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THE FUNCTIONAL EQUIVALENCE OF STABLE AND UNSTABLE INVERSIONS OF THE MELLIN TRANSFORM

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Abstract: We use the term "stable" to mean numerical stability, that the formal solution of an integral equation can be well-approximated using standard methods, including but not limited to hypergeometric functions or numerical integration. Thus, valid trial solutions may be found by symbolic integration or computerized numerical integration. We specifically analyze a class of forward problems that can be expressed as Meijer G-Functions. In Euclidean spaces $\mathbb{R}^n$, the conditions under which an unstable inverse of the Mellin Transform has an equivalent stable inverse are established. These inverses are formally and analytically equivalent, so that any closed form solution of one is also a solution of the other. For this reason, it is a *functional equivalence*. Mathematically, at least, this non-uniqueness is benign, much like the indeterminacy of a square root or phase. But the mathematical formalism and feasibility of numerical modeling of these inverses may be radically different. As such, we have shown that a linear inverse may not be unique. The inverse harmonic Radon Transform is an example of this difficulty. We demonstrate that, under certain easily satisfied conditions, a stable inversion does exist.

More importantly, a stable low pass inversion is an accurate picture of the underlying physics if the forward problem exists and is a Hermitian operator. We demonstrate these results with the n-dimensional Radon Harmonic Transform. We also provide the reader with brief introductions to ultra-harmonic functions, the Funk-Hecke Theorem, and the methods of the Mellin Transform applied to generalized hypergeometric functions.

AMS Subject Classification: 45Q05, 44A12, 47A52, 35R30, 33C20, 33C55, 42A38
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1. Introduction

The use of the Mellin Transform to solve Mellin convolutions of integral equations often results in a numerical inversion that is unstable, in the sense that any discretization of the solution fails to converge. Random errors in the data and in truncated computer arithmetic are amplified so much that a numerical solution is meaningless. As a result, the formal analytical result is considered flawed. The difficulty seems to be "intrinsic" to the solution. When this difficulty is encountered, this avenue of attack is usually abandoned. A primary example of this difficulty is the harmonic inversion of the Radon Transform in n-dimensional spaces. In particular, the 2-D and 3-D Radon Transforms are of much practical interest.

This inversion is based on the ultra-spherical, spherical, and circular harmonic transforms of the Radon Transform. From these transforms, researchers have been able to determine important information that is utilized in practical and commercial applications of tomography. Cormack's early work with the circular harmonic transform [5], [6] was critical in demonstrating the practicability of tomographic reconstruction [1].

Information about sampling rates and the null space of the Radon Transform have been obtained from the harmonic form and have been used to obtain optimal sampling rates.

The harmonic transforms yield symmetries not available by any other means. Such symmetries were put to good use. See the method for restoring the sinograms of data from the PENN-PET scanner [14]. From the harmonic transform of 2D sinogram data, it is known that certain frequencies do not contribute to the sinogram. Therefore, a sizable signal coming from this region may introduce artifacts by aliasing into valid frequencies. This method is also known as "the elimination of null space functions from the 2D frequency map of the sinogram," aka the consistency conditions.

There are other symmetries that lead to useful artifact reduction or noise reduction in tomographic reconstruction. The radial moderator is the intermediate result of the angular transform of the sinogram only. The angular FFT and the radial FFT commute, so one may use the FFT in either order. Conventional 2-D FBP takes the 1-D radial transform and dispenses with the angular transform. A ramp filter is applied to radial FFT of the data as a step in the reconstruction algorithm. The angular transform is not used. The reconstruction is completed in real space by a process known as back-projection.

[1]However, he too was stymied by unstable numerical modeling and was only able to reconstruct the first 14 radial harmonics with good accuracy.

In node filtering, the sinogram is first transformed with respect to angle only. The result is the radial modulators with angular frequencies $\ell = 0, 1, 2, ..., \ell_M$, where M is defined below. It can be shown that the data of the untransformed radial dimension is a complex polynomial with at least $\ell + 1$ nodes or zeros, counting endpoints. The current node filter is simply to accept only those modulators that had nodes equal to or in excess of $\ell + 1$. Otherwise, the radial modulator is rejected by setting it to zero. Noticeable improvement in extremely noisy SPECT data (exponential Radon Transform) was found by modifying node filtering to include the effects of uniform attenuation [9]. Given an attenuation coefficient of μ and a maximum reconstruction radius of r_{max}, the effect of attenuation is to increase the parameter ℓ' with

$$\ell' = \left[\sqrt{\ell^2 + (\mu r_{max})^2} \right],$$

where $[x]$ denotes the integer value less than or equal to x. For a realistic value of $\mu r_{max} > 2$, this leads to a nonlinear increase in the numbers of nodes, especially for ℓ small. This correction was found to have a beneficial effect on image quality with negligible computational effort. The effect of node filtering is shown in Figure **??**.

Practical reconstruction algorithms based on harmonic transforms, though several exist (for example, see [12], and [4]) have been too late arriving on the scene to have any impact on commercial tomographic reconstruction. Part of the reason for this lack of use was that the inversion of the harmonic solution, in the form of a Mellin convolution integral equation, first obtained for $\mathbb{R}^n$ in [15], was extremely unstable as an exercise in numerical integration. The inverse harmonic solution to the Radon Transform was thought to be as unstable as limited angle tomography. It was thought for a long time that any solution based upon harmonic analysis must also be unstable [15] and [7].

It has been shown for 2-D reconstruction that conventional filtered back-projection and reconstruction by circular harmonic transforms for N^2 pixels are both $\mathcal{O}\left(cN^3\right)$ in the number of floating point operations. The scale factor c for harmonic reconstruction is significantly smaller than it is for filtered back projection and the image quality of harmonic reconstruction might be slightly better [10, Ch. 6]. For J samples each for M projections in 360° and N^2 reconstructed image points, the execution time for the CHT algorithm is approximately $9/32J^2M$ whereas for FBP it is approximately $2MN^2$. These estimates are not directly comparable, but if $J \simeq N$, which is usually the case, then the CHT algorithm offers a decided speed advantage. If given a count rate of n_0 cm^{-2} photons per study, and an estimated linear attenuation coefficient

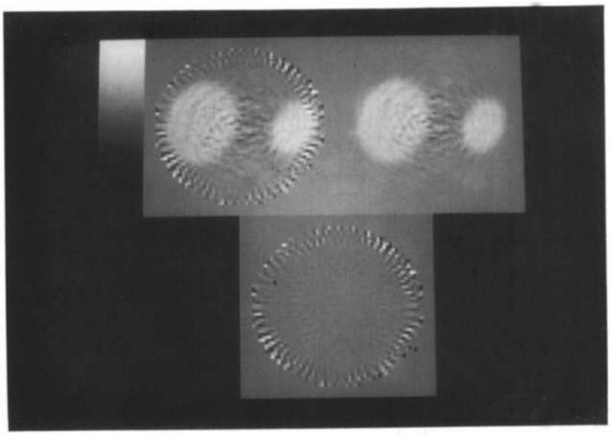

Figure 1: Reconstructions of a typical slice of the liver-spleen phantom. (Upper left) the reconstruction without the node filter. (Upper right) reconstruction with the node filter. (Center) the difference image, stretched to emphasize differences, in order to emphasize any errors in the center of the FOV. Isotope used is Tc-99m. 128 projections in 360 deg. of size 64x64 were acquired on a GE 400 Starcam. Pixel size 6.4mm. *To demonstrate the utility of the node filter, there were only 50,000 total counts in the study, or 392 cts/projection.*

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of μ_0 then for an attenuation-scaled count rate of $\mu_0\sqrt{n_0} = 10^4 cm^{-2}$ the SNR is 5.8 for CHT reconstruction versus 6.9 for FBP [10, Ch. 6].

Hawkins [12] developed a computerized circular harmonic reconstruction algorithm based on stable recursion that was free of numerical error and it was first recognized in peer-reviewed literature. This algorithm was based on the Chebyshev/Zernike polynomial solution to the unstable harmonic inverse found by Alan Cormack [5]. Cormack's work in turn rested upon the exhaustive work of Erdelyi, et. al. [8].

Both of the algorithms [12] and [4] produced image quality equivalent to filtered back-projection. Hawkins' algorithm utilized the analytical solution directly, bypassing the unstable harmonic back-projection by applying the FFT to the orthogonal Chebyshev Transform of the data. The second algorithm was an optimized numerical method based on the stable harmonic inverse. These two results were the first examples of the stable/unstable conundrum upon which this manuscript is based. P. E. Mijnarends [17] found the stable harmonic inverse for the 3-D spherical harmonic inverse, to complement the unstable version

[15]. Hawkins [10] independently found the same stable harmonic inverse.

We must emphasize that *stable* is not an absolute numerical quantity. The Radon filtered-back-projection inversion is still unstable, but can be nicely stabilized with Fourier frequency space filtering techniques. There is some noise amplification but it is manageable. There are many places to start in the literature, but the most extensive review and best organized for medical imaging and optical devices is found in Barrett and Myers' "Fundamentals of Imaging Science" [2]. For optical instrumentation, see "Principles of Optics" by Born and Wolf [3]. In some sense, these are renormalization problems as well. For truly ill-posed problems, this is the best we can hope for. These methods were later applied to the attenuated Radon Transform and the first medically recognized quantitative SPECT protocol that was allowed to be used in a Phase II study on human volunteers in the Johns Hopkins Hospital Oncology Clinic [21],[22].

Despite the good news that these stable harmonic inverses brought, harmonic analysis is still viewed skeptically in some quarters. It appeared that for the Radon Transform, at least, harmonic analysis was at a dead end. Why this issue was not dealt with more urgently is not known. As we have seen, Harmonic Transforms are rich in symmetry conditions that often improve reconstruction image quality.

In 2007, Hawkins proved that the Chebyshev/Zernike polynomial solution, well established as a solution of the unstable inverse is also a solution to the stable inverse [11]. Therefore, for this polynomial pair, the stable and unstable circular harmonic inverses are formally equivalent, as indeed they must be; the inverse of a linear integral transform, if it exists, is unique. If inverses differ formally, it can be only by a multiple that is equivalent to unity or by the addition of a function equal to zero. In that article [11], it was again demonstrated that the method of reconstruction with orthogonal polynomials is essentially equivalent to filtered back-projection in both execution speed (numerical complexity) and image quality.

The harmonic Radon Transform and the harmonic X-Ray Transform are examples of Mellin convolutions. Initially, the purpose of this manuscript was to address the stability issues of the Radon Transform. It is clear now that stability issues can be quite common, and not confined to the Radon Transform.

In this work, we take up the issue of integral transforms that may be represented as Mellin convolutions. This is a very large category that includes Fredholm equations of the first kind. The Fourier convolution integral is an important Fredholm equation of the 1^{st} kind. In fact, Fourier convolutions may be written as Mellin convolutions, and vice-versa.

An important goal of this manuscript is to prove under what general conditions these transforms can have stable or unstable inversions. With these tools, we do not need to investigate each solution in detail or calculate the actual inverse.

We illustrate this by proving the equivalence of the unstable and stable harmonic inverses for the Radon Transform in the n-dimensional space $\mathbb{R}^1 \times S^{n-1}$. We restrict ourselves to a particular pair of stable and unstable inverses that differ by a term equal to unity in Mellin Transform frequency space.This equivalence is independent of any particular solution pair of orthogonal functions since any solution pair satisfies all possible inverses, some stable and others unstable.

We emphasize that many equivalent inverses may be found. We have focused on only a particular choice of stable and unstable inverses.

Please consult [16] and Appendix A for the notation that is to follow. For example, the stable inverse can be found by interchanging the pair of parameters (a) and (c) of the rational fraction of Gamma functions that comprise the Mellin Transforms of generalized hypergeometric functions. See Section 3.3 for the complete list. These pairs have the desirable property of making the parameters A, B, C, D and global parameter η invariant. See equations (51), (52), (53), and (55). There are many potential possibilities.

We must make clear that the inverses, in general, are not *identically* equivalent for all of the values of the parameter space on which the solution depends. For the Radon Transform, this equivalence requires that the value of n, the dimension of Euclidean space, and the value of ℓ, the degree of the ultra-spherical polynomial, both be *positive integers* such that $n \geq 2$ and $\ell \geq 0$. The problem is physically meaningful for these values of n and ℓ. We show equivalence by utilizing Mellin Transforms. The results generally are Mellin Transforms that can be analytically continued to all complex values of n and ℓ. We must restrict their values to obtain equivalence, because the manipulations of the transforms we employ are not valid except for the restrictions on n and ℓ.

The non-uniqueness of the inverse of a Mellin convolution, up to a factor of unity, should not be surprising. The fact that the inverses have such different representations and numerical implementations is surprising. On the surface, this finding seems to be impossible, since a linear inverse, if it exists, must be unique. However, if we treat the inverse as a black box, then all of these inverses appear identical, since any solution of one is a solution of all the others.

An important new development herein is that a stable harmonic inverse is the ultimate form of regularization for inverse problems that produce unstable inverses via the Mellin Transform. Also important is that perturbation analysis for these transforms now makes sense, both computationally and analytically.

If there exists two inverses, one stable and the other unstable, how do we know that the stable inverse can be selected over the unstable inverse? For mathematical models involving data or instrumentation, the forward problem often shall be represented by a direct and stable numerical algorithm, as it must be for any kind of real-world measuring device. There are no infinities in measurement or detection. The integral equation representing the device will usually be easy to "code" as a numerical approximation for computerized investigations. Therefore we consider only forward problems that represent realizable devices and consist of any arrangement of measuring rods, clocks, scales, diffraction gratings, thermometers, and so on.

With the Radon Transform, we will show that there exists an *unstable* forward transform. We always choose the stable forward problem on physical grounds. This is an extra degree of freedom granted us by the physical situation. Therefore, we can base stability on physical grounds. If a stable forward problem exists and is Hermitian, we feel obliged, based on symmetry, to see if a stable inverse is possible and if it has any predictive value.

1.1. Outline of the Paper

In Section 2, we present the Mellin Transform and the inverse Mellin Transform. In Section 3 we outline Slater's Theorem and show how it can be applied to find either stable or unstable solutions of a Mellin convolution. We show how the theorem can be augmented to find a stable inversion. We then apply this result to obtain a stable inversion of the harmonic Radon Transform.

In Section 4, we state the n-dimensional Radon Transform, introduce the harmonic polynomials, surface harmonics, and the Funk-Hecke Theorem. This powerful theorem is central to our work. We also discuss the symmetries of the spaces $\mathbb{R}^1 \times S^{n-1}$. In Section 5 we derive the stable harmonic inverse and its Mellin Transform. In Section 6, we review the unstable harmonic inverse of the Radon Transform, and discuss why the solution is unsuitable for numerical integration. We show that the n-dimensional harmonic form of the stable inverse can be derived directly from the well-known stable version of the (Euclidian) inverse Radon Transform in $\mathbb{R}^n$ [18]. In Section 7 we apply elementary transformations to the Mellin Transform to prove its equivalence to the unstable inverse in Mellin Transform frequency space. There are two distinct cases that must be treated separately, one for odd dimensional spaces, and another for even dimensional spaces.

In Section 8, we show how the stable inverse yields an unstable forward transform. In Section 9 we show that multiple inverses exist for the Fox H-functions. In Section 10 we discuss the ramifications of this work and possible new directions. Section 11 is acknowledgments.

In Section 12, we provide the notation and background for Slater's Theorem. If you are not familiar with the notation of Slater's Theorem, we suggest that you read this Appendix first. In Section13 we provide a sample calculation of the transform methods we use.

2. The Mellin Transform

The Mellin Transform $\mathbf{M}$ of a function f of one variable is

$$\tilde{f}(\mu) = \mathbf{M}f(\mu) = \int_0^\infty f(x)\, x^{\mu-1} dx, \tag{1}$$

provided that there are values for $\mathrm{Re}(\mu)$ for which the integral converges. With a change of variables, it is equivalent to the Fourier Transform.

As a Hilbert space operator, it is a unitary operator, meaning that it is Hermitian and the inverse is equivalent to the adjoint. See [2] for an in-depth discussion of this transform and others used in imaging science.

The inverse Mellin Transform is given by

$$f(x) = \mathbf{M}^{-1}\left[\tilde{f}\right](x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{f}(\mu)\, x^{-\mu} d\mu, \tag{2}$$

where γ is chosen so that the integral converges. As discussed in Section (12), this integral is closed to the left (to contain the poles of the series (a) or to the right (to contain the poles of the series (b). There is a list of conditions given in (12) by (51), (52), and (53). For our discussion, the most important of these is

$$A + B \geq C + D. \tag{3}$$

If this condition is satisfied, Eqn. (2) converges with closure either to the right or left. Otherwise, the integral may fail to converge at all.

For a very large class of functions f (the generalized hypergeometric functions), the Mellin Transform $\tilde{f}$ is a rational function of Gamma functions. The Gamma function is meromorphic because it has isolated poles of order 1 and no zeros. The ratio of meromorphic functions is a meromorphic function as long

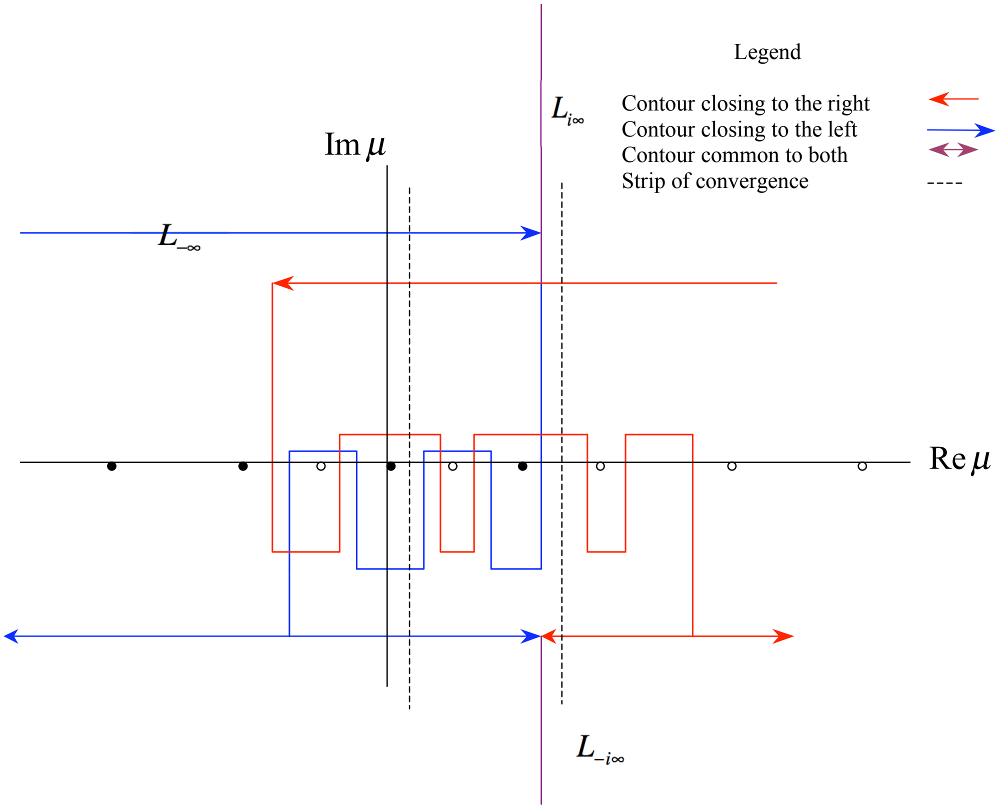

Figure 2: The Mellin-Barnes Contours for separating the left series of poles from the right series of poles

as the poles are isolated and are a countable set. The Mellin Transforms of this class consist only of otherwise analytic functions with isolated poles and zeros.

A pole in the denominator function will produce a zero. If the poles of $\tilde{f}$ coincide, the function f has logarithmic singularities. If a pole and a zero coincide, they cancel, and the singularity is removed. See figure **??**.

The Mellin convolution is defined by

$$\mathbf{K}_2(y) = \int_0^\infty \mathbf{K}_1\left(\frac{y}{x}\right)\mathbf{K}\left(x\right)\frac{dx}{x}. \tag{4}$$

The Mellin Transform of the Mellin convolution is

$$\tilde{\mathbf{K}}_2\left(\mu\right) = \tilde{\mathbf{K}}_\mathbf{1}\left(\mu\right)\tilde{\mathbf{K}}\left(\mu\right). \tag{5}$$

We may solve for $\tilde{\mathbf{K}}_{\mathbf{1}}$ by division:

$$\tilde{\mathbf{K}}_{\mathbf{1}}(\mu) = \frac{\tilde{\mathbf{K}}_2(\mu)}{\tilde{\mathbf{K}}(\mu)}. \tag{6}$$

3. Determining Stability from Slater's Theorem

This is a very abridged discussion of the main points of Theorems 18 and 19, [16, Pt. 1, §4]. Whether a particular solution is stable will depend on a careful sorting out of the parameters. The advantage gained is that one can tell just from the Mellin Transform of the kernel that the inversion is stable. We can also find the Mellin Transform of a stable inverse, if it exists. We will see how this works to our advantage with the inversion of the Radon Harmonic Transform. Even if the outcome is a kernel consisting of two or more operators applied as successive Mellin convolutions, we will know beforehand if it is stable. In this preliminary presentation, we assume that none of the parameters a_j, b_k, c_l, d_i differ by an integer, e.g., $a_j \neq b_k + m, m \in \mathbb{Z}$ for all pairs $\{j, k\}$ & etc.. Two merged poles of 1^{st} order become a 2^{nd} order pole. We specifically exclude the logarithmic cases. Let $\tilde{\mathbf{K}}$ be represented by (50). Then the inversion is given by eqn. (48) or eqn. (49). If a solution exists, it is necessary that the general conditions (51) and (52) be met. These conditions may be relaxed under certain easily satisfied conditions, to wit:

If

$$A + D > B + C, \tag{7}$$

Cf. eqn. (55) then

$$\mathbf{K}(x) = \sum\nolimits_A (x), \quad \text{for } 0 \leq x \leq 1.$$

As $x = |z| \to 0$,

$$\sum\nolimits_A (z) = O\left(|z|^{\alpha}\right), \quad \alpha = \min\left(\operatorname{Re}\left(a_j\right)\right), \;\; j = 1, \ldots, A. \tag{8}$$

The inversion kernel will be bounded if $\alpha \geq 0$ and stable, or physically useful, if $\alpha \geq -1$, since the singularity for $\alpha = -1$ can be removed by contour integration.

If

$$A + D < B + C, \tag{9}$$

then

$$\mathbf{K}(x) = \sum\nolimits_B (1/x), \quad \text{for } 1 \leq x < \infty. \tag{10}$$

As $x = |z| \to \infty$, $\sum_B (1/z)$ has order

$$\sum\nolimits_B (1/z) = O\left(|z|^{-\beta}\right), \beta = \min\left(\mathrm{Re}\left(b_k\right)\right), \quad k = 1, \dots, B$$

This solution will be bounded if $\beta > 0$, and stable if $\beta \geq -1$.

For the case (7) we choose γ so that the contour encloses the poles of $\Gamma((a) + \mu)$ but excluded the poles of $\Gamma((b) - \mu)$ and close the contour to the left by using either $L_{-i\infty}$ or $L_{-\infty}$. For the case (9), we enclose the poles of $\Gamma((b) - \mu)$, exclude the poles of $\Gamma((a) + \mu)$, and complete the contour to the right by using either $L_{+i\infty}$ or $L_{+\infty}$.

If $A + D = B + C$, it is necessary that $\arg z = 0$, that is, $x = \mathrm{Re}\,(z)$ is real and positive and $z = x > 0$. Then under the conditions

$$\gamma\,(A + D - B - C) < -1 - \mathrm{Re}\,(\eta)\,,$$

the solution is

$$K\,(x) = \begin{cases} \sum_A (x), & 0 \leq x < 1, \\ \sum_B (1/x), & 1 \leq x < \infty. \end{cases}$$

If $A + B = C + D$ and $A + D = B + C$, then it is true that

$$A = D \text{ and } B = C. \tag{11}$$

The constraint on η can be relaxed to

$$\mathrm{Re}\,(\eta) < 0. \tag{12}$$

Lemma 3.1. *If $B \geq 1$, $A + D = B + C$, $A + B > C + D$, and $|\arg z| < \pi/2\,(A + B - C - D)$, $\sum_A (z)$ and $\sum_B (1/z)$ have the asymptotic expansion as $z \to \infty$:*

$$\sum\nolimits_A (z) \sim \sum\nolimits_B (1/x) = O\left(|z|^{-\beta}\right), \quad \beta = \min\left(\mathrm{Re}\left(b_k\right)\right), \quad k = 1, 2, \dots, B.$$

We will show how the conditions in Theorems 18, 19, and 20 of [16, §4] allow us to determine stability of the different inverses of the harmonic Radon Transform.

3.1. The Mellin Transform and the Hilbert Transform

There is a relationship between the Mellin Transform and the Hilbert Transform that we will need. Specifically, we shall need the Mellin Transform of the Hilbert

Transform, not the Hilbert Transform as a Mellin Transform via a change of variable.

There are two cases here, one for $f(y)$ even, and one for $f(y)$ odd.

Let $f\colon \mathbb{R}^1 \to \mathbb{R}^1$. The Hilbert Transform of a function f, $\mathbf{H}f$, is defined as

$$\mathbf{H}f(x) \triangleq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}dy.$$

Let the operator $*$ define Fourier convolution:

$$(h*f)(x) = \int_{-\infty}^{\infty} h(x-y)\, f(y)\, dy. \tag{13}$$

We can always convert a convolution to a Mellin Transform by using the substitutions $x = \log\xi$ and $y = \log\psi$. We then obtain

$$(h*f)(\log\xi) = \int_{0}^{\infty} h\left((\log\frac{\xi}{\psi})\right) f(\log\psi)\frac{1}{\psi}d\psi.$$

But this is not what we want.

When the function f is separated into even and odd components (which can always be done for functions of bounded variation), the Hilbert Transform is a sum of Mellin convolutions, as can be easily seen:

Lemma 3.2. *By rewriting* $\mathbf{H}$*, we obtain* $\mathbf{H}$ *as a sum of Mellin convolutions:*

$$\mathbf{H}f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{f(-y)}{\frac{x}{y}+1}\frac{dy}{y} + \frac{1}{\pi}\int_{0}^{\infty} \frac{f(y)}{\frac{x}{y}-1}\frac{dy}{y} \tag{14}$$

This property of the Hilbert Transform allows us to invert the Hilbert Transform using the Mellin Transform.

Lemma 3.3. *If f is even, then $f(y) = f(-y)$ and*

$$\mathbf{H}f(s) = \frac{2s}{\pi}\int_{0}^{\infty} f(y)\,\frac{1}{s^2-y^2}dy,$$

or, equivalently, as a Mellin convolution:

$$\mathbf{H}f(s) = \frac{2}{\pi}\int_{0}^{\infty} f(y)\,\frac{s}{y}\frac{1}{\frac{s^2}{y^2}-1}\frac{dy}{y}.$$

If f is odd, then $f(y) = -f(-y)$ and

$$\mathbf{H}f(s) = \frac{2}{\pi}\int_0^{\infty} f(y)\frac{y}{s^2 - y^2}dy,$$

and as

$$\mathbf{H}f(s) = \frac{2}{\pi}\int_0^{\infty} f(y)\frac{1}{\frac{s^2}{y^2} - 1}\frac{dy}{y}.$$

For n an even integer, we will need the Hilbert convolution. If $\ell \in \mathbb{Z}^+$ is even , then from eqn. (28) the harmonic g_ℓ is even, so

$$\begin{aligned}
\mathbf{M}[g_\ell(s)](\mu) &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} g_\ell(y)\frac{1}{s-y}dy\, s^{\mu-1}ds \\
&= \frac{2}{\pi}\int_0^{\infty} g_\ell(y)\int_0^{\infty}\frac{1}{s^2-y^2}s^{\mu}ds\, dy \qquad (15)\\
&= -\mathbf{M}[g_\ell](s)\,\Gamma\left[\begin{array}{cc}\frac{1-\mu}{2}, & \frac{\mu+1}{2}\\ 2-\frac{\mu}{2}, & \frac{\mu}{2}-1\end{array}\right] \\
&= \mathbf{M}[g_\ell](\mu)\tan\frac{\pi\mu}{2}
\end{aligned}$$

We see immediately that this transformation is of type (11), and that $\alpha = 1/2$ and $\beta = 1/2$, so the result will be a stable transformation, provided, of course, that the Mellin Transform of g_ℓ is stable.

For $\ell \in \mathbb{Z}^+$ odd, then from eqn(28) g_ℓ is odd, for which we have similarly as in (15):

$$\begin{aligned}
\mathbf{M}[g_\ell(s)](\mu) &= \frac{1}{\pi}\int_0^{\infty} g_\ell(y)\,y^{-1}\int_{-\infty}^{\infty}\frac{1}{s^2/y^2-1}s^{\mu-1}dsdy \\
&= \frac{1}{\pi}\int_0^{\infty} g_\ell(y)\,y^{-1}y^{\mu-1}\int_0^{\infty}\frac{t^{\mu/2-1}}{t-1}dtdy \qquad (16)\\
&= -\mathbf{M}[g_\ell](s)\,\Gamma\left[\begin{array}{cc}\frac{\mu}{2}, & 1-\frac{\mu}{2}\\ \frac{\mu}{2}+\frac{1}{2}, & \frac{1}{2}-\frac{\mu}{2}\end{array}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\mathbf{M}\left[g_\ell\right](\mu)\,\frac{\sin\left(\pi\left(\frac{\mu}{2}+\frac{1}{2}\right)\right)}{\sin\left(\pi\frac{\mu}{2}\right)} \\
&= -\mathbf{M}\left[g_\ell\right](\mu)\cot\left(\frac{\pi\mu}{2}\right).
\end{aligned}
$$

This transform is of type (11). Furthermore $\alpha = 0$ and $\beta = 1$. So this transform is stable. We will need these results below.

3.2. An Example of Other Functionally Equivalent Solutions

The inversion of (46) has more than one answer. We can use the identity

$$
\Gamma(\mu)\,\Gamma(1-\mu)\,\frac{\sin\pi\mu}{\pi} \equiv 1, \tag{17}
$$

true for all complex values of μ, to generate solutions that have different characteristics. Let us say that in (50), $A = C = \kappa, B = D = 0$, and $\eta < -1$ so that

$$
\tilde{K}(\mu) = \Gamma\left[\begin{array}{c}(a)+\mu \\ (c)+\mu\end{array}\right]. \tag{18}
$$

In addition, let there exist some $a_1 \in \{(a) \mid a_1 < -1\}$ and $c_1 \in \{(c) \mid c_1 \neq a_1\}$, and suppose that the rest of conditions in Section 3 are satisfied then the inverse exists but is unstable. In other words, the solution is of the form $\sum_A(z)$, and the solution is unstable because $\alpha < -1$ (8). We can generate another trial solution of the form (50). Consider just the rational function of Gamma functions. Suppose that in addition, $c_1 \leq 2$. This part of the solution is stable because now, $\alpha = 1 - a_1 \geq 2$ and $\beta = 1 - c_1 \geq -1$.

$$
\begin{aligned}
\tilde{K}^{trial}(\mu) &= \Gamma\left[\begin{array}{cc} a_2+\mu, \ldots a_j+\mu, & 1-c_1-\mu \\ c_2+\mu, \ldots c_l+\mu, & 1-a_1-\mu \end{array}\right] \\
&\times \frac{\sin\pi\left(c_1+\mu\right)}{\sin\pi\left(a_1+\mu\right)}.
\end{aligned} \tag{19}
$$

The factor on the right of the top equation is equivalent to multiplying or dividing the unstable solution by 1, e.g., eqn. (17). We now have a solution composed of two Mellin convolutions, one being a trigonometric fraction and one of the form (50). It appears that we have simply transferred the instability to the trigonometric fraction. For certain values of n and ℓ, this will not be true. These, of course, are the cases not covered by Slater's Theorem. If a_1 is an integer multiple of c_1, then the trigonometric factor will be a multiple of -1.

It is still true that $A+B=C+D$ and $A+D=B+C$. We must simultaneously transform a numerator/denominator pair, in order to maintain this relationship. Now, however, $B=C=1$, and $\sum_B (1/z)$ is no longer identically zero.

We note that in the original problem, eqn. (18), from the condition (3), that $A=C, D=B$. Under the transformation, $B=D=1$ and $A=C=\kappa-1$. With the new coefficients, η is invariant.

$$b_1^{trial} = 1 - c_1,$$
$$d_1^{trial} = 1 - a_1$$

and

$$\eta^{trial} = \eta - a_1 + c_1 + b_1^{trial} - d_1^{trial} = \eta.$$

By using the identity we end up with $b_1^{trial} > -1$. Nothing else changes and we obtain an inversion that is stable $\forall x \geq 0$.

There is the additional symmetry of the trigonometric term. Something interesting happens if a_1 and c_1 differ by integer multiples of $1/2$, i. e.,

$$a_1 - c_1 = \frac{j}{2}, \quad j \in \mathbb{Z}. \tag{20}$$

If j is even, the poles and zeros cancel out and the trigonometric term evaluates to $(-1)^{j/2}$. If j is odd, then the trigonometric term becomes the Mellin Transform of the Hilbert operator, as defined in Section 3.1:

$$\frac{\sin \pi (c_1 + \mu)}{\sin \pi (a_1 + \mu)} = \begin{cases} (-1)^{j/2}, & if\ j\ is\ even, \\ -(-1)^{(j-1)/2} \tan \pi (c_1 + \mu) & \\ (-1)^{(j-1)/2} \cot \pi (a_1 + \mu), & if\ j\ is\ odd. \end{cases} \tag{21}$$

For the solution to (18) to exist it is necessary that $A=C$. Now, $A^{trial} = A-1$, $C^{trial} = C-1$, $B^{trial} = B+1$, and $D^{trial} = D+1$. This condition holds each time we transfer elements of (a) to (d), and elements of (c) to (b). We can therefore find a multiplicity of solutions. Since $A+D=B+C$, then so is $A^{trial} + D^{trial} = B^{trial} + C^{trial}$ and the inversion has the form (58).

Let $\tilde{g}^{trial}(\mu)$ be just the fraction of Gamma functions in (19) and let j in (21) be odd. If $g(t)$ is an *even* function, then the trigonometric fraction to use is the *tangent* which corresponds to the Hilbert Transform for even functions. If $\tilde{g}^{trial}(\mu)$ is *odd*, the appropriate trigonometric fraction is the *cotangent*, as this is the Mellin Transform of the Hilbert operator for odd functions. So regardless if g_ℓ is even or odd, the appropriate Hilbert Transform can be found.

3.3. Comments on the Multiplicity of Equivalent Inverses

We have used one of the most straightforward substitutions to find a stable inverse. For the example above, $A = C$ and $B = D$. This integral exists because conditions (52) are valid. And some of the substitutions can possibly lead to difficulties that have not yet been explored. Let us denote a replacement of

$$\Gamma\left(a_1 + \mu\right) = \pi / \left(\sin(\pi\left(a_1 + \mu\right))\Gamma(1 - a_1 - \mu)\right)$$

by $A \to D$. Then the only possible substitutions are $A \to D, D \to A, B \to C$,and $C \to B$. These substitutions can be taken in any order. Here are a list of substitutions:

1. $A \to D$. This single substitution is possible because the zeros of the sine term cancel all of the poles of the $\Gamma(1-a_1-\mu)$ term. The simple poles left come from the zeros of the sine function that are not cancelled and these will be equivalent to the poles of the $\Gamma(a+\mu)$, or $\mu = k - a_1, k = 0, -1, ...$ So nothing has changed and the instability is not removed.

2. This is also true for other single substitutions $D \to A, B \to C$, or $C \to B$

3. To remove an instability we must use a *pair* of substitutions such as $A \to D$ **and** $C \to B$. Thus the convergence criteria (51), (52), (55) and η are all invariant. And $d_{trial} = 1 - a_{trial} > -1$. If the value of $c_{trial} < 2$, then $b_{trial} = 1 - c_{trial} > -1$ and the instability is removed.

4. The pair of substitutions $A \to D$ **and** $D \to A$ or $B \to C$ **and** $C \to B$ also result in the invariance of conditions (51), (52), (55) and η. Of course, $a_{trial} \neq 1 - d_{trial}$ nor $b_{trial} \neq 1 - c_{trial}$.

4. The Harmonic Polynomials in $\mathbb{R}^n$

A polynomial H_ℓ of a real variable $\mathbf{x} = [x_1, x_2, ..., x_n]$ and degree ℓ is said to be *homogeneous of degree* ℓ if, for any real number λ,

$$H_\ell\left(\lambda \mathbf{x}\right) = \lambda^\ell H_\ell\left(\mathbf{x}\right).$$

A *harmonic polynomial* H_ℓ in $\mathbb{R}^n$ is a polynomial of a real variable x and degree ℓ such that

- $H_\ell\left(\mathbf{x}\right)$ is homogeneous.

- $H_\ell(\mathbf{1}) = 1$.
- $\Delta_n H_\ell(\mathbf{x}) = 0$, where Δ_n is the Laplace operator $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.
- For every vector $\zeta \in S^{n-1}$, and $\mathbf{x} = r\theta \in \mathbb{R}^1 \times S^{n-1}$,

$$H_\ell(\mathbf{x}) = r^\ell \frac{C_\ell^{(n-2)/2}(\zeta \cdot \theta)}{C_\ell^{(n-2)/2}(1)}$$

  where $C_\ell^{(n-2)/2}$ is a Gegenbauer polynomial and where $\zeta \cdot \theta$ is the standard inner product of ζ and θ. For more information, see [8].
- There are two different normalizations of the Gegenbauer polynomials that are widely used. We use the conventional notation $p = n - 2$, also used in [1], [8] and [16], In [7], the normalization is given by

$$C_\ell^{p/2}(1) = \frac{(p)_\ell}{\ell!}. \tag{22}$$

  For $n = 2$, the normalized polynomials are the Chebyshev polynomials of the 1^{st} kind:

$$\frac{C_\ell^0(x)}{C_\ell^0(1)} = T_\ell(x).$$

- Both the object and the Radon projection have expansions in terms of the (ultra) spherical harmonic functions $S_{\ell\mathbf{m}}(\theta)$. They are polynomials of degree ℓ in θ. The vector $\mathbf{m}$ is a vector of $n-1$ positive numbers such that

$$\mathbf{m} = [m_0, m_1, \ldots, m_{n-2}], \ell = m_0 \geq m_1 \geq \ldots m_{n-2} \geq 0.$$

  An element $\mathbf{x}$ of S^{n-1} can also be written in terms of angles $\{\psi_1, \psi_1, \ldots, \psi_{n-2}, \pm\phi\}$ as

$$\mathbf{x} = \begin{bmatrix} \cos\psi_1 \\ \sin\psi_1 \cos\psi_2 \\ \sin\psi_1 \sin\psi_2 \cos\psi_3 \\ \vdots \\ \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-2} \cos\phi \\ \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-2} \sin\phi \end{bmatrix}, \tag{23}$$

  with $|\mathbf{x}| = 1$.

- The differential volume element and differential surface element of the sphere are

$$\begin{aligned} dx^n &= r^{n-1} dr d\Omega^{n-1}(\theta) \\ &= r^{n-1} \sin^p(\psi_1) \sin^{p-1}(\psi_2) ... \sin(\psi_p)\, dr d\theta_1 ... d\theta_p d\phi. \end{aligned}$$

To emphasize the importance of the largest index $\ell = m_0$, we use the compound subscript $\ell\mathbf{m}$. We therefore investigate the Radon Transformation of the components $z_\ell S_{\ell\mathbf{m}} \to g_\ell S_{\ell\mathbf{m}}$. A surface harmonic $S_{\ell\mathbf{m}} : S^{n-1} \to \mathbb{C}$ has the form

$$S_{\ell\mathbf{m}}(\theta_k; \pm\phi) = e^{\pm i m_p \phi} \prod_{k=0}^{n-3} (\sin\theta_{k+1})^{m_{k+1}} C_{m_k - m_{k+1}}^{m_{k+1}+(n-2-k)/2} (\cos\theta_{k+1}), \qquad (24)$$

where the functions $C_{m_k - m_{k+1}}^{m_{k+1}+(n-2-k)/2}(x)$ are the Ultraspherical polynomials [1]. Let $g(\theta, s)$ be the Radon Transform of $f(\mathbf{x})$. Both g and f may be represented as linear combinations of ultraspherical polynomials:

$$g(\theta, s) = \sum_{\mathbf{m}} g_\ell(s) S_{\ell\mathbf{m}}(\theta) \text{ and } f(\mathbf{x}) = \sum_{\mathbf{m}} z_\ell(r) S_{\ell\mathbf{m}}(\theta), \qquad (25)$$

where $\sum \mathbf{m}$ is a multi-index sum over $\mathbf{m}$:

$$\sum_{\mathbf{m}} = \sum_{\ell=0}^{\infty} \sum_{m_1=0}^{\ell} \sum_{m_2=0}^{m_1} \cdots \sum_{m_{n-2}=0}^{m_{n-1}}.$$

We will show below that the radial harmonic functions f_ℓ and z_ℓ are functions only of ℓ, the fundamental order of the harmonic functions. The ultra-spherical harmonic functions are orthonormal and complete on S^{n-1}, i. e.,

$$\int_{S^{n-1}} S_{\ell\mathbf{m}}(\theta)\, S_{j\mathbf{m}'}(\theta)\, d\Omega^{n-1}(\theta) = \delta_{\ell j}. \qquad (26)$$

[8, §11.4].

4.1. Symmetries of the Harmonic Transform

We will consider circular, spherical, and ultra spherical coordinate systems in which $\mathbb{R}^n$ is replaced with $\mathbb{R}^1 \times S^{n-1}$. This topology has important symmetries that simplify the harmonic transforms. We take the radial system given in (23) and find the representation of f by orthonormal surface harmonics:

$$f(r,\theta) = \sum_{\mathbf{m}} f_\ell(r)\, S_{\ell\mathbf{m}}(\theta) \tag{27}$$

An important symmetry arises from the fact that

$$(r, \theta_1, ...\theta_p, \varphi) = (-r, \theta_1 + \pi, \theta_2, ...\theta_p, \varphi).$$

There is also another symmetry for

$$(r, \theta_1, \ldots, \theta_p, \varphi) = (-r, \pi - \theta_1, \ldots, \pi - \theta_p, \phi + \pi).$$

From (24) and for both of these symmetries, we see that

$$S_{\ell\mathbf{m}}(\pm\phi, \theta_1 + \pi, \theta_2, ...\theta_{n-2}) = (-1)^\ell S_{\ell\mathbf{m}}(\pm\phi, \theta_1, \theta_2, ...\theta_{n-2}),$$

$$\begin{aligned} S_{\ell\mathbf{m}}(&\pm\phi + \pi, \pi - \theta_1, \pi - \theta_2, ...\pi - \theta_{n-2}) \\ &= (-1)^{m_p} \prod_{k=0}^{n-3} (-1)^{m_k - m_{k+1}} S_{\ell\mathbf{m}}(\pm\phi, \theta_1, \theta_2, ...\theta_{n-2}) \\ &= (-1)^{m_p} (-1)^{m_k - m_p} S_{\ell\mathbf{m}}(\pm\phi, \theta_1, \theta_2, ...\theta_{n-2}) \\ &= (-1)^\ell S_{\ell\mathbf{m}}(\pm\phi, \theta_1, \theta_2, ...\theta_{n-2}) \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{S}_{\ell\mathbf{m}}(\pm\phi, \theta_1 + \pi, ...\theta_p) &= \mathrm{S}_{\ell\mathbf{m}}(\pm\phi + \pi, \pi - \theta_1, ...\pi - \theta_p) \\ &= (-1)^\ell \mathrm{S}_{\ell\mathbf{m}}(\pm\phi, \theta_1, ...\theta_p). \end{aligned}$$

This implies that for any radial function $f_\ell(r)$ as represented in eqn. (25) that

$$f_\ell(r) \text{ is } \left\{ \begin{array}{c} odd\ for\ \ell\ odd \\ even\ for\ \ell\ even. \end{array} \right\} \quad \Box \tag{28}$$

so that the harmonic representation is not multivalued on the hypersphere. There is another symmetry, or rather a set of symmetries, that shall be mentioned:

$$\begin{aligned} \mathrm{S}_{\ell\mathbf{m}}(\pm\varphi, \theta_1, ...\theta_p) &= \mathrm{S}_{\ell\mathbf{m}}(2\pi \mp \varphi, \theta_1, ...\theta_p) \\ &= \mathrm{S}_{\ell\mathbf{m}}(\pm\varphi, 2\pi - \theta_1, ...\theta_p) \\ &= \mathrm{S}_{\ell\mathbf{m}}(\pm\varphi, \theta_1, 2\pi - \theta_2, ...\theta_p), \end{aligned}$$

where any or all of the angles may be replaced by 2π minus the angle.

4.2. The n-Dimensional Radon Transform

The unknown object to be reconstructed is the function f, with $f : \mathbb{R}^n \to \mathbb{R}$. We will consider bounded functions of L_2 spaces.

There are two cases that must be kept in mind:

- Functions of compact support, which are spanned by the radial Jacobi polynomials. These are often associated with tomographic reconstruction. That is because they have finite support. For $\mathbb{R}^1 \times S^1$, these radial polynomials are the Zernike Polynomials.
- functions that are nonzero on $\mathbb{R}^n$, which are spanned by the radial Hermite wave polynomials. These polynomials require that amplitude and first derivatives of the amplitude be continuous. These functions are also solutions of the Schrödinger equation, as well as many classical radio equations.

The n-dimensional Radon Transform has several equivalent representations. We show two of them below:

$$\begin{aligned} g(\theta, s) &= \mathbf{R}f(\theta, s) \\ &= \int\limits_{\mathbf{x}\cdot\theta=\mathbf{s}} f(\mathbf{x})\, d^n\mathbf{x} \\ &= \int\limits_{\mathbb{R}^n} f(\mathbf{x})\, \delta(s - \mathbf{x}\cdot\theta)\, d^n\mathbf{x}. \end{aligned} \tag{29}$$

See [7].

A key theorem underlying the investigation of integral transforms is the Funk-Hecke Theorem [8]. We state it now for $n >= 2$.

Theorem 4.1. *Funk-Hecke Theorem: Let $F(x)$ be a function of the real variable x which is continuous for $-1 \le x \le 1$ and let $S_{\ell\mathbf{m}}(\xi)$ be a surface harmonic of degree ℓ. Then for any unit vector $\zeta \in S^{n-1}$*

$$\int\limits_{S^{n-1}} F(\xi\cdot\zeta)\, S_{\ell\mathbf{m}}(\xi)\, d\Omega^{n-1}(\xi) = \lambda_\ell S_{\ell\mathbf{m}}(\zeta), \tag{30}$$

where the integral in (30) is taken of the whole area of the unit hypersphere S^{n-1}, and where

$$\lambda_\ell = \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \int_{-1}^{1} F(t)\, C_\ell^{(n-2)/2}(t)\left(1-t^2\right)^{(n-3)/2} dt.$$

Here, ω' denotes the total area *of the unit hypersphere S^{n-2}, viz.*

$$\omega' = \left|S^{n-2}\right| = \frac{2\pi^{n/2-1/2}}{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}.$$

It is sufficient that $|F(x)|$ and $|F(x)|^2$ are Lebesgue-integrable for $-1 \le x \le 1$.

Proof. Since the $C_n^{p/2}(\zeta \cdot \xi)$ are a complete, orthogonal set on $S^{(n-1)}$ for ζ fixed, we can express F as a linear combination of these functions:

$$F(\xi \cdot \zeta) = \sum_{j=0}^{\infty} a_j C_j^{p/2}(\xi \cdot \zeta),$$

with

$$a_j = \frac{h(n,\ell)}{\omega\left[C_\ell^{p/2}(1)\right]^2}\delta_{\ell j} \int\limits_{S^{n-1}} F(\xi \cdot \zeta)\, C_\ell^{p/2}(\xi \cdot \zeta)\, d\Omega^{n-1}(\xi).$$

The function $h(n,\ell)$ is defined in [8, ch. 11].

The integral over S^{n-1} may be integrated using Cartesian coordinates

$$\zeta = [t, t_1, \ldots, t_{n-2}], \quad t^2 + t_1^2 + \cdots + t_{n-2}^2 = 1.$$

Because the dot product, along with the Jacobian of the Cartesian coordinate system, are both invariant under rotations, the integrand is also invariant under rotations. At most, only one of the parameters of $\mathbf{m}$ affect the harmonic transform of integrals over hyperplanes of dimension $n-1$. Only $\ell = m_0$ is needed. Consider an $n-1$-fold integral over S^{n-1}: Let R be a rotation in S^{n-1}. The invariance of the dot product under rotation implies that for some t, $|t| \le 1$,

$$R\zeta \cdot R\theta = \zeta \cdot \theta = t.$$

Let R be a rotation such that

$$\zeta' = R\zeta = [1, 0, 0, \cdots, 0];$$

then

$$\zeta \cdot \theta = \zeta' \cdot \theta' = t.$$

The integrand now depends only on the fundamental $\cos\psi_1{}' = t$ and r. This proves that the radial harmonics depend only on the fundamental ℓ, i. e., $F_{\ell\mathbf{m}} \equiv F_\ell$ and $z_{\ell\mathbf{m}} \equiv z_\ell$ are parameterized only by $\ell = m_0$.

We see that

$$\int_{S^{n-1}} F(\xi\cdot\zeta)\, S_{\ell\mathbf{m}}(\xi)\, d\Omega^{n-1}(\xi) = \frac{\omega' S_{\ell\mathbf{m}}(\zeta)}{C_\ell^{p/2}(1)} \int_{-1}^{1} F(t)\, C_\ell^{p/2}(t)\left(1-t^2\right)^{(p-1)/2} dt. \tag{31}$$

where we have used the result that

$$\int_{S^{n-1}} S_{\ell\mathbf{m}}(\xi)\, C_j^{p/2}(\xi\cdot\zeta)\, d\Omega^{n-1}(\xi) = \frac{\omega}{h(n,\ell)}\delta_{\ell j} C_\ell^{p/2}(1)\, S_{\ell\mathbf{m}}(\zeta),$$

From [8, *A. Erdelyi, et. al.*] □

4.3. An Application of the Funk-Hecke Theorem to the Radon Transform

Consider the Radon Transform in Cartesian coordinates, equation (29). We apply the theorem to obtain the harmonic form. Using representations (25), we see that

$$\begin{aligned} g(s,\theta) &= \sum_{\ell\mathbf{m}} g_\ell(s)\, S_{\ell\mathbf{m}}(\theta) \\ &= \int_{\mathbb{R}^n} f(\mathbf{x})\,\delta(s-\mathbf{x}\cdot\theta)\, d^n\mathbf{x} \\ &= \int_0^\infty \sum_{j\mathbf{m}'} f_j(r) \int_{S^{n-1}} S_{j\mathbf{m}'}(\zeta)\,\delta\left(\frac{s}{r}-\zeta\cdot\theta\right) d\Omega^{n-1}(\zeta) r^{n-2} dr \\ &= \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \sum_{j\mathbf{m}'} S_{j\mathbf{m}'}(\theta) \int_0^\infty f_j(r)\, C_j^{(n-2)/2}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)^{(n-3)/2} r^{n-2} dr. \end{aligned}$$

We have used the property that the δ function is homogeneous of order -1. Taking the inner product with $S_{\ell\mathbf{m}}(\theta)$ over $\Omega(\theta)$, and using (26), we obtain

$$g_\ell(s) = \frac{\omega'}{C_j^{p/2}(1)} \int_0^\infty f_\ell(r)\, C_\ell^{p/2}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)_+^{(n-3)/2} r^{n-2} dr \tag{32}$$

We make the observation that integrals over hyperplanes actually simplify the analysis of n-dimensional spaces. The two-dimensional circular harmonic transform of integrals over lines of S^1 captures all of the essential features of integrals over the hyperplanes of $\mathbb{R} \times S^{n-1}$.

If the integral is over hyperplanes of $\mathbb{R}^2 \times S^{n-1}$, (K-planes, [13]), then the result will depend on two parameters, and so on, down to integrals over $\mathbb{R}^{n-1} \times S^{n-1}$, the family of straight lines, a.k.a the X-Ray Transform, that depend upon $n-1$ parameters of $\mathbf{m}$.

5. The Stable Inverse

We begin with the inversion of the Radon Transform given in [18, Thm. 2.1] and show how this transforms to a stable harmonic inverse.

Theorem 5.1. *The Mellin convolution of the stable harmonic inverse of the Radon Transform is*

$$\begin{aligned} z_\ell(r) &= \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \frac{1}{r} \times \\ &\times \int_0^\infty \left\{ \begin{array}{c} (-1)^{(n-1)/2} \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \text{ odd} \\ (-1)^{(n-2)/2} H(s) * \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \text{ even} \end{array} \right\} \times \\ &\times C_\ell^{(n-2)/2}(s/r) \left(1 - \left(\frac{s}{r}\right)^2\right)_+^{(n-3)/2} ds. \end{aligned} \tag{33}$$

Proof. We derive the stable harmonic inverse from the Cartesian inverse in [18] for Euclidean spaces $\mathbb{R}^n$. We make the substitution $f(\mathbf{r}) = z_\ell(r) S_{\ell\mathbf{m}}(\theta)$ and $g(s,\theta) = g_\ell(s) S_{\ell\mathbf{m}}(\theta)$:

$$f(\mathbf{r}) = \frac{1}{2}(2\pi)^{1-n} \int_{S^{n-1}} \left[\begin{array}{c} (-1)^{(n-1)/2} \frac{\partial^{n-1}}{\partial s^{n-1}}, n \text{ odd} \\ (-1)^{(n-2)/2} \mathbf{H} \frac{\partial^{n-1}}{\partial s^{n-1}}, n \text{ even} \end{array} g(\theta, s) \right]_{s=\theta\cdot\mathbf{r}} d\Omega^{n-1}(\theta) \tag{34}$$

By use of the Funk-Hecke Theorem, the harmonic components satisfy

$$\begin{aligned} z_\ell(r) S_{\ell\mathbf{m}}(\omega) &= \lambda_{\ell\mathbf{m}} S_{\ell\mathbf{m}}(\omega), \\ \lambda_{\ell\mathbf{m}} &= \frac{1}{2}(2\pi)^{1-n} \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \\ &\times \int_{-1}^{1} \left\{ \begin{array}{c} (-1)^{(n-1)/2} \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \text{ odd} \\ (-1)^{(n-2)/2} H(s) * \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \text{ even} \end{array} \right|_{s=rt} C_\ell^{(n-2)/2}(t) \end{aligned}$$

$$
\begin{aligned}
& \left(1-t^2\right)^{(n-3)/2} dt \\
= \; & (2\pi)^{1-n} \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \frac{1}{r} \qquad (35) \\
\times \; & \int_0^\infty \left\{ \begin{array}{c} (-1)^{(n-1)/2} \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \; odd \\ (-1)^{(n-2)/2} H(s) * \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s), n \; even \end{array} \right. C_\ell^{(n-2)/2} \\
& \left(\frac{s}{r}\right) \left(1 - \frac{s^2}{r^2}\right)_+^{(n-3)/2} ds.
\end{aligned}
$$

The solution is stable because the kernel $\left| C_\ell^{(n-2)/2}(s/r)\left(1-s^2/r^2\right)_+^{(n-3)/2} \right|$ is bounded for $0 \leq s/r \leq 1$. □

6. The Unstable Inverse and its Mellin Transform

The inversion of integral equation (32) yields the unstable harmonic inverse [15]:

$$
z_\ell(r) = \frac{c_n}{C_\ell^{(n-2)/2}(1)} \frac{1}{r} \int_0^\infty g_\ell^{(n-1)}(s) C_\ell^{(n-2)/2}\left(\frac{s}{r}\right)\left(\frac{s^2}{r^2}-1\right)_+^{(n-3)/2} ds, \quad (36)
$$

where c_n is defined in [19, Thm. 2.7], and where, as usual,

$$
g_\ell^{(n-1)}(s) \triangleq \frac{\partial^{n-1}}{\partial s^{n-1}} g_\ell(s).
$$

We will later use the identity

$$
\frac{1}{2}(2\pi)^{1-n}\left|S^{n-2}\right| = \frac{(-1)^n}{2} c_n \qquad (37)
$$

This inverse is unstable because the argument of the kernel, $s/r > 1$, over the domain of integration. The ultra-spherical polynomials increase like $(s/r)^\ell$ as $s/r \to \infty$. So this solution is severely unstable. We take the Mellin Transform of the forward transform equation (32). The result is

$$
\tilde{g}_\ell(\mu) \;\; = \;\; \int_0^\infty g_\ell(s) s^{\mu-1} ds
$$

$$
\begin{aligned}
&= \frac{\omega'}{C_\ell^{(n-2)/2}(1)} \int_0^\infty s^{\mu-1} \int_0^\infty z_\ell(r)\, C_\ell^{n/2-1}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)_+^{(n-3)/2} r^{n-2}\,dr\,ds \\
&= \frac{\omega'\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)(n-2)_\ell}{2C_\ell^{(n-2)/2}(1)\,\ell!}\tilde{z}_\ell(\mu+n-1)\times \qquad (38)\\
&\times\ \Gamma\left[\begin{array}{cc} \frac{\mu}{2}, & \frac{\mu}{2}+\frac{1}{2} \\ \frac{\mu}{2}+\frac{n}{2}+\frac{\ell}{2}-\frac{1}{2}, & \frac{\mu}{2}-\frac{\ell}{2}+\frac{1}{2}\end{array}\right].
\end{aligned}
$$

([16], Pt. 2, §10, 47(1)). This Mellin Transform is stable because $0 \leq s/r \leq 1$. Hence, the ultra spherical polynomials are bounded in that interval and a numerical solution is feasible.

With the substitution $\mu \to \mu - n + 1$, we obtain

$$
\tilde{z}_\ell(\mu) = \tilde{g}_\ell(\mu-n+1)\frac{2C_\ell^{(n-2)/2}(1)\,\ell!}{\omega'\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)(n-2)_\ell}\Gamma\left[\begin{array}{cc} \frac{\mu}{2}+\frac{\ell}{2}, & \frac{\mu}{2}-\frac{n+\ell}{2}+1 \\ \frac{\mu-n+1}{2}, & \frac{\mu-n}{2}+1\end{array}\right) \qquad (39)
$$

We see immediately that this inversion is unstable, because

$$
\alpha = -\frac{n+\ell}{2}+1,
$$

and instability commences at $n+\ell > 4$.

This representation is not the most useful for another reason. Note that the Mellin Transform of $\tilde{g}_\ell(\mu-n+1)$ is

$$
\begin{aligned}
\tilde{g}_\ell(\mu-n+1) &= (-1)^{n-1}\Gamma\left[\begin{array}{c}\mu-n+1\\ \mu\end{array}\right]\left(\mathbf{M}\frac{\partial^{n-1}}{\partial s^{n-1}}g_\ell(s)\right)(\mu)\\
&= (-1)^{n-1}\frac{2^{\mu-n}\Gamma\left(\frac{\mu-n+1}{2}\right)\Gamma\left(\frac{\mu-n}{2}+1\right)}{2^{\mu-1}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}\left(\mathbf{M}\frac{\partial^{n-1}}{\partial s^{n-1}}g_\ell(s)\right)(\mu),
\end{aligned}
$$

where we have used the Gamma multiplication formula. We combine this last result with (39) and obtain for the Mellin Transform of the unstable inverse:

$$
\begin{aligned}
\tilde{z}_\ell(\mu) &= (-1)^{n-1}\frac{2^{-n+2}C_\ell^{(n-2)/2}(1)\ell!}{\omega'\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)(n-2)_\ell}\Gamma\left[\begin{array}{cc}\frac{\mu}{2}+\frac{\ell}{2}, & \frac{\mu}{2}-\frac{n+\ell}{2}+1\\ \frac{\mu}{2} & \frac{\mu+1}{2}\end{array}\right]. \qquad (40)\\
&\times\mathbf{M}\left[\frac{\partial^{n-1}}{\partial s^{n-1}}\tilde{g}_\ell(s)\right](\mu)
\end{aligned}
$$

The inversion contour and imaging parameters must satisfy

$$
\mathrm{Re}\,(n/2-1) \quad > \quad -1/2,
$$

$$\mathrm{Re}\frac{\mu}{2} \quad > \quad \frac{n+\ell}{2}-1, \quad \ell=0,1,2,\ldots$$

The inverse Mellin Transform exists because sufficient conditions (51), (52), and (55) are satisfied. The transform is unstable because $\alpha < -1$ for $n+\ell > 4$. By noting the following, that

$$\mathbf{M}\left[2t^{-1}\frac{\ell!}{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)(p)_\ell}C_\ell^{(p)/2}\left(t^{-1}\right)\left(\frac{1}{t^2}-1\right)_+^{(n-3)/2}\right](\mu)$$
$$=\Gamma\left[\begin{array}{cc}\frac{\mu}{2}+\frac{\ell}{2}, & \frac{\mu}{2}-\frac{n+\ell}{2}+1\\ \frac{\mu}{2} & \frac{\mu+1}{2}\end{array}\right].$$

[16, Pt. 2, §10, 48(1)], we see that this reduces to

$$z_\ell(r) = (-1)^{n-1}\frac{2^{-n+2}}{\omega'\Gamma^2\left(\frac{n}{2}-\frac{1}{2}\right)}\frac{1}{r^{n-2}}\times$$
$$\times\int_0^\infty \frac{\partial^{n-1}}{\partial s^{n-1}}g_\ell(s)\,C_\ell^{(n-2)/2}(s/r)\left(s^2-r^2\right)_+^{(n-3)/2}ds, \tag{41}$$

which agrees with [7]. The kernel is also unstable for $n+\ell>4$, since

$$C_\ell^{(n-2)/2}(s/r)\left(s^2-r^2\right)_+^{(n-3)/2} \to \infty \text{ as s} \to \infty.$$

7. The Derivation of the Mellin Transform of the Stable Inverse from the Mellin Transform of the Unstable Inverse

Theorem 7.1. *Test*

Theorem 7.2. *The stable and unstable inverses for the Radon Harmonic Transform for $n>2$ are equivalent.*

We begin with the Mellin Transform of the unstable inverse (40) with the goal of obtaining the Mellin Transform of a stable inverse.

$$\tilde{z}_\ell(\mu) = (-1)^{n-1}\frac{2^{-n+2}C_\ell^{(n-2)/2}(1)\ell!}{\omega'\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)(n-2)_\ell}\Gamma\left[\begin{array}{cc}\frac{\mu}{2}+\frac{\ell}{2}, & \frac{\mu}{2}-\frac{n+\ell}{2}+1\\ \frac{\mu}{2} & \frac{\mu+1}{2}\end{array}\right]$$
$$\times\mathbf{M}\left[\frac{\partial^{n-1}}{\partial s^{n-1}}\tilde{g}_\ell(s)\right](\mu).$$

We multiply/divide by a factor of one. This yields the stable inverse

$$\tilde{z}_\ell(\mu) = (-1)^{n-1}2^{-n+2}\frac{C_\ell^{(n-2)/2}(1)}{\omega'}\left[\Gamma\left(\frac{n-1}{2}\right)\frac{(n-2)_\ell}{\ell!}\right]^{-1}$$

$$\times \quad \Gamma\left[\begin{array}{cc} 1-\frac{\mu}{2}, & \frac{1-\mu}{2} \\ 1-\frac{\mu}{2}-\frac{\ell}{2} & \frac{n+\ell-\mu}{2} \end{array}\right]$$
$$\times \quad \frac{\sin\pi\frac{\mu}{2}\sin\pi\frac{\mu+1}{2}}{\sin\pi\frac{\mu+\ell}{2}\sin\pi\left(\frac{\mu}{2}-\frac{n+\ell}{2}+1\right)}\mathbf{M}\left[\frac{\partial^{n-1}}{\partial s^{n-1}}\tilde{g}_\ell(s)\right](\mu).$$

The trigonometric fraction combined with the factor $(-1)^{n-1}$ simplifies to

$$(-1)^{n-1}\frac{\sin\pi\frac{\mu}{2}\sin\pi\frac{\mu+1}{2}}{\sin\pi\frac{\mu+\ell}{2}\sin\pi\left(\frac{\mu}{2}-\frac{n+\ell}{2}+1\right)}= \tag{42}$$
$$= \quad (-1)^{n-1}\frac{\sin\pi\frac{\mu}{2}\cos\pi\frac{\mu}{2}}{\left(\sin\pi\frac{\mu}{2}\cos\pi\frac{\ell}{2}+\cos\pi\frac{\mu}{2}\sin\pi\frac{\ell}{2}\right)\left(\sin\pi\frac{\mu}{2}\cos\pi\left(\frac{-n-\ell+2}{2}\right)+\cos\pi\frac{\mu}{2}\sin\pi\left(\frac{-n-\ell+2}{2}\right)\right)}$$
$$= \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd}, \ell \text{ odd}, \\ (-1)^{(n-1)/2}, & n \text{ odd}, \ell \text{ even}, \\ (-1)^{(n-2)/2}\left(\tan\pi\frac{\mu}{2}\right), & n \text{ even}, \ell \text{ odd}, \\ (-1)^{(n-2)/2}\left(-\cot\pi\frac{\mu}{2}\right), & n \text{ even}, \ell \text{ even}. \end{cases}$$

Using the result for the gamma function fraction from Appendix 13, we see that from eqn. (4) the Mellin convolution variable is r/s with respect to s, so $x^{-1}=s/r$. Taking the inverse Mellin Transform, we obtain

$$z_\ell(r)=\frac{2^{-n+2}C_\ell^{(n-2)/2}(1)}{\omega'}\left[\frac{(n-2)_\ell\,\Gamma\left(\frac{n-1}{2}\right)}{\ell!}\right]^{-2}\times \tag{43}$$
$$\times\frac{1}{r}\int_0^\infty C_\ell^{(n-2)/2}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)_+^{(n-3)/2}$$
$$\left\{\begin{array}{l} (-1)^{(n-1)/2}\frac{\partial^{n-1}}{\partial s^{n-1}}\tilde{g}_\ell(s), n \text{ odd}, \\ (-1)^{(n-2)/2}\mathbf{H}\left(\frac{\partial^{n-1}}{\partial s^{n-1}}\tilde{g}_\ell(s)\right), n \text{ even} \end{array}\right. ds.$$

This inverse must be bounded because

$$\sum\nolimits_B(1/t)=O\left(|t|^{-1/2}\right)\to 0, \quad \text{as } t\to\infty,$$

as we can see from eqn. (42)

Using fact that

$$\frac{C_\ell^{(n-2)/2}(1)\,2^{-n+2}}{\omega'}\left[\frac{(n-2)_\ell\,\Gamma\left(\frac{n-1}{2}\right)}{\ell!}\right]^{-2}=\left|S^{n-2}\right|\frac{1}{2}(2\pi)^{1-n},$$

and the Funk-Hecht Theorem we show that

$$Z(\mathbf{x}) = \frac{1}{2}(2\pi)^{1-n}$$
$$\times \int\limits_{S^{n-1}} \left[\begin{array}{ll} (-1)^{(n-1)/2} \frac{\partial^{n-1}}{\partial s^{n-1}} \tilde{g}_\ell(s), & n \text{ odd}, \\ (-1)^{(n-2)/2} \mathbf{H}\left(\frac{\partial^{n-1}}{\partial} s^{n-1} \tilde{g}_\ell(s)\right), & n \text{ even} \end{array} \right]_{s=\mathbf{x}\cdot\zeta} d^{n-1}\zeta \qquad \square$$

We see that, for n even, the unstable inverse transform (40) is equal to (43) and also to (34). This proves the main result: The stable and unstable inverses for the Radon Harmonic Transform for $n > 2$ are equivalent.

8. An Unstable Forward Problem

As might be expected, a stable transform of a forward problem can possess an unstable representation. As an example, consider our harmonic Radon Transform. We start with eqn. (39):

$$\tilde{g}_\ell(\mu) = \frac{\omega' \Gamma\left(\frac{n}{2} - \frac{1}{2}\right)}{2} \tilde{z}_\ell(\mu + n - 1)\, \Gamma\left[\begin{array}{cc} \frac{\mu}{2}, & \frac{\mu}{2} + \frac{1}{2} \\ \frac{\mu}{2} + \frac{n}{2} + \frac{\ell}{2} - \frac{1}{2}, & \frac{\mu}{2} - \frac{\ell}{2} + \frac{1}{2} \end{array} \right].$$

We use the identity $\tilde{z}_\ell(\mu + n - 1) = \mathbf{M}\left[r^{n-1} z_\ell(r)\right](\mu)$ The kernel is of course the rational function of gamma functions. This kernel is stable, since by Slater's Theorem,

$$\begin{aligned} A &= 2, B = 0, \\ C &= 2, D = 0, \\ \eta &= 0 + \frac{1}{2} - \frac{n}{2} - \frac{\ell}{2} + \frac{1}{2} + \frac{\ell}{2} - \frac{1}{2} \\ &= \frac{1}{2} - \frac{n}{2}. \end{aligned}$$

The solution is of form (11) with $A = 2$ and $B = 0$.The parameter $\alpha = 0$. To find an equivalent unstable kernel, we see that

$$\Gamma\left[\begin{array}{cc} \frac{\mu}{2}, & \frac{\mu}{2} + \frac{1}{2} \\ \frac{\mu}{2} + \frac{n}{2} + \frac{\ell}{2} - \frac{1}{2}, & \frac{\mu}{2} - \frac{\ell}{2} + \frac{1}{2} \end{array} \right] = \frac{\sin(\pi \mu/2) \sin\left(\pi\left(\frac{\mu}{2} + \frac{1}{2}\right)\right)}{\sin\left(\pi\left(\frac{\mu+n+\ell-1}{2}\right)\right) \sin\left(\pi \frac{\mu-\ell+1}{2}\right)} \times$$
$$\times\ \Gamma\left[\begin{array}{cc} \frac{3-n-\ell-\mu}{2} & \frac{\ell+1-\mu}{2} \\ 1 - \frac{\mu}{2}, & \frac{1}{2} - \frac{\mu}{2}, \end{array} \right].$$

This kernel exists, because the inversion integral exists for interesting values of n, but it is unstable because

$$\beta = \frac{3-n-\ell}{2} < -1.$$

The trigonometric fraction evaluates to

$$\frac{\sin\left(\pi\frac{\mu}{2}\right)\sin\left(\pi\left(\frac{\mu}{2}+\frac{1}{2}\right)\right)}{\sin\left(\pi\frac{\mu+n+\ell-1}{2}\right)\sin\left(\pi\frac{\mu-\ell+1}{2}\right)} = \begin{cases} \frac{1}{(-1)^{(n-1)/2}}, & n \text{ odd}, \ell \text{ odd} \\ \frac{1}{(-1)^{(n-1)/2}}, & n \text{ odd}, \ell \text{ even} \\ \frac{-\cot\left(\pi\frac{\mu}{2}\right)}{(-1)^{(n-2)/2}}, & n \text{ even}, \ell \text{ odd} \\ \frac{\tan\left(\pi\frac{\mu}{2}\right)}{(-1)^{(n-2)/2}} & n \text{ even}, \ell \text{ even} \end{cases}$$

These are the appropriate Mellin-Hilbert operators for odd and even functions, respectively equations (17) and (15).

$$\mathbf{M}^{-1}\left[\Gamma\left[\begin{array}{cc} \frac{3-n-\ell-\mu}{2}, & \frac{\ell+1-\mu}{2} \\ 1-\frac{\mu}{2}, & \frac{1}{2}-\frac{\mu}{2} \end{array}\right]\right]\left(\frac{r}{s}\right) =$$
$$\frac{1}{2}\mathbf{M}^{-1}\left[\Gamma\left[\begin{array}{cc} \frac{3-n-\ell}{2}-\mu, & \frac{\ell+1}{2}-\mu \\ 1-\mu, & \frac{1}{2}-\mu \end{array}\right]\right]\left(\frac{r^2}{s^2}\right).$$

We see that $\lambda = \frac{n}{2} - 1$ in [16, pt. 2, §10, 48(1)]. The Mellin convolution of the forward problem can be written as

$$g_\ell(s) = \begin{cases} \frac{\omega^{\square} 2^{\lambda-1/2}\ell!}{4(2\lambda)_\ell} \int_0^\infty C_\ell^{(n-2)/2}\left(\frac{r}{s}\right)\left(\frac{r^2}{s^2}-1\right)_+^{\lambda-1/2} z_\ell(r)\, r^{n-2} dr, & n \text{ odd}, \\ \frac{\omega^{\square} 2^{\lambda-1/2}\ell!}{4(2\lambda)_\ell} \int_0^\infty C_\ell^{(n-2)/2}\left(\frac{r}{s}\right)\left(\frac{r^2}{s^2}-1\right)_+^{\lambda-1/2} \mathbf{H}[z_\ell](r)\, r^{n-2} dr, & n \text{ even}. \end{cases}$$

And we see immediately that the kernel is unbounded for $r/s > 1$. Thereby, we have generated an equivalent forward transform for which the harmonic form is exponentially unstable.

9. Application to Fox H-Functions

Definition 9.1. The Fox-H function may be thought of as the Mellin Transform of a two dimensional function

$$H\left[x, y; (\alpha, a, A)_m^n; L_s, L_t\right]$$
$$= \frac{1}{(2\pi i)^2} \int_{L_t}\int_{L_s} \frac{\prod_{j=1}^m \Gamma(\alpha_j + a_j s + A_j t)}{\prod_{j=m+1}^n \Gamma(\alpha_j + a_j s + A_j t)} x^s y^t ds dt$$

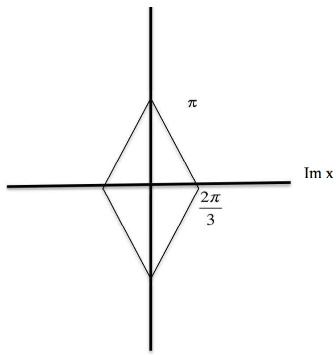

Figure 3: The region of convergence for the Fox-H function

or

$$\frac{\prod_{j=1}^{m}\Gamma\left(\alpha_j+a_j s+A_j t\right)}{\prod_{j=m+1}^{n}\Gamma\left(\alpha_j+a_j s+A_j t\right)}$$
$$=\int_0^\infty\int_0^\infty H\left[x,y;(\alpha,a,A)_m^n;L_s,L_t\right]x^{s-1}y^{t-1}dx\,dy$$

where $\alpha_j\in\mathbb{C},a_j,A_j\in\mathbb{R},j=1,2,...,n$; n,m are nonnegative integers with $m\leq n$. Here L_s and L_t are infinite contours in the complex s- and t- planes, respectively such that $\alpha_j+a_js+A_{j)t}\neq 0,-1,-2,...$ for $j=1,...,n$.

Consider the function

$$\theta\left(s,t\right)=\frac{\Gamma\left(1+s+t\right)\Gamma\left(1-4t\right)}{\Gamma\left(1+2s+3t\right)}\frac{\Gamma\left(2+s\right)\Gamma\left(1-3s\right)}{\Gamma\left(s\right)},$$

with $n=6$ and $m=4$ in definition (9.1). From [20, Consequence 2], the inverse exists in the region

$$\frac{3\left|\arg x\right|}{2}+\left|\arg y\right|<\pi. \tag{44}$$

. This region is a parallelogram, as is shown in figure (**??**):

There are several functionally equivalent transforms, but consider this one, where we have replaced $\Gamma\left(2+s\right)/\Gamma\left(s\right)$ with $\Gamma\left(1-s\right)/\Gamma\left(-1-s\right)$:

$$\theta\left(s,t\right)=\frac{\Gamma\left(1+s+t\right)\Gamma\left(1-4t\right)}{\Gamma\left(1+2s+3t\right)}\frac{\Gamma\left(1-s\right)\Gamma\left(1-3s\right)}{\Gamma\left(-1-s\right)}.$$

Although the inverse is formally a different Fox H-function, it exists and converges in the same region as (44), as shown in figure **??**. Therefore, care must be taken to select the inversion that best conforms to the physical setting.

10. Concluding Remarks

We have found the conditions by which a Mellin inversion is stable or unstable. Although the inverse of a linear system, if it exists, must be unique, uniqueness is required only in a formal or analytical sense. The numerical properties of these inverses may differ radically, as we have demonstrated with the Radon Transform. Also, for a given forward problem, there may exist mathematically equivalent forms that have radically different numerical properties.

Although the details of creating functionally equivalent transforms can be time-consuming, the implementation is often straightforward in Fourier or Mellin frequency space by repeated use of the identity, eqn. (17). We stress that a seemingly intractable problem can have a simple remedy.

The stabilization of the inverse is a very effective, if not the most effective way of regularizing or tempering ill-posed problems. If the researcher's investigation leads to an unstable solution, the methods outlined here potentially provide an alternative. Many unstable inverse problems of this type should be revisited in order to determine if a stable solution exists. Different equivalent inverses with different properties of numerical stability will also result.

The problem of adding a term equal to zero to find an equivalent inverse has not been fully investigated. For the Radon Transform, at least, the functions spanning the "ghosts" has been found by A. Louis [15]. For the Radon Transform of dimension two, the ghost functions are the Chebychev polynomials of the 2nd kind and these have been extensively studied [4], [7], and [10].

11. Acknowledgments

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12. Appendix A: Slater's Theorem and Notation

We begin with the definition of the Gamma function, which is also the Mellin Transform of e^{-x}:

$$\Gamma(s) \triangleq \int_0^\infty e^{-x} x^{s-1} dx$$

From this definition and integration by parts, we obtain the well-known *Gamma reduction formula* $\Gamma(s+1) = s\Gamma(s)$. This leads to the general formula and the definition of *Pochammer's symbol*, $(s)_k$:

$$\Gamma(s+k) = (s)_k \Gamma(s), \quad (s)_k = s(s+1)\dots(s+k-1),$$

By setting $s = 1$ we obtain $\Gamma(1) = 1$ and

$$\Gamma(k+1) = k! = (1)_k .$$

Next, we define the generalized hypergeometric functions; they are functions represented by series whose coefficients contain the expressions $(a)_k$.

$$\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!} = {}_pF_q((a),(b);z), \tag{45}$$

$$(a) = a_1, a_2, \dots, a_p, \quad (b) = b_1, b_2, \dots, b_q.$$

We shall need to define some additional symbols:

$$\begin{aligned} \Gamma\left[\begin{array}{c} a_1, a_2, \dots, a_A \\ b_1, b_2, \dots, b_B \end{array}\right] &\equiv \Gamma[(a);(b)] \\ &= \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_B)}, \end{aligned} \tag{46}$$

$$\begin{aligned} (a)+s &= a_1+s, a_2+s, \dots a_A+s, \\ (b)'-b_k &= b_1-b_k, b_2-b_k, \dots, b_{k-1}-b_k, b_{k+1}-b_k, \dots b_B-b_k, \end{aligned} \tag{47}$$

$$\sum\nolimits_A (z) = \sum_{j=1}^{A} z^{a_j} \Gamma\left[\begin{array}{cc} (a)'-a_j, & (b)+a_j \times \\ (c)-a_j, & (d)+a_j \end{array}\right]$$

$$\times \quad {}_{B+C}F_{A+D-1}\left(\begin{array}{c} (b)+a_j, \quad 1+a_j-(c)\,; \quad (-1)^{C-A}\,z \\ 1+a_j-(a)', \quad (d)+a_j \end{array} \right), \tag{48}$$

$$\begin{aligned} \sum\nolimits_B (1/z) \quad &= \quad \sum_{k=1}^{B} z^{b_k}\Gamma\left[\begin{array}{cc} (b)'-b_k, & (a)+b_k \\ (d)-b_k, & (d)+b_k \end{array} \right] \times \\ &\times \quad {}_{A+D}F_{B+C-1}\left(\begin{array}{c} (a)+b_k, \quad 1+b_k-(d)\,; \quad \frac{(-1)^{D-B}}{z} \\ 1+b_k-(b)', \quad (c)+b_k \end{array} \right), \end{aligned} \tag{49}$$

$$|\arg z| < \pi.$$

Theorem 12.1. *Slater's Theorem*
The Mellin Transform of a sum of generalized hypergeometric functions (45) represented as a rational fraction of Gamma functions. Let $\sum_A (z)$ *or* $\sum_B (1/z)$, $z \in \mathbb{C}$ *be hypergeometric functions as defined in Appendix (12). See (46), (48), (49). Then the Mellin Transform is*

$$\tilde{K}(\mu) = \Gamma\left[\begin{array}{cc} (a)+\mu, & (b)-\mu \\ (c)+\mu, & (d)-\mu \end{array} \right] \tag{50}$$

where the vectors (a), (b), (c) *and* (d) *have, respectively, A,B,C and D components* a_j, b_k, c_l, d_m. *Let*

$$\eta = \sum_{j=1}^{A} a_j + \sum_{k=1}^{B} b_k - \sum_{l=1}^{C} c_l - \sum_{m=1}^{D} d_m.$$

Then if the following two groups of conditions hold:

$$-\mathrm{Re}\,(a_j) < \mathrm{Re}\,(\mu) < \mathrm{Re}\,(b_k) \quad (j = 1,2,...,A, \quad k = 1,2,...,B)\,, \tag{51}$$

$$\begin{aligned} &A+B > C+D, \\ &A+B = D+C,\ \mathrm{Re}\,(\mu)\,(A+D-B-C) < -\mathrm{Re}\,(\eta)\,, \\ &A = C,\ B = D, \mathrm{Re}\,(\eta) < 0, \end{aligned} \tag{52}$$

then for these μ we have

$$\tilde{K}(\mu) = \begin{cases} \int\limits_0^\infty x^{\mu-1} \sum_A (x) dx & \text{if } A+D > B+C \\ \int\limits_0^1 x^{\mu-1} \sum_A (x) dx + \int\limits_1^\infty x^{\mu-1} \sum_B (x) dx & \text{if } A+D = B+C \\ \int\limits_0^\infty x^{\mu-1} \sum_B (x) dx & \text{if } A+D < B+C \end{cases} \quad (53)$$

When $A+D \geq B+C$ and when the closure of the contour is taken as a semi-circle to the right, i. e., $L_{i\infty}$, the absolute convergence of the integral is preserved on the sector

$$0 \leq |\arg z| < \pi/2\,(A+B-C-D)\,, \quad (54)$$

provided that

$$\gamma\,(A+D-B-C) < -1 - \operatorname{Re}(\eta) + \frac{1}{2}\,(A+B-C-D)\,.$$

Condition (51) can be weakened if the poles of the numerator cancel with zeros coming from the denominator, i. e. if $a_j + m = c_l, m \in \mathbb{Z}$, then it may be dropped from list of constraints (51). Thus we have for these Mellin-Barns Integrals, the following list of solutions:

$$K(z) = \begin{cases} \sum_A (z), & A+D > B+C, \quad L = L_{-\infty}, \quad |z| < \infty, \\ & A+D = B+C, \quad L = L_{-i\infty}, \quad |z| < 1, \\ \sum_B (1/z), & A+D < B+C, \quad L = L_{+\infty}, \quad |z| < \infty, \\ & A+D = B+C, \quad L = L_{+i\infty}, \quad |z| > 1, \end{cases} \quad (55)$$

For $A+D = B+C$ if in addition,

$$\operatorname{Re}(\eta) + C - A + 1 < 0, \quad (56)$$

then

$$\sum\nolimits_A (1) = \sum\nolimits_B (1) \quad (57)$$

The functions $\sum_A (z)$ and $\sum_B (1/z)$ are not analytic continuations of each other. See [16, §4] for conditions that ensure analytic continuation. For $A+B = C+D$ and $A+D = B+C$, then $A = D$ and $B = C$, the inversion of the Mellin Transform must be restricted to the x-axis

$$\mathbf{K}(x) = \begin{cases} \sum_A (x), & 0 \leq x \leq 1 \\ \sum_B (1/z), & 1 \leq x < \infty \end{cases}, \quad x = \operatorname{Re}(z) \quad (58)$$

In general, $\sum_A (z)$ and $\sum_B (1/z)$ are multivalued: We restrict $|\arg z| < \pi$ so that $\sum_A (z)$ and $\sum_B (1/z)$ are not multi-valued. .

13. Appendix B: Expansion of the Gamma Function Fraction in Equation

We apply a series of substitutions, $\mu' = \mu/2, \to \mu' = \mu - 1/2, \to \mu' = -\mu$. From (51) the initial value of γ is $\gamma < \frac{1}{2}$. We have

$$\mu' = \mu/2, \to$$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left[\begin{array}{cc} 1-\frac{\mu}{2}, & \frac{1-\mu}{2} \\ 1-\frac{\mu}{2}-\frac{\ell}{2}, & \frac{n+\ell-\mu}{2}\end{array}\right] x^{-\mu} d\mu notag$$

$$= \frac{2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left[\begin{array}{cc} 1-\mu, & \frac{1}{2}-\mu \\ 1-\mu-\frac{\ell}{2} & \frac{n+\ell}{2}-\mu\end{array}\right]\left(x^2\right)^{-\mu} d\mu,$$

$$\mu' = \mu - 1/2, \to \tag{59}$$

$$= \frac{2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left[\begin{array}{cc} \frac{1}{2}-\mu, & -\mu \\ -\mu+\frac{1}{2}-\frac{\ell}{2}, & \frac{n+\ell-1}{2}-\mu\end{array}\right]\left(x^2\right)^{-\mu-1/2} d\mu,$$

$$\mu' = -\mu, \to$$

$$= \frac{2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left[\begin{array}{cc} \mu, & \frac{1}{2}+\mu \\ \mu+\frac{1-\ell}{2}, & \frac{n+\ell-1}{2}+\mu\end{array}\right]\left(x^2\right)^{\mu-1/2} d\mu$$

$$= \frac{2}{2\pi i}x^{-1}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left[\begin{array}{cc} \mu, & \frac{1}{2}+\mu \\ \mu+\frac{1-\ell}{2}, & \frac{n+\ell-1}{2}+\mu\end{array}\right]\left(x^{-2}\right)^{-\mu} d\mu$$

$$= 2\frac{\ell!}{\Gamma\left(\frac{n-1}{2}\right)(n-2)_\ell}x^{-1}C_\ell^{(n-2)/2}\left(\frac{1}{x}\right)\left(1-x^{-2}\right)_+^{(n-3)/2}.$$

[16, pt. 2, §10, 47(1)] The final value of γ is $\gamma > -\frac{1}{2}$.

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