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NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH GENERALIZED KROPINA METRIC

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Abstract: The purpose of present paper to determine the Finsler spaces due to deformation of generalized Kropina metric. Consequently, we obtain the non-holonomic frame for generalized Kropina metric with condition $n \geq 1$, such as $L(\alpha, \beta) = \frac{\beta^{(n+1)}}{\alpha^n}$.

AMS Subject Classification: 53C60

Key Words: Finsler space with (α, β)-metrics, generalized Kropina metric, GL-metric, nonholonomic Finsler frame

1. Introduction

In 1982, P.R. Holland [1], [2], studies a unified formalism that uses a nonholonomic frame on space time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden [3] was the first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space. The author R.G. Beil [5], [6], have studied a gauge transformation viewed as a nonholonomic frame

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on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries. In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M.

In the present paper, the fundamental tensor field might be taught as the result of two Finsler deformations. Then we can determine a corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with Generalized Kropina Metric and discuss two special cases when $n = 1$ and $n = 2$. This is an extension work of Ioan Bucataru and Radu Miron [10], Tripathi [14] and Narasimhamurthy [15].

Consider, $a_{ij}(x)$ the components of a Riemannian metric on the base manifold M, $a\,(x, y) > 0$ and $b\,(x, y) \geq 0$ Two functions on TM and $B\,(x, y) = B_i\,(x, y)\,(dx^i)$ a vertical 1-form on TM. Then

$$g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x, y)B_j(x, y) \tag{1.1}$$

is a generalized Lagrange metric, called the Beil metric. The metric tensor g_{ij} is also known as a Beil deformation of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in General Relativity for $a(x, y) = \exp(2\sigma(x, y))$ and b = 0. The case a(x, y) = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6].

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with(α,β)-metrics [11]. The first Finsler spaces with (α,β)–metrics were introduced by the physicist G. Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested a more general case and considered the class of Lagrange spaces with (α,β)–metric, which was discussed in [12]. A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland [1], [2]. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a Lagrange space with (α,β)-metric. Next we defined some different Finsler space with (α,β)-metrics.

Definition 2.1. A Finsler space $F^n = \{M, F(x, y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the

Finsler metric $F : TM \to R$ is given by

$$F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\} \tag{2.1}$$

where $\alpha^2(x, y) = a_{ij}(x)y^i y^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M

Consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space(M,F). Taking into account the homogeneity of a and F we have the following formulae:

$$\begin{gathered}
p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial\alpha}{\partial y^j}; \quad p_i = a_{ij}p^j = \frac{\partial\alpha}{\partial y^i};\\
l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^i}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i \\
l^i = \frac{1}{L}p^i; l^i l_i = p^i p_i = 1; l^i p_i = \frac{\alpha}{L}; p^i l_i = \frac{L}{\alpha};\\
b_i P^i = \frac{\beta}{\alpha}; b_i l^i = \frac{\beta}{L}
\end{gathered} \tag{2.2}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x, y) = \frac{L}{\alpha}a_{ij} + b_i P_j + P_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j \tag{2.3}$$

Theorem 2.1. [10]*: For a Finsler space (M,F) consider the matrix with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j) \tag{2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i})$, $\quad j \in 1, 2, 3, \ldots, n$ *is a non holonomic frame.*

Theorem 2.2. [7]*: With respect to frame the holonomic components of the Finsler metric tensor* $a_{\alpha\beta}$ *is the Randers metric* g_{ij}*, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)–metric $F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}$ we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial\alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial\beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}) \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have,

$$\rho_{-1}\beta+\rho_{-2}\alpha^2=0 \tag{2.7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [13].

$$g_{ij}(x,y)=\rho a_{ij}(x)+\rho_0 b_i(x)+\rho_{-1}\{b_i(x)y_j+b_j(x)y_i\}+\rho_{-2}y_iy_j \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij}\rightarrow h_{ij}=\rho a_{ij}+\frac{1}{\rho_{-2}}(\rho_{-1}b_i+\rho_{-2}y_i)(\rho_{-1}b_j+\rho_{-2}y_j)$$

$$II. \qquad h_{ij}\rightarrow g_{ij}=h_{ij}+\frac{1}{\rho_{-2}}(\rho_0\rho_{-1}-\rho_{-1}^2)b_ib_j \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the theorem (7.9.1) in [10], given by,

$$X_j^i=\sqrt{\rho}\delta_j^i-\frac{1}{\beta^2}\{\sqrt{\rho}+\sqrt{\rho+\frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i+\rho_{-2}y^i)(\rho_{-1}b_j+\rho_{-2}y_j) \tag{2.10}$$

where $B^2=a_{ij}(\rho_{-1}b^i+\rho_{-2}y^i)(\rho_{-1}b_j+\rho_{-2}y_j)=\rho_{-1}^2b^2+\beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij}=X_i^kX_j^la_{kl} \tag{2.11}$$

Again the frame that corresponds to the II_{nd} deformation (2.9) given by,

$$Y_j^i=\delta_j^i-\frac{1}{C^2}\{1\pm\sqrt{1+\left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2}-\rho_{-1}^2}\right)}\}b^ib_j \tag{2.12}$$

where $C^2=h_{ij}b^ib^j=\rho b^2+\frac{1}{\rho_{-2}}(\rho_{-1}b^2+\rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn}=Y_m^iY_n^jh_{ij} \tag{2.13}$$

Theorem 2.3. [10]: *Let $F^2(x,y)=L\{\alpha(x,y),\beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition (2.7) is true. Then*

$$V_j^i=X_k^iY_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

3. Nonholonmic Frames for Finsler Space with Generalized Kropina Metric

In this section we consider the case of nonholonomic Finlser frame with Generalized Kropina metric $L(\alpha,\beta) = \frac{\beta^{2n+2}}{\alpha^{2n}}$

$$\rho_1 = -\frac{n\beta^{2n+2}}{\alpha^{2n+2}}, \quad \rho_0 = \frac{(n+1)(2n+1)\beta^{2n}}{\alpha^{2n}}, \tag{3.1}$$
$$\rho_{-1} = -\frac{2n(n+1)\beta^{2n+1}}{\alpha^{2n+2}}, \quad \rho_{-2} = \frac{2n(n+1)\beta^{2n+2}}{\alpha^{2n+4}},$$
$$B^2 = \frac{4n^2(n+1)^2\beta^{4n+2}(\alpha^2 b^2 - \beta^2)}{\alpha^(4n+6)}$$

Using (3.1) in (2.10) we have,

$$X^i_j = \sqrt{-n}\frac{\beta^{n+1}}{\alpha^{n+1}}\delta^i_j - 4n^2(n+1)^2\frac{\beta^{4n}}{\alpha^{4n+4}}\{\sqrt{-n}\frac{\beta^{n+1}}{\alpha^{n+1}} + \sqrt{(-n\frac{\beta^{2n+2}}{\alpha^{2n+2}} + \frac{\alpha^{2n+4}}{2n(n+1)\beta^{2n}})}\}(b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{3.2}$$

Again using (3.1) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{\alpha^{2n}C^2}{(n+1)\beta^{2n}}}\}b^i b_j \tag{3.3}$$

where $C^2 = -\frac{n\beta^{2n+2}}{\alpha^{2n+2}}b^2 + \frac{2n(n+1)\beta^{2n}}{\alpha^{2n+4}}(\alpha^2 b^2 - \beta^2)^2$

Theorem 3.1. *Let $F^2(x,y) = \frac{\beta^{2n+2}}{\alpha^{2n}}$ be the metric function of a Finsler space with generalized Kropina metric, for which the condition (2.7) is true, Then*

$$V^i_j = X^i_k Y^k_j$$

is non holomic Finsler Frame with X^i_k and Y^k_j are given by (3.2) and (3.3) respectively.

Case I: Consider $n = 1$ the case of nonholonomic Finlser frame with Kropina Metric $L(\alpha,\beta) = \frac{\beta^4}{\alpha^2}$

$$\rho_1 = -\frac{\beta^4}{\alpha^4} \quad \rho_0 = \frac{6\beta^2}{\alpha^2}, \quad \rho_{-1} = -\frac{4\beta^3}{\alpha^4}, \quad \rho_{-2} = \frac{4\beta^4}{\alpha^6}, \tag{3.4}$$

$$B^2 = \frac{16\beta^6(\alpha^2 b^2 - \beta^2)}{\alpha^{10}}$$

Using (3.4) in (2.10) we have,

$$X_j^i = \sqrt{-1}\frac{\beta^2}{\alpha^2}\delta_j^i - 16\frac{\beta^4}{\alpha^8}\{\sqrt{-1}\frac{\beta^2}{\alpha^2} + \sqrt{(-1\frac{\beta^4}{\alpha^4} + \frac{\alpha^6}{4\beta^2})}\}(b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{3.5}$$

Again using (3.4) in (2.12) we have

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{\alpha^2 C^2}{2\beta^2}}\}b^i b_j \tag{3.6}$$

where $C^2 = -\frac{\beta^4}{\alpha^4}b^2 + \frac{4\beta^2}{\alpha^6}(\alpha^2 b^2 - \beta^2)^2$

Corollary 3.1. *Let $F^2(x,y) = \frac{\beta^4}{\alpha^2}$ be the metric function of a Finsler space with special generalized Kropina metric, for which the condition (2.7) is true, Then*

$$V_j^i = X_k^i Y_j^k$$

is non holomic Finsler Frame with X_k^i and Y_j^k are given by (3.5) and (3.6) respectively.

Case II: Consider $n = 2$ the case of nonholonomic Finlser frame with Kropina Metric $L(\alpha,\beta) = \frac{\beta^6}{\alpha^4}$

$$\rho_1 = -\frac{\beta^6}{\alpha^6} \quad \rho_0 = \frac{10\beta^4}{\alpha^4}, \quad \rho_{-1} = -\frac{12\beta^5}{\alpha^6}, \quad \rho_{-2} = \frac{8\beta^6}{\alpha^8}, \tag{3.7}$$

$$B^2 = \frac{144\beta^{10}(\alpha^2 b^2 - \beta^2)}{\alpha^{14}}$$

Using (3.7) in (2.10) we have,

$$X_j^i = \sqrt{-2}\frac{\beta^3}{\alpha^3}\delta_j^i - 144\frac{\beta^8}{\alpha^{12}}\{\sqrt{-2}\frac{\beta^3}{\alpha^3} + \sqrt{(-2\frac{\beta^6}{\alpha^6} + \frac{\alpha^8}{12\beta^4})}\}(b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \tag{3.8}$$

Again using (3.7) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{\alpha^4 C^2}{3\beta^4}}\}b^i b_j \tag{3.9}$$

where $C^2 = -\frac{2\beta^6}{\alpha^6} b^2 + \frac{12\beta^4}{\alpha^8} (\alpha^2 b^2 - \beta^2)^2$.

Corollary 3.2. *Let $F^2(x, y) = \frac{\beta^6}{\alpha^4}$ be the metric function of a Finsler space with special generalized Kropina metric, for which the condition (2.7) is true, Then*

$$V_j^i = X_k^i Y_j^k$$

is non holomic Finsler Frame with X_k^i and Y_j^k are given by (3.8) and (3.9) respectively.

4. Conclusions

Non-holonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru (see [7], [8]), has been determined such a non-holonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with(α,β)–metric, it appears a natural question: Does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α, β)-metrics.

In this work, we consider the Generalized Kropina Finsler metrics and we determine the nonholonomic Finsler frames. Further we discuss two cases for n = 1 and n = 2 in Generalized Kropina Finsler metrics and we determine also the non holonomic Finsler frames for them.we found here induces a Finsler connection on TM with torsion and no curvature. But, in Finsler geometry, there are many (α,β)-metrics, in future work we can determine the frames for them also.

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