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# AN IMPLICIT VISCOSITY TECHNIQUE OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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**Abstract:** In this paper, we present a new viscosity technique of nonexpansive mappings in Hilbert spaces. The strong convergence theorems of the proposed technique is proved under certain assumptions imposed on the sequence of parameters. We also give applications of the proposed viscosity technique to a more general system of variational inequalities, the constrained convex minimization problem and the $K$-mapping.

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## 1. Introduction

In this paper, we shall take $H$ as a real Hilbert space, $\langle \cdot, \cdot \rangle$ as inner product, $\| \cdot \|$ as the induced norm, and $C$ as a nonempty closed subset of $H$.

**Definition 1.1.** Let $T : H \to H$ be a mapping. Then $T$ is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

**Definition 1.2.** A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta \|x - y\|.$$

**Definition 1.3.** $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in $C$, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following theorem gives the condition for a projection mapping to be nonexpansive.

**Theorem 1.4.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $P_c : H \to H$ a metric projection. Then:*

*(a) $\|P_c x - P_c y\|^2 \leq \langle x - y, P_c x - P_c y \rangle$ for all $x, y \in H$.*

*(b) $P_c$ is a nonexpansive mapping, that is, $\|x - P_c x\| \leq \|x - y\|$ for all $y \in C$.*

*(c) $\langle x - P_c x, y - P_c x \rangle \leq 0$ for all $x \in H$ and $y \in C$.*

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

**Theorem 1.5.** [2] (The demiclosedness principle) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T : C \to C$ such that $x_n \rightharpoonup x^* \in C$ and $(I - T)x_n \to 0$. Then $x^* = Tx^*$. (Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively).*

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

**Theorem 1.6.** [9] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad \forall n \geq 0,$$

*where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(2) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

*Then* $a_n \to 0$ *as* $n \to \infty$.

The following strong convergence theorem, which is also called the *viscosity approximation method* for nonexpansive mappings in real Hilbert spaces is given by Moudafi [6] in 2000.

**Theorem 1.7.** *Let* $C$ *be a nonempty closed convex subset of a real Hilbert space* $H$. *Let* $T$ *be a nonexpansive mapping of* $C$ *into itself such that* $F(T) := \{x \in H : T(x) = x\}$ *is nonempty. Let* $f$ *be a contraction of* $C$ *into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} T(x_n), \quad n \ge 0,$$

*where the sequence* $\{\epsilon_n\} \in (0,1)$ *satisfies:*

(1) $\lim\limits_{n\to\infty} \epsilon_n = 0$,

(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$,

(3) $\lim\limits_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

*Then* $\{x_n\}$ *converges strongly to a fixed point* $x^*$ *of the mapping* $T$, *which is also the unique solution of the variational inequality*

$$\langle (I-f)x, y-x \rangle \ge 0, \quad \forall \in F(T).$$

In 2015, Xu et al. [9] applied viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \ge 0.$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of $T$. Ke and Ma [5], motivated and inspired by the idea of Xu et al. [9], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(s_n x_n + (1-s_n) x_{n+1}\right),$$

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}).$$

Our contribution in this direction is the following new viscosity rule:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n T(w_n), \\ w_n = \frac{x_n + x_{n+1}}{2}. \end{cases} \tag{1.1}$$

## 2. The Main Result

**Theorem 2.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by the condition (1.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ *and* $\lim_{n\to\infty} \gamma_n = 1$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, (v) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

*Then $\{x_n\}$converges strongly to a fixed point $x^*$ of the mapping $T$, which is also the unique solution of the variational inequality*

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in F(T).$$

*In other words, $x^*$ is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.*

*Proof.* We divide the proof into the following four steps.

*Step 1.* Firstly, we show that $\{x_n\}$ is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|T(y_n)-p\| \\
&= \|T(\alpha_n(w_n)+\beta_n f(w_n)+\gamma_n T(w_n))-p\| \\
&\le \|\alpha_n(w_n)+\beta_n f(w_n)+\gamma_n T(w_n)-p\| \\
&= \|\alpha_n(w_n)-\alpha_n p+\beta_n f(w_n)-\beta_n p+\gamma_n T(w_n)+(\alpha_n+\beta_n-1)p\| \\
&\le \alpha_n\|(w_n)-p\|+\beta_n\|f(w_n)-p\|+\gamma_n\|T(w_n)-p\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\beta_n\|f(w_n)-f(p)\| \\
&\quad +\beta_n\|f(p)-p\|+\gamma_n\|w_n-p\| \\
&\le \frac{\alpha_n}{2}\|x_n-p\|+\frac{\alpha_n}{2}\|x_{n+1}-p\|+\theta\beta\|w_n-p\|+\beta\|f(p)-p\| \\
&\quad +\gamma_n\left(\frac{1}{2}\|x_n-p\|+\frac{1}{2}\|x_{n+1}-p\|\right) \\
&= \left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{\alpha_n+\gamma_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\| \\
&= \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\quad +\frac{\gamma_n}{2}\|x_{n+1}-p\|+\beta_n\|f(p)-p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left(1-\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_{n+1}-p\| \\
&\qquad\qquad \le \left(\frac{1-\beta_n+\theta\beta_n}{2}\right)\|x_n-p\|+\beta_n\|f(p)-p\|
\end{aligned}$$

implies

$$(1+\beta_n(1-\theta))\|x_{n+1}-p\| \le (1-\beta_n(1-\theta))\|x_n-p\|+2\beta_n\|f(p)-p\|. \quad (2.1)$$

Since $\beta_n, \theta \in (0,1)$, $1-\beta_n(1-\theta) \ge 0$. Moreover, by (2.1) and $\alpha_n+\beta_n+\gamma_n = 1$, we get

$$\begin{aligned}
\|x_{n+1}-p\| &\le \frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\|x_n-p\|+\frac{2\beta_n}{1+\beta_n(1-\theta)}\|f(p)-p\| \\
&\le \left[1-\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\right]\|x_n-p\|+\frac{2\beta_n(1-\theta)}{1+\beta_n(1-\theta)}\left(\frac{1}{1-\theta}\|f(p)-p\|\right).
\end{aligned}$$

Thus, we have

$$\|x_{n+1}-p\| \leq \max\left\{\|x_n-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.$$

By applying induction, we obtain

$$\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}.$$

Hence, we concluded that $\{x_n\}$ is bounded. Consequently, we deduce immediately from it that $\{f(w_n)\}$ and $\{T(w_n)\}$ are bounded.

*Step 2.* Now, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|T(y_n)-T(y_{n-1})\| \\
&= \|T\left(\alpha_n(w_n)+\beta_n f(w_n)+\gamma_n T(w_n)\right) \\
&\quad - T\left(\alpha_{n-1}(w_{n-1})+\beta_{n-1}f(w_{n-1})+\gamma_{n-1}T(w_{n-1})\right)\| \\
&\leq \|\alpha_n(w_n)+\beta_n f(w_n)+\gamma_n T(w_n) \\
&\quad - [\alpha_{n-1}(w_{n-1})+\beta_{n-1}f(w_{n-1})+\gamma_{n-1}T(w_{n-1})]\| \\
&= \Bigg\|\frac{\alpha_n}{2}(x_{n+1}-x_n)+\frac{\alpha_n}{2}(x_n-x_{n-1})+\frac{1}{2}(\alpha_n-\alpha_{n-1})x_n \\
&\quad + \frac{1}{2}(\alpha_n-\alpha_{n-1})x_{n-1}+\beta_n\left(f(w_n)-f(w_n)\right)+(\beta_n-\beta_{n-1})f(w_{n-1}) \\
&\quad + \gamma_n\left[T(w_n)-T(w_{n-1})\right]+(\gamma_n-\gamma_{n-1})T(w_{n-1})\Bigg\| \\
&= \Bigg\|\frac{\alpha_n}{2}(x_{n+1}-x_n)+\frac{\alpha_n}{2}(x_n-x_{n-1})+\frac{1}{2}(\alpha_n-\alpha_{n-1})(x_n+x_{n-1}) \\
&\quad + \beta_n\left(f(w_n)-f(w_{n-1})\right)+(\beta_n-\beta_{n-1})f(w_{n-1}) \\
&\quad + \gamma_n\left[T(w_n)-T(w_{n-1})\right]-\left[(\alpha_n-\alpha_{n-1})+(\beta_n-\beta_{n-1})\right]T(w_{n-1})\Bigg\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{2}\|x_n-x_{n-1}\| \\
&\quad + \frac{1}{2}|\alpha_n-\alpha_{n-1}|\,\|x_{n-1}+x_n-2T(w_{n-1})\|+\beta_n\|f(w_n)-f(w_{n-1})\| \\
&\quad + |\beta_n-\beta_{n-1}|\,\|f(w_{n-1})-T(w_{n-1})\|+\gamma_n\|T(w_n)-T(w_{n-1})\| \\
&\leq \frac{\alpha_n}{2}\|x_{n+1}-x_n\|+\frac{\alpha_n}{2}\|x_n-x_{n-1}\|+\left(\frac{1}{2}|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|\right)M \\
&\quad + \theta\beta_n\|w_n-w_{n-1}\|+\gamma_n\|w_n-w_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) M \\
&\quad + \frac{\theta\beta_n}{2}\|x_{n+1} - x_n\| + \frac{\theta\beta_n}{2}\|x_n - x_{n-1}\| + \frac{\gamma_n}{2}\|x_{n+1} - x_n\| + \frac{\gamma_n}{2}\|x_n - x_{n-}\| \\
&= \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_{n+1} - x_n\| + \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_n - x_{n-1}\| \\
&\quad + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) M,
\end{aligned}
$$

where $M > 0$ is a constant such that

$$
M \geq \max\left\{\sup_{n\geq 0}\|x_n + x_{n-1} - 2T(w_{n-1})\|, \sup_{n\geq 0}\|f(w_{n-1}) - T(w_{n-1})\|\right\}.
$$

It give

$$
\begin{aligned}
&\left(1 - \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\right)\|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_n + \theta\beta_n + \gamma_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) M
\end{aligned}
$$

implies

$$
\begin{aligned}
&\left(1 - \frac{1 - \beta_n + \theta\beta_n}{2}\right)\|x_{n+1} - x_n\| \\
&\leq \frac{1 - \beta_n + \theta\beta_n}{2}\|x_n - x_{n-1}\| + \left(\frac{1}{2}|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\right) M
\end{aligned}
$$

implies

$$
\begin{aligned}
&(1 + \beta_n(1-\theta))\|x_{n+1} - x_n\| \\
&\leq (1 - \beta_n(1-\theta))\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}|)M.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq &\left(\frac{1 - \beta_n(1-\theta)}{1 + \beta_n(1-\theta)}\right)\|x_n - x_{n-1}\| \\
&+ \frac{M}{1 + \beta_n(1-\theta)}(|\alpha_n - \alpha_{n-1}| - 2|\beta_n - \beta_{n-1}|).
\end{aligned}
$$

Since $\theta, \beta_n \in (0,1)$, $1 + \beta_n(1-\theta) \geq 1$, and $\frac{1-\beta_n(1-\theta)}{1+\beta_n(1-\theta)} \leq 1 - \beta_n(1-\theta)$. Thus

$$
\begin{aligned}
\|x_{n+1} - x_n\| \leq &\left[1 - \beta_n(1-\theta)\right]\|x_n - x_{n-1}\| \\
&+ \frac{M}{1 + \beta_n(1-\theta)}(|\alpha_n - \alpha_{n-1}| - 2|\beta_n - \beta_{n-1}|).
\end{aligned}
$$

Since $\sum_{n=0}^{\infty} \beta_n = \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, by Theorem 1.6, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

*Step 3.* In this step, we claim that $\limsup_{n\to\infty} \langle x^* - f(x^*), x^* - x_n \rangle \leq 0$, where $x^* = P_{F(T)} f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point $p$ of $T$. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and Theorem 1.5 we have $p = Tp$. This together with the property of the metric projection implies that

$$\begin{aligned}
\limsup_{n\to\infty} \langle x^* - f(x^*), x^* - x_n \rangle &= \limsup_{n\to\infty} \langle x^* - f(x^*), x^* - x_{n_i} \rangle \\
&= \langle x^* - f(x^*), x^* - p \rangle \\
&\leq 0.
\end{aligned}$$

*Step 4.* Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Now we again take $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)} f$. Consider

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|T(y_n) - x^*\|^2 \\
&= \|T(\alpha_n (w_n) + \beta_n f(w_n) + \gamma_n T(w_n)) - x^*\|^2 \\
&\leq \|\alpha_n (w_n) + \beta_n f(w_n) + \gamma_n T(w_n) - x^*\|^2 \\
&= \|\alpha_n (w_n) - \alpha_n x^* + \beta_n f(w_n) - \beta_n x^* + \gamma_n T(w_n) + (\alpha_n + \beta_n - 1) x^*\|^2 \\
&= \alpha_n^2 \|(w_n) - x^*\|^2 + \beta_n^2 \|f(w_n) - x^*\|^2 + \gamma_n^2 \|T(w_n) - x^*\|^2 \\
&\quad + 2\alpha_n \beta_n \langle w_n - x^*, f(w_n) - x^* \rangle + 2\alpha_n \gamma_n \langle w_n - x^*, T(w_n) - x^* \rangle \\
&\quad + 2\beta_n \gamma_n \langle f(w_n) - x^*, T(w_n) - x^* \rangle \\
&\leq \alpha_n^2 \|(w_n) - x^*\|^2 + \beta_n^2 \|f(w_n) - x^*\|^2 + \gamma_n^2 \|w_n - x^*\|^2 \\
&\quad + 2\alpha_n \beta_n \langle (w_n) - x^*, f(w_n) - x^* \rangle + 2\alpha_n \gamma_n \|x_n - x^*\| \|T(w_n) - x^*\| \\
&\quad + 2\beta_n \gamma_n \langle f((w_n)) - f(x^*), T(w_n) - x^* \rangle \\
&\quad + 2\beta_n \gamma_n \langle f(x^*) - x^*, T(w_n) - x^* \rangle \\
&\leq (\alpha_n^2 + \gamma_n^2) \|w_n - x^*\|^2 + 2\alpha_n \gamma_n \|w_n - x^*\|^2 \\
&\quad + 2\beta_n \gamma_n \|f(w_n) - f(x^*)\| \|w_n - x^*\| + K_n \\
&\leq (\alpha_n + \gamma_n)^2 \|w_n - x^*\|^2 + 2\theta \beta_n \gamma_n \|w_n - x^*\|^2 + K_n \\
&\leq \left((\alpha_n + \gamma_n)^2 + 2\theta \beta_n \gamma_n\right) \|w_n - x^*\|^2 + K_n \\
&\leq \left((1 - \beta_n)^2 + 2\theta \beta_n \gamma_n\right) \|w_n - x^*\|^2 + K_n,
\end{aligned}$$

where

$$K_n = \beta_n^2 \left\| f(w_n) - x^* \right\|^2 + 2\alpha_n \beta_n \left\langle (w_n) - x^*, f(w_n) - x^* \right\rangle \\ + 2\beta_n \gamma_n \left\langle f(x^*) - x^*, T(w_n) - x^* \right\rangle.$$

It become

$$\left[(1-\beta_n)^2 + 2\theta\beta_n\gamma_n\right] \|w_n - x^*\|^2 \geq \|x_{n+1} - x_n\|^2 - K_n$$

implies

$$\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n}\, \|w_n - x^*\| \geq \sqrt{\|x_{n+1} - x_n\|^2 - K_n}$$

implies

$$\frac{1}{2}\sqrt{(1-\beta_n)^2 + 2\theta\beta_n\gamma_n}(\|x_{n+1} - x^*\| + \|x_n - x^*\|) \\ \geq \sqrt{\|x_{n+1} - x_n\|^2 - K_n}$$

implies

$$\frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \\ + 2\|x_{n+1} - x^*\|\|x_n - x^*\|) \\ \geq \|x_{n+1} - x_n\|^2 - K_n$$

implies

$$\frac{1}{4}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \\ + (\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2)) \\ \geq \|x_{n+1} - x_n\|^2 - K_n$$

implies

$$\left[1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right] \|x_{n+1} - x^*\|^2 \\ \leq \left[\frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right] \|x_n - x^*\|^2 + K_n.$$

Thus, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq \frac{\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\|x_n-x^*\|^2+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\\
&=\frac{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)-1+((1-\beta_n)^2+2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\|x_n-x^*\|^2\\
&\quad+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\\
&=\left[1-\frac{1-((1-\beta_n)^2+2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\right]\|x_n-x^*\|^2\\
&\quad+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}.
\end{aligned}$$

Note that

$$0<1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)<1$$

implies

$$\frac{1-((1-\beta_n)^2+2\theta\beta_n\gamma_n)}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\geq 1-((1-\beta_n)^2+2\theta\beta_n\gamma_n).$$

Thus, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq [1-(1-((1-\beta_n)^2+2\theta\beta_n\gamma_n))]\|x_n-x^*\|^2\\
&\quad+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\\
&=[(1-\beta_n)^2-2\theta\beta_n\gamma_n]\|x_n-x^*\|^2+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}\\
&\leq (1-\beta_n)^2\|x_n-x^*\|^2+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}.
\end{aligned}$$

Since $0<1-\beta_n<1$, this give $(1-\beta_n)^2<(1-\beta_n)$ and

$$\|x_{n+1}-x^*\|^2\leq (1-\beta_n)\|x_n-x^*\|^2+\frac{K_n}{1-\frac{1}{2}((1-\beta_n)^2+2\theta\beta_n\gamma_n)}. \tag{2.2}$$

By $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \gamma_n = 1$ we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{K_n}{\beta_n \left(1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)\right)} \\
&= \limsup_{n\to\infty} \left( \frac{\beta_n \|f(w_n) - x^*\|^2 + 2\alpha_n \langle w_n - x^*, f(w_n) - x^* \rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \right. \\
&\quad \left. + \frac{2\gamma_n \langle f(x^*) - x^*, T(w_n) - x^* \rangle}{1 - \frac{1}{2}((1-\beta_n)^2 + 2\theta\beta_n\gamma_n)} \right) \\
&\le 0.
\end{aligned} \tag{2.3}$$

From (2.2), (2.3) and Theorem 1.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0,$$

which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

## 3. Applications

### 3.1. A More General System of Variational Inequalities

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. In [1], Cai and Bu considered the problem of finding $x_1^*, x_2^*, ...., x_N^* \in C \times C \times \cdots \times C$ such that

$$\begin{cases}
\langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \ge 0, \\
\langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \ge 0, \\
\vdots \\
\langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \ge 0, \\
\langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \ge 0, \quad \forall x \in C.
\end{cases} \tag{3.1}$$

The equation (3.1) can be written as

$$\begin{cases}
\langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \ge 0, \\
\langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \ge 0, \\
\vdots \\
\langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \ge 0, \\
\langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \ge 0, \forall x \in C,
\end{cases}$$

which is a more general system of variational inequalities in Hilbert spaces, where $\lambda_i > 0$ for all $i \in \{1, 2, 3, ......, N\}$. We also have following lemmas.

**Lemma 3.1.** [1] *Let $C$ be a nonempty closed convex subject of a real Hilbert space $H$. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be $\delta_i$-inverse-strongly monotone for some positive real number $\delta_i$, namely,*

$$\langle A_i x - A_i y, x - y\rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C.$$

*Let $G : C \to C$ be a mapping defined by*

$$\begin{aligned} G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned} \tag{3.2}$$

*If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, \ldots, N\}$, then $G$ is nonexpansive.*

**Lemma 3.2.** [4] *Let $C$ be a nonempty closed convex subject of a real Hilbert space $H$. Let $A_i : C \to H$ be a nonlinear mapping, where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem* (3.1) *if and only if*

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N)x_N^*, x_i^* \\ = P_C(I - \lambda_{i-1}A_{i-1})x_{i-1}^*, \quad i = 2, 3, 4, ...., N, \end{aligned} \tag{3.3}$$

*that is,*

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 3.2, we know that $x_1^* = G(x_1^*)$, that is, $x_1^*$ is a fixed point of the mapping $G$, where $G$ is defined by (3.2). Moreover, if we find the fixed point $x_1^*$, it is easy to get the other points by (3.3). Applying Theorem 2.1 we get the result

**Theorem 3.3.** *Let $C$ be a nonempty closed convex subject of a real Hilbert space $H$. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be $\delta_i$-inverse-strongly monotone for some positive real number $\delta_i$ with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$\begin{aligned} G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned}$$

*Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = G(y_n), \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n G(w_n), \\ w_n = \frac{x_n + x_{n+1}}{2}, \end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the conditions* (i)-(iv) *and*

(vi) $\lim_{n\to\infty} \|x_n - G(x_n)\| = 0.$

*Then $\{x_n\}$ converges strongly to a fixed point $x^*$ of the nonexpansive mapping $G$ which is also the unique solution of the variational inequality*

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in F(G).$$

*In other words, $x^*$ is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.*

### 3.2. The Constrained Convex Minimization Problem

Now, we consider the following constrained convex minimization problem:

$$\min_{x\in C} \phi(x), \tag{3.4}$$

where $\phi : C \to \mathbb{R}$ is a real-valued convex function and assumes that the problem (3.4)is consistent. Let $\Omega$ denote its solution set. For the minimization problem (3.4), if $\phi$ is (Fréchet) differentiable, then we have the following lemma.

**Lemma 3.4.** (Optimality Condition) [7] *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (3.4) *is that $x^*$ solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{3.5}$$

*Equivalently, $x^* \in C$ solves the fixed point equation*

$$x^* = P_C (x^* - \lambda\nabla\phi(x^*))$$

*for every constant $\lambda > 0$. If, in a addition $\phi$ is convex, then the optimality condition* (3.5) *is also sufficient.*

It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping $A$ is $\delta$-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following result.

**Theorem 3.5.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For the minimization problem (3.4), assume that $\phi$ is (Fréchet) differentiable and the gradient $\nabla\phi$ is a $\delta$-inverse-strongly monotone mapping for some positive real number $\delta$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = P_C(I - \lambda\nabla\phi)(y_n) \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n P_C(I - \lambda\nabla\phi)(w_n) \\ w_n = \frac{x_n + x_{n+1}}{2}, \end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the conditions* (i)-(iv) *and*

(vii) $\lim_{n\to\infty} \|x_n - P_C(I - \lambda\nabla\phi)(x_n)\| = 0.$

*Then $\{x_n\}$ converges strongly to a solution $x^*$ of the minimization problem* (3.4), *which is also the unique solution of the variational inequality*

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in \Omega.$$

*In other words, $x^*$ is the unique fixed point of the contraction $P_\Omega f$, that is, $P_\Omega f(x^*) = x^*$.*

### 3.3. The $K$-Mapping

Kangtunyakarn and Suantai [3] in 2009 gave $K$-mappings generated by $T_1, T_2, T_3, \ldots, T_N$ and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ as follows.

**Definition 3.6.** [3] Let $C$ be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of $C$ into itself and let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, 3, \ldots, N$. We define a mapping $K : C \to C$ as follows:

$$\begin{cases} U_1 = \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ \vdots \\ U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a *K-mapping* generated by $T_1, T_2, T_3, ......, T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

In 2014, Suwannaut amd Kangtunyakarn [8] established the following result for $K$-mappings generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

**Lemma 3.7.** [8] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For $i = 1, 2, 3, ..., N$, let $\{T_i\}_{i=1}^N$ be a finite family of $K_i$-strictly pseudo-contractive mappings of $C$ into itself with $K_i \leq \omega_i$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, namely there exist constants $K_i \in [0, 1)$ such that*

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + K_i \|(I - T_i)x - (I - T_i)y\|^2, \quad \forall x, y \in C.$$

*Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let $K$ be the $K$-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. Then the following properties hold:*

$(a)$ $F(K) = \bigcap_{i=1}^N F(T_i)$.

$(b)$ *$K$ is a nonexpansive mapping.*

On the bases of above lemma, we have the following result.

**Theorem 3.8.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For $i = 1, 2, 3, ..., N$, let $\{T_i\}_{i=1}^N$ be a finite family of $K_i$-strictly pseudo-contractive mappings of $C$ into itself with $K_i \leq \omega_i$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let $K$ be the $K$-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be sequence generated by*

$$\begin{cases} x_{n+1} = K(y_n), \\ y_n = \alpha_n(w_n) + \beta_n f(w_n) + \gamma_n K(w_n), \\ w_n = \frac{x_n + x_{n+1}}{2}, \end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the conditions* (i)-(iv) *and*

(viii) $\lim_{n\to\infty} \|x_n - K(x_n)\| = 0$.

*Then $\{x_n\}$ converges strongly to a fixed point $x^*$ of the mappings $\{T_i\}_{i=1}^N$, which is also the unique solution of the variational inequality*

$$\langle (I - f)x, y - x\rangle, \quad \forall y \in F(K) = \bigcap_{i=1}^{N} F(T_i).$$

*In other words, $x^*$ is the unique fixed point of the contraction $P_{\bigcap_{i=1}^N F(T_i)} f$, that is, $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.*

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