1 d(Tx_n, Tx_{n+2}) + \lambda_2 d(Tx_n, Sx_n) + \lambda_3 d(Tx_{n+2}, Sx_{n+2}) \\ =& \lambda_1 d(y_{n-1}, y_{n+1}) + \lambda_2 d(y_{n-1}, y_n) + \lambda_3 d(y_{n+1}, y_{n+2}) \\ \preceq& \lambda_1 s\left[d(y_{n-1}, y_n) + d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+1})\right] \\ &+ \lambda_2 d(y_{n-1}, y_n) + \lambda_3 d(y_{n+1}, y_{n+2}), \end{aligned}$$

which implies that,

$$
\begin{aligned}
d\left(y_n,\, y_{n+2}\right) &\preceq \left[\frac{s\lambda_1+\lambda_2}{1-s\lambda_1}\right] d\left(y_{n-1},\, y_n\right) + \left[\frac{s\lambda_1+\lambda_3}{1-s\lambda_1}\right] d\left(y_{n+1},\, y_{n+2}\right)\\
&\preceq \left[\frac{s\lambda_1+\lambda_2}{1-s\lambda_1}\right] \lambda^{n-1} d\left(y_0, y_1\right) + \left[\frac{s\lambda_1+\lambda_3}{1-s\lambda_1}\right] \lambda^{n} d\left(y_0, y_1\right)\\
&\preceq \left[\frac{s\lambda_1+\lambda_2}{1-s\lambda_1} + \frac{s\lambda_1+\lambda_3}{1-s\lambda_1}\lambda\right] \lambda^{n-1} d\left(y_0, y_1\right)\\
&\preceq \left[\frac{s\lambda_1+\lambda_2}{1-s\lambda_1} + \frac{s\lambda_1+\lambda_3}{1-s\lambda_1}\right] \lambda^{n-1} d\left(y_0, y_1\right)\\
&= \left[\frac{2s\lambda_1+\lambda_2+\lambda_3}{1-s\lambda_1}\right] \lambda^{n-1} d\left(y_0, y_1\right)
\end{aligned}
$$

Hence

$$d\left(y_n,\, y_{n+2}\right) \preceq \alpha\lambda^{n-1} d\left(y_0, y_1\right), \tag{2.4}$$

for all $n \in \mathbb{N}$, where $\alpha = \frac{2s\lambda_1+\lambda_2+\lambda_3}{1-s\lambda_1} \geq 0$.

For the sequence $\{yn\}$ we will separate $d(yn, yn+p)$ in to two cases.

Suppose p is odd say $2m+1$, for $m \geq 1$, then by using b-rectangular inequality, (2.2), (2.3) and the fact that $s\lambda^2 < 1$, we get

$$
\begin{aligned}
d(y_n,&y_{n+2m+1}) \preceq s\left[d\left(y_n, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right) + d\left(y_{n+2}, y_{n+2m+1}\right)\right]\\
&\preceq s\left[d\left(y_n, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right)\right] + s^2\left[d\left(y_{n+2}, y_{n+3}\right) + d\left(y_{n+3}, y_{n+4}\right)\right.\\
&\quad \left. + d\left(y_{n+4}, y_{n+2m-1}\right)\right]\\
&\preceq s\left[d\left(y_n, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right)\right] + s^2\left[d\left(y_{n+2}, y_{n+3}\right) + d\left(y_{n+3}, y_{n+4}\right)\right]\\
&\quad + \cdots + s^m d\left(y_{n+2m}, y_{n+2m+1}\right)\\
&\preceq s\left[\lambda^n d\left(y_0, y_1\right) + \lambda^{n+1} d\left(y_0, y_1\right)\right] + s^2\left[\lambda^{n+2} d\left(y_0, y_1\right) + \lambda^{n+3} d\left(y_0, y_1\right)\right]\\
&\quad + \cdots + s^m \lambda^{n+2m} d\left(y_0, y_1\right)]\\
&\preceq s\lambda^n\left[1 + s\lambda^2 + \cdots\right] d\left(y_0, y_1\right) + s\lambda^{n+1}\left[1 + s\lambda^2 + \cdots\right] d\left(y_0, y_1\right)\\
&= (1+\lambda)\, s\lambda^n\left[1 + s\lambda^2 + \cdots\right] d\left(y_0, y_1\right).
\end{aligned}
$$

Hence

$$d\left(y_n, y_{n+2m+1}\right) \left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(y_0, y_1\right),$$

for all $n \in \mathbb{N}$.

Let $\theta \ll c$ be given. Since, $s\lambda^2 < 1$, let us note that

$$\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(y_0, y_1\right) \to \theta, \quad \text{as } n \to \infty.$$

By Remark 1.2, for any $c \in int(P)$, we find $N_1 \in \mathbb{N}$ such that for each $n > N_1$, we have

$$\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(y_0, y_1\right) \ll c.$$

Thus,

$$d\left(y_n, y_{n+2m+1}\right)\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(y_0, y_1\right) \ll c,$$

for all $n > N_1$ and $m \geq 1$.

Suppose p is even, say 2m, for $m \geq 1$. Then by the using of b-rectangular property, (2.2), (2.3) and (2.4) we obtain

$$\begin{aligned}
d(y_n, y_{n+2m}) &\preceq s\left[d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+d\left(y_{n+2}, y_{n+2m}\right)\right] \\
&\preceq s\left[d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)\right]+s^2\left[d\left(y_{n+2}, y_{n+3}\right)+d\left(y_{n+3}, y_{n+4}\right)\right. \\
&\quad \left.+d\left(y_{n+4}, y_{n+2m}\right)\right] \\
&\preceq s\left[d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)\right]+s^2\left[d\left(y_{n+2}, y_{n+3}\right)+d\left(y_{n+3}, y_{n+4}\right)\right] \\
&\quad +\cdots+s^{m-1}\left[d\left(y_{n+2m-4}, y_{n+2m-3}\right)+d\left(y_{n+2m-3}, y_{n+2m-2}\right)\right. \\
&\quad \left.+d\left(y_{n+2m-2}, y_{n+2m}\right)\right] \\
&\preceq s\left[d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)\right]+s^2\left[d\left(y_{n+2}, y_{n+3}\right)+d\left(y_{n+3}, y_{n+4}\right)\right] \\
&\quad +\cdots+s^{m-1}\left[d\left(y_{n+2m-4}, y_{n+2m-3}\right)+d\left(y_{n+2m-3}, y_{n+2m-2}\right)\right] \\
&\quad + s^{m-1} d\left(y_{n+2m-2}, y_{n+2m}\right) \\
&\preceq s\left[\lambda^n d\left(y_0, y_1\right)+\lambda^{n+1} d\left(y_0, y_1\right)\right]+s^2\left[\lambda^{n+2} d\left(y_0, y_1\right)+\lambda^{n+3} d\left(y_0, y_1\right)\right] \\
&\quad +\cdots+s^{m-1}\left[\lambda^{n+2m-4} d\left(y_0, y_1\right)+\lambda^{n+2m-3} d\left(y_0, y_1\right)\right] \\
&\quad + s^{m-1}\alpha\lambda^{n+2m-3} d\left(y_0, y_1\right) \\
&\preceq s\lambda^n\left[1+s\lambda^2+\cdots\right] d\left(y_0, y_1\right)+s\lambda^{n+1}\left[1+s\lambda^2+\cdots\right] d\left(y_0, y_1\right) \\
&\quad + s^{m-1}\alpha\lambda^{n+2m-3} d\left(y_0, y_1\right) \\
&= (1+\lambda) s\lambda^n\left[1+s\lambda^2+\cdots\right] d\left(y_0, y_1\right)+s^{m-1}\alpha\lambda^{n+2m-3} d\left(y_0, y_1\right).
\end{aligned}$$

That is,

$$d(y_n, y_{n+2m})\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(y_0, y_1\right)+s^{m-1}\alpha\lambda^{n+2m-3} d\left(y_0, y_1\right)$$
$$\left(s\frac{1+\lambda}{1-s\lambda^2}+s^{m-1}\alpha\lambda^{2m-3}\right)\lambda^n d\left(y_0, y_1\right),$$

for all $n \in \mathbb{N}$, where $\alpha \geq 0$.

Let $\theta \ll cbegiven.$
Since, $s\lambda^2 < 1$, we notice that

$$\left(s\frac{1+\lambda}{1-s\lambda^2} + s^{m-1}\alpha\lambda^{2m-3}\right)\lambda^n d\left(y_0, y_1\right) \to \theta, \quad \text{as } n \to \infty.$$

By Remark 1.2 for any $c \in int(P)$, we find $N_2 \in \mathbb{N}$ such that

$$\left(s\frac{1+\lambda}{1-s\lambda^2} + s^{m-1}\alpha\lambda^{2m-3}\right)\lambda^n d\left(y_0, y_1\right) \ll c,$$

for all $n > N_1$ and $m \geq 1$. Thus,

$$d\left(y_n, y_{n+2m}\right)\left(s\frac{1+\lambda}{1-s\lambda^2} + s^{m-1}\alpha\lambda^{2m-3}\right)\lambda^n d\left(y_0, y_1\right) \ll c,$$

for all $n > N_1$ and $m \geq 1$.

Let

$$N_0 = \max\{N1, N2\}.$$

Thus for each $c \in int(P)$ we have,

$$d\left(y_n, y_{n+p}\right) \ll c, \tag{2.5}$$

for all $n > N_0$ and $p \geq 1$. Therefore, $\{yn\}$ is a Cauchy sequence in X. Suppose, $T(X)$ is a complete subspace of X, there exists $z \in T(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{n+1} = z. \tag{2.6}$$

Consequently, we can find $x \in X$ such that

$$z = Tx. \tag{2.7}$$

Using b-rectangular inequality, (2.1), (2.2), (2.7) and the fact that $s\lambda 2 < 1$, we receive

$$\begin{aligned} d\left(Sx,\, z\right) \preceq & s\left[d\left(Sx,\, y_n\right) + d\left(y_n,\, y_{n+1}\right) + d\left(y_{n+1},\, z\right)\right] \\ = & s\left[d\left(Sx,\, Sx_n\right) + d\left(y_n,\, y_{n+1}\right) + d\left(y_{n+1},\, z\right)\right] \\ \preceq & s\left[\lambda_1 d\left(Tx,\, Tx_n\right) + \lambda_2 d\left(Tx,\, Sx\right) + \lambda_3 d\left(Tx_n,\, Sx_n\right)\right] \\ & + sd\left(y_n,\, y_{n+1}\right) + sd\left(y_{n+1},\, z\right) \\ = & s\lambda_1 d\left(z,\, y_{n-1}\right) + s\lambda_2 d\left(z,\, Sx\right) + s\lambda_3 d\left(y_{n-1},\, y_n\right) \\ & + sd\left(y_n,\, y_{n+1}\right) + sd\left(y_{n+1},\, z\right). \end{aligned}$$

which implies

$$d\left(Sx,\, z\right) \frac{1}{1-s\lambda_2}\left[s\lambda_1 d\left(z,\, y_{n-1}\right)+s\lambda_3 d\left(y_{n-1},\, y_n\right)+sd\left(y_n,\, y_{n+1}\right)+sd\left(y_{n+1},\, z\right)\right],$$

for all $n \in \mathbb{N}$.

It follows from (2.5) and (2.6) that, $d(Sx, z) = \theta$, i.e., $Sx = z$.

Hence,

$$Sx = Tx = z. \tag{2.8}$$

Thus z is a point of coincidence of S and T in X.

Suppose z^* is another point of coincidence of S and T. Then it follows from (2.1) that

$$\begin{aligned} d\left(z,\, z^*\right) =& d\left(Sx,\, Sx^*\right) \\ \preceq& \lambda_1 d\left(Tx,\, Tx^*\right)+\lambda_2 d\left(Tx,\, Sx\right)+\lambda_3 d\left(Tx^*,\, Sx^*\right) \\ =& \lambda_1 d\left(z, z^*\right)+\lambda_2 d\left(z, z\right)+\lambda_3 d\left(z^*, z^*\right) \\ =& \lambda_1 d\left(z, z^*\right) \\ \prec& \frac{1}{s} d\left(z, z^*\right), \end{aligned}$$

which is a contradiction.

Therefore we must have $z = z^*$. Hence, S and T have a unique point of coincidence in X.

Suppose that S and T are weakly compatible mappings. Then from (2.8) we have, $Sz = STx = TSx = Tz = w$ (say). That is w is another point of coincidence in X, therefore by the uniqueness of point of coincidence we must have $w = z$. Hence,

$$Sz = Tz = z,$$

i.e., z is a unique common fixed point in X. □

With the suitable value of λ_1, λ_2 and λ_3 we obtain the following generalizations of Theorem 3.1 and Theorem 3.3 of R. George, et al. [15] on generalized cone b-metric spaces.

Theorem 2.2 *Let (X, d) be a generalized cone b-metric space with $s > 1$, P be a solid cone and let the maps $S, T : X \to X$ be such that $S(X) \subset T(X)$ and $T(X)$ is complete. Suppose that the following condition satisfies:*

$$d(Sx, Sy) \preceq \lambda d(Tx, Ty),$$

for all $x, y \in X$, where $\lambda \in [0, 1/s)$.

Then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible then S and T have a unique common fixed point in X.

Theorem 2.3. *Let (X, d) be a generalized cone b-metric space with $s > 1$, P be a solid cone and let the maps $S, T : X \to X$ be such that $S(X) \subset T(X)$ and $T(X)$ be complete. Suppose that the following condition satisfies:*

$$d(Sx, Sy) \preceq \lambda \left[d(Tx, Sx) + d(Ty, Sy)\right],$$

for all $x, y \in X$, where $\lambda \in [0, 1/s + 1)$.

Then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible then S and T have a unique common fixed point in X.

Taking $S = I$ (the identity mapping of X) in Theorem 2.1, Theorem 2.2 and Theorem 2.3, we get an analogue of Reich contraction principle [13], Kannan contraction principle [15] and Banach contraction principle [15] on generalized cone b-metric spaces as follows.

Corollary 2.4. *Let (X, d) be a complete generalized cone b-metric space with $s > 1$, P be a solid cone and $S : X \to X$ be a map satisfying:*

$$d(Sx, Sy) \preceq \lambda_1 d(x, y) + \lambda_2 d(x, Sx) + \lambda_3 d(y, Sy),$$

for all $x, y \in X$, and $\lambda_1, \lambda_2, \lambda_3 \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1/s$.

Then S has a unique fixed point in X.

Corollary 2.5. *Let (X, d) be a complete generalized cone b-metric space with $s > 1$, P be a solid cone and $S : X \to X$ be map satisfying:*

$$d(Sx, Sy) \preceq \lambda[d(x, Sx) + d(y, Sy)],$$

for all $x, y \in X$, where $\lambda \in [0, 1/s + 1)$.

Then S has a unique fixed point in X.

Corollary 2.6. *Let (X, d) be a complete generalized cone b-metric space with $s > 1$, P be a solid cone and $S : X \to X$ be a map satisfying:*

$$d(Sx, Sy) \preceq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1/s)$.

Then S has a unique fixed point in X.

The following example support Theorem 2.1.

Example 2.7. Let $X = \{a, b, c, d\}$, where $a, b, c, d \in \mathbb{R}$, $E = \mathcal{M}_{n\times n}(\mathbb{R})$ be the space of all real matrices of order $n \geq 1$ and

$$P = \{M = (a_{ij})_{1\leq i,j\leq n} : a_{ij} \geq 0, \forall i, j\}$$

is a cone in E.

Define $d : X \times X \to E$ such that,

$$\begin{cases} d(x,y) = d(y,x), & \text{for all } x, y \in X; \\ d(x,x) = \mathbb{O}_{n\times n}, & \text{for all } x \in X; \\ d(a,b) = 0.1\, I_n; & \\ d(a,c) = d(b,c) = 0.01\, I_n; & \\ d(a,d) = d(b,d) = d(c,d) = 0.02\, I_n, & \end{cases}$$

where I_n is the identity matrix. In this case, (X, d) is not a cone metric space with respect to P since,

$$d(a,b) = 0.1I_n > d(a,c) + d(c,b) = 0.01I_n + 0.01I_n = 0.02I_n$$

and (X, d) is not a cone rectangular metric space with respect to P since,

$$d(a,b) = 0.1I_n > d(a,c) + d(c,d) + d(d,b) = 0.01I_n + 0.02I_n + 0.02I_n = 0.05I_n.$$

However, it is easy to see that (X, d) is a complete generalized cone b-metric space with coefficient $s = 2 > 1$.

Further, let S and $T : X \to X$ be two mapps defined by:

$$S(x) = \begin{cases} c, & \text{if } x \in \{a, b, c\}, \\ b, & \text{if } x = d; \end{cases}$$

and

$$T(x) = \begin{cases} c, & \text{if } x \in \{a, c\}, \\ b, & \text{if } x = b, \\ d, & \text{if } x = d. \end{cases}$$

It is clear that, $S(X) \subset T(X)$ and the pair (S, T) is weakly compatible. Moreover, we have S and T satisfy the contraction condition (2.1) of Theorem 2.1 with $\lambda_1 = 1/8$, $\lambda_2 = 1/6$, $\lambda_3 = 1/8$ and hence $x = c$ is a unique common fixed point of the mappings S and T.

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