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DOUBLE POWER METHOD ITERATION FOR PARALLEL EIGENVALUE PROBLEM

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Abstract: In this paper, we introduce the double power iteration method witch can be seen as an extension of the classical power iteration in the sense that we calculate the two dominants eigenvalues at each stage. This work aims to propose a solution of slow convergence problem to the power iteration method and the calculation of the second dominant eigenvalue. We develop a parallel iterative procedure for the calculation of eigenvalues of a given matrix and we can expressed this method as a Quadrant Interlocking Factorization (QIF) which introduced by [7] and studied by [8]-[9] in other works.

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1. Introduction

Although the power iteration method approximates only one eigenvalue of a matrix, it remains useful for certain computational problems. For instance, Google uses it to calculate the PageRank of documents in their search engine [1], and Twitter uses it to show users recommendations of who to follow [2].

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For matrices that are well-conditioned and as sparse as the Web matrix, the power iteration method can be more efficient than other methods of finding the dominant eigenvector. But when the two dominants eigenvalues are of approximately the same magnitude this method may converge slowly or fail.

The central to the field of matrix computations is the problem of solving a system of linear equations $Ax = b$ and the numerical solutions of the eigenvalues and the corresponding eigenvectors of a large and dense matrix plays an important role in numerous scientific applications. Rutishauser [5] proposed the LR algorithm for the calculation of the eigenvalues of a A. In this procedure, we obtain a sequence of matrices $A^{(1)}$, $A^{(2)}$, . . . which in general reduces to an upper triangular matrix. The most popular methods developed to solve this problem after [5] by factorization such as the QR algorithm, the Givens method, the Housholder transformation [10] [11] and by projection techniques on appropriate subspaces such as Lanczos and Davidson methods [13] [12].

In this paper, we present a double power iteration as iterative procedure to calculate the two dominants eigenvalues at each stage. This procedure still a variants of LR algorithm and the eigenvalues are obtained from simple 2×2 matrices derived from the main and cross diagonals of the limit matrix. So we can compute the eigenvalues of this limit matrix in parallel. We can express our approach as a Quadrant Interlocking Factorization and then we can give the solution of the linear equation $Ax = b$.

The paper is organized as follows. We start by giving mathematical framework for double power iteration algorithm for computing the two dominant eigenelements of a matrix A. Next, we prove the convergence of the algorithm. Finally, we factorize A in the form like those in [7] to evaluate all eigenvalues and we give an easy way to the solution of linear equation.

2. Double Power Iteration Method

Let $A \in \mathcal{L}(\mathbb{R}^n)$ be a real matrix, I the identity matrix and $\{e_i\}_{1\leq i\leq n}$ the standard basis of $\mathbb{R}^n$. We denote by u^t the transpose of vector u , B^T the transpose of matrix B and by δ_{ij} for $i, j \in \{1, ..., n\}$, the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

The proposed double power iteration method is the solution of the following problem:

Find x, y, z and t the real elements such that: for u and v two vectors on $\mathbb{R}^n$, we have

$$\begin{cases} Au = xu + yv \\ Av = zu + tv \end{cases} \tag{1}$$

This problem is equivalent to transform

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,n} \\ \vdots & \ddots & & \cdot^{\cdot^{\cdot}} & \vdots \\ a_{i,1} & & a_{i,i} & & a_{i,n} \\ \vdots & \cdot^{\cdot^{\cdot}} & & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,n} \end{pmatrix}$$

into

$$A^{(1)} = \begin{pmatrix} a^{(1)}_{1,1} & \cdots & a^{(1)}_{1,i} & \cdots & a^{(1)}_{1,n} \\ 0 & \ddots & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & a^{(1)}_{i,i} & & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & & \ddots & 0 \\ a^{(1)}_{n,1} & \cdots & a^{(1)}_{n,i} & \cdots & a^{(1)}_{n,n} \end{pmatrix} \tag{2}$$

Let define a matrix Q applied for this transformation by
$Q = I + u.e_1^t + v.e_n^t$ where $u_1 = u_n = 0$ and $v_1 = v_n = 0$.

Then $Q^{-1} = I - u.e_1^t - v.e_n^t$.
So we write $A^{(1)}$ in the form $A^{(1)} = Q^{-1}AQ$.

Then for $1 \leq i, j \leq n$, we have:

$$a^{(1)}_{i,j} =< A^{(1)}e_j, e_i >=< Q^{-1}AQe_j, e_i >=< AQe_j, \left(Q^{-1}\right)^T e_i >$$

$$=< A(I + u.e_1^t + v.e_n^t)e_j, (I - u.e_1^t - v.e_n^t)e_i >$$

$$=< A(e_j + \delta_{1j}u + \delta_{1n}v), e_i - u_ie_1 - v_ie_n >$$

$$= a_{i,j} - u_ia_{1,j} - v_ia_{n,j} + \delta_{1j}(< Au, e_i > -u_i < Au, e_1 > -v_n < Au, e_n >)$$

$$+\delta_{nj}(< Av, e_i > -u_i < Av, e_1 > -v_i < Av, e_i >)$$

By (2.2), we have

$$\forall i, 1 < i < n \; a^{(1)}_{i,1} = a^{(1)}_{i,n} = 0$$

then, for $j = 1$ and $j = n$, we get the following system:

$$(E)\begin{cases} a_{i,1} - u_i a_{1,1} - v_i a_{n,1} + < Au, e_i > - u_i < Au, e_1 > - v_i < Au, e_n >= 0 \\ a_{i,n} - u_i a_{1,n} - v_i a_{n,n} + < Av, e_i > - u_i < Av, e_1 > - v_i < Av, e_i >= 0 \end{cases}$$

$$(E)\begin{cases} u_i(a_{1,1} + < Au, e_1 >) + v_i(a_{n,1} + < Au, e_n >) = a_{i,1} + < Au, e_i > \\ u_i(a_{1,n} + < Av, e_1 >) + v_i(a_{n,n} + < Av, e_n >) = a_{i,n} + < Av, e_i > \end{cases}$$

Let:

$$\bar{u} = u + e_1$$
$$\bar{v} = v + e_n$$

Then $\forall i, 1 \leq i \leq n$, we have:

$$(E)\begin{cases} \bar{u}_i < A\bar{u}, e_1 > + \bar{v}_i < A\bar{u}, e_n >=< A\bar{u}, e_i > \\ \bar{u}_i < A\bar{v}, e_1 >) + \bar{v}_i < A\bar{v}, e_n >=< A\bar{v}, e_i > \end{cases}$$

so

$$(E)\begin{cases} \bar{u} < A\bar{u}, e_1 > + \bar{v} < A\bar{u}, e_n >= A\bar{u} \\ \bar{u} < A\bar{v}, e_1 >) + \bar{v} < A\bar{v}, e_n >= A\bar{v} \end{cases}$$

Assume that the matrix A is diagonalizable, then there exist a basis $\{z_i\}_{1 \leq i \leq n}$ formed by the eigenvectors of A such that:

$$\bar{u} = \sum_{i=1}^{n} \alpha_i z_i$$
$$\bar{v} = \sum_{i=1}^{n} \beta_i z_i$$

We get then,

$$(E)\begin{cases} \sum_{i=1}^{n} \alpha_i z_i < A\bar{u}, e_1 > + \sum_{i=1}^{n} \beta_i z_i < A\bar{u}, e_n >= A\sum_{i=1}^{n} \alpha_i z_i = \sum_{i=1}^{n} \alpha_i \lambda_i z_i \\ \sum_{i=1}^{n} \alpha_i z_i < A\bar{v}, e_1 >) + \sum_{i=1}^{n} \beta_i z_i < A\bar{v}, e_n >= A\sum_{i=1}^{n} \beta_i z_i = \sum_{i=1}^{n} \beta_i \lambda_i z_i \end{cases}$$

$$(E)\begin{cases} \alpha_i < A\bar{u}, e_1 > + \beta_i < A\bar{u}, e_n >= \alpha_i \lambda_i \\ \alpha_i < A\bar{v}, e_1 >) + \beta_i < A\bar{v}, e_n >= \beta_i \lambda_i \end{cases}$$

Under suitable conditions on A, the solution of this system can be obtained using iterative method like Jacobi or Gauss-Seidel method as follow:

Algorithm for solving the system:

For $p = 0, 1, \ldots$ until convergence solve the following i set of 2×2 linear systems

$$(E_p)\begin{cases} \alpha_i^{p+1} < A\bar{u}^p, e_1 > + \beta_i^{p+1} < A\bar{u}^p, e_n >= \alpha_i^p \lambda_i \\ \alpha_i^{p+1} < A\bar{v}^p, e_1 > + \beta_i^{p+1} < A\bar{v}^p, e_n >= \beta_i^p \lambda_i \end{cases}$$

Convergence analysis:

(E_p) is a $2x2$ system equation who have a simple expression as follow:

$$\Delta_p =< A\bar{u}^p, e_1 >< A\bar{v}^p, e_n > - < A\bar{u}^p, e_n >< A\bar{v}^p, e_1 >$$

is independent of the index i and can be computed only once. Then

$$\begin{cases} \alpha_i^{p+1} = \frac{1}{\Delta_p} \begin{vmatrix} \alpha_i^p \lambda_i & < A\bar{u}^p, e_n > \\ \beta_i^p \lambda_i & < A\bar{v}^p, e_n > \end{vmatrix} & = \frac{\lambda_i}{\Delta_p} \left[\alpha_i^p < A\bar{v}^p, e_n > -\beta_i^p < A\bar{u}^p, e_n > \right] \\ \beta_i^{p+1} = \frac{1}{\Delta_p} \begin{vmatrix} < A\bar{u}^p, e_1 > & \alpha_i^P \lambda_i \\ < A\bar{v}^p, e_1 > & \beta_i^P \lambda_i \end{vmatrix} & = \frac{\lambda_i}{\Delta_p} \left[\beta_i^p < A\bar{u}^p, e_1 > -\alpha_i^p < A\bar{v}^p, e_1 > \right] \end{cases}$$

Let

$$\gamma_{i,j}^{p+1} = \alpha_i^{p+1}\beta_j^{p+1} - \alpha_j^{p+1}\beta_i^{p+1}$$

$$= \frac{\lambda_i\lambda_j}{\Delta_p^2} \left(\alpha_i^p < A\bar{v}^p, e_n > -\beta_i^p < A\bar{u}^p, e_n > \right) \left(\beta_j^p < A\bar{u}^p, e_1 > -\alpha_j^p < A\bar{v}^p, e_1 > \right)$$

$$-\frac{\lambda_i\lambda_j}{\Delta_p^2} \left(\alpha_j^p < A\bar{v}^p, e_n > -\beta_j^p < A\bar{u}^p, e_n > \right) \left(\beta_i^p < A\bar{u}^p, e_1 > -\alpha_i^p < A\bar{v}^p, e_1 > \right)$$

so

$$\gamma_{i,j}^{p+1} = \frac{\lambda_i\lambda_j}{\Delta_p^2} \left[\left(\alpha_i^p\beta_j^p - \alpha_j^p\beta_i^p \right) \left(< A\bar{u}^p, e_1 >< A\bar{v}^p, e_n > - < A\bar{u}^p, e_n >< A\bar{v}^p, e_1 > \right) \right]$$

$$= \frac{\lambda_i\lambda_j}{\Delta_p^2} \gamma_{i,j}^p \Delta_p = \frac{\lambda_i\lambda_j}{\Delta_p} \gamma_{i,j}^p.$$

Lemma 1. a.

$$\Delta_p = \sum_{l<k} \lambda_i\lambda_i\gamma_{i,j}^{p+1} \left(< z_l, e_1 >< z_k, e_n > - < z_k, e_1 >< z_l, e_n > \right).$$

b.

$$\gamma_{i,j}^{p+1} = \frac{(\lambda_i\lambda_j)^{p+1}\gamma_{i,j}^0}{\sum_{l<k} (\lambda_l\lambda_k)^{p+1}\gamma_{l,k}^0\sigma_{l,k}},$$

where $\sigma_{l,k} =< z_l, e_1 >< z_k, e_n > - < z_k, e_1 >< z_l, e_n >$

Proof. Easy compute gives the results. □

Lemma 2. $\forall i, j > 2$, $\lim_{p\to\infty} \gamma_{i,j}^{p} = 0$

Proof. With Lemma1-b, we have

$$\gamma_{i,j}^{p+1} = \frac{(\lambda_i\lambda_j)^{p+1}\,\gamma_{i,j}^{0}}{(\lambda_1\lambda_2)^{p+1}\,\gamma_{1,2}^{0}\sigma_{1,2} + \sum_{1<l<k} (\lambda_l\lambda_k)^{p+1}\,\gamma_{l,k}^{0}\sigma_{l,k}}$$

then

$$\gamma_{i,j}^{p+1} = \frac{(\lambda_i\lambda_j)^{p+1}\,\gamma_{i,j}^{0}}{(\lambda_1\lambda_2)^{p+1}\left(\gamma_{1,2}^{0}\sigma_{1,2} + \sum_{1<l<k} \left(\frac{\lambda_l\lambda_k}{\lambda_1\lambda_2}\right)^{p+1}\gamma_{l,k}^{0}\sigma_{l,k}\right)}$$

Suppose that $\gamma_{1,2}^{0}$ and $\sigma_{1,2}$ are both not null, then
$\lim_{p\to\infty} \gamma_{i,j}^{p} = 0$ □

Lemma 3. For $(i, j) = (1, 2)$,we have $\lim_{p\to\infty} \gamma_{1,2}^{p} = \frac{1}{\sigma_{1,2}}$

Proof. From Lemma 1,

$$\gamma_{1,2}^{p+1} = \frac{(\lambda_1\lambda_2)^{p+1}\,\gamma_{1,2}^{0}}{(\lambda_1\lambda_2)^{p+1}\left(\gamma_{1,2}^{0}\sigma_{1,2} + \sum_{1<l<k} \left(\frac{\lambda_l\lambda_k}{\lambda_1\lambda_2}\right)^{p+1}\gamma_{l,k}^{0}\sigma_{l,k}\right)}$$

$$= \frac{\gamma_{1,2}^{0}}{\gamma_{1,2}^{0}\sigma_{1,2} + \sum_{1<l<k} \left(\frac{\lambda_l\lambda_k}{\lambda_1\lambda_2}\right)^{p+1}\gamma_{l,k}^{0}\sigma_{l,k}}$$

with the same hypothesis that $\gamma_{1,2}^{0}$ and $\sigma_{1,2}$ are both not null, then

$$\lim_{p\to\infty} \gamma_{1,2}^{p} = \frac{\gamma_{1,2}^{0}}{\gamma_{1,2}^{0}\sigma_{1,2}} = \frac{1}{\sigma_{1,2}}$$

□

Error Criteria

We denote by ε_p^1 and $\varepsilon_p^2$the error calculation defined by

$$\begin{cases} \varepsilon_p^1 = \bar{u}^{p+1} - \bar{u}^{p} & = \sum_{i=1}^{n} \alpha_i^{p+1} z_i - \sum_{i=1}^{n} \alpha_i^{p} z_i = \sum_{i=1}^{n} A_i z_i \\ \varepsilon_p^2 = \bar{v}^{p+1} - \bar{v}^{p} & = \sum_{i=1}^{n} \beta_i^{p+1} z_i - \sum_{i=1}^{n} \beta_i^{p} z_i = \sum_{i=1}^{n} B_i z_i \end{cases}$$

Let us give the expression of $A_i = \alpha_i^{p+1} - \alpha_i^p$ and $B_i = \beta_i^{p+1} - \beta_i^p$,

$$A_i = \frac{\lambda_i}{\Delta_p}\left[\alpha_i^p < A\bar{v}^p, e_n > -\beta_i^p < A\bar{u}^p, e_n >\right]$$

$$-\frac{\lambda_i}{\Delta_{p-1}}\left[\alpha_i^{p-1} < A\bar{v}^{p-1}, e_n > -\beta_i^{p-1} < A\bar{u}^{p-1}, e_n >\right]$$

$$= \frac{\lambda_i}{\Delta_p}\left[\sum_{j=1}^{n} \alpha_i^p \lambda_j \beta_j^p < z_j, e_n > - \sum_{j=1}^{n} \alpha_j^p \lambda_j \beta_i^p < z_j, e_n >\right] -$$

$$\frac{\lambda_i}{\Delta_{p-1}}\left[\sum_{j=1}^{n} \alpha_i^{p-1} \lambda_j \beta_j^{p-1} < z_j, e_n > - \sum_{j=1}^{n} \alpha_j^{p-1} \lambda_j \beta_i^{p-1} < z_j, e_n >\right]$$

$$= \frac{\lambda_i}{\Delta_p}\left[\sum_{j=1}^{n} \lambda_j < z_j, e_n > \left(\alpha_i^p \beta_j^p - \alpha_j^p \beta_i^p\right)\right]$$

$$-\frac{\lambda_i}{\Delta_{p-1}}\left[\sum_{j=1}^{n} \lambda_j < z_j, e_n > \left(\alpha_i^{p-1} \beta_j^{p-1} - \alpha_j^{p-1} \beta_i^{p-1}\right)\right]$$

then

$$A_i = \sum_{j=1}^{n} < z_j, e_n > \left(\gamma_{i,j}^{p+1} - \gamma_{i,j}^p\right)$$

and by the Lemma 2, we get

$$\forall i, j > 2 \; A_i \to 0 \text{ where } p \to \infty$$

then for the large p, we get for $i < j \leq 2$:

$$\varepsilon_p^1 = \left(\frac{\lambda_1^2}{\Delta_p} - 1\right)\gamma_{1,1}^p < z_1, e_n > + \left(\frac{\lambda_1 \lambda_2}{\Delta_p} - 1\right)\gamma_{1,2}^p < z_2, e_n >$$

so

$$\gamma_{1,1}^P = \alpha_1^p \beta_1^p - \alpha_1^p \beta_1^p = 0$$

we write

$$\varepsilon_p^1 = \left(\frac{\lambda_1 \lambda_2}{\Delta_p} - 1\right)\gamma_{1,2}^p < z_2, e_n > = \left(\gamma_{1,2}^{p+1} - \gamma_{1,2}^p\right) < z_2, e_n >$$

then

$$\begin{array}{c} \varepsilon_p^1 \to 0 \\ p \to \infty \end{array}$$

with the same reasoning, we get :

$$\begin{array}{c} \varepsilon_p^2 \to 0 \\ p \to \infty \end{array}$$

Then, the sequence $\{\bar{u}^p\}_p$ and $\{\bar{v}^p\}_p$ converge respectively to $\bar{u} = \alpha_1 z_1 + \alpha_2 z_2$ and $\bar{v} = \beta_1 z_1 + \beta_2 z_2$.

3. Matrix Factorization

In this section, we will expressed the double power iteration method as a Quadrant Interlocking Factorization (QIF) which introduced by [7]. For that, we need to repeat our previous work to the transformed matrix $A^{(1)}$ recursively $\lfloor p = \frac{n}{2} \rfloor$ time step.

Having computing $\bar{u}$ and $\bar{v}$ previously, we compute the matrices W and Z as follow :

$$\begin{array}{llll} Z_{1,1}^{(1)} = a_{1,1} + < Au, e_1 > & , & Z_{n,1}^{(1)} = a_{n,1} + < Au, e_n > & , \\ Z_{1,n}^{(1)} = a_{1,n} + < Av, e_1 > & , & Z_{n,n}^{(1)} = a_{n,n} + < Av, e_n > & , \\ W_{.,1}^{(1)} = \bar{u} & W_{.,n}^{(1)} = \bar{v} & , & \\ W_{i,i}^{(1)} = 1 & W_{i,j}^{(1)} = 0 & 2 \le i, j \le n-1 & i \ne j \end{array}$$

and

$$Z_{i,j}^{(1)} = a_{i,j} - u_i a_{1,j} - v_i a_{n,j} \quad , \quad 2 \le i, j \le n-1 \ .$$

The method is clearly recursive and results after p steps in finals matrices W of the form (1)

$$W = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ w_{2,1} & \ddots & 0 & \iddots & w_{2,n} \\ \vdots & w_{.,.} & 1 & w_{.,.} & \vdots \\ w_{n-1,1} & \iddots & 0 & \ddots & w_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and Z of the form (2)

$$Z = \begin{pmatrix} z_{1,1} & \cdots & z_{i,1} & \cdots & z_{1,n} \\ 0 & \ddots & z_{.,.} & \iddots & 0 \\ \vdots & 0 & z_{i,i} & 0 & \vdots \\ 0 & \iddots & z_{.,.} & \ddots & 0 \\ z_{1,n} & \cdots & z_{i,n} & \cdots & z_{n,n} \end{pmatrix}$$

Then, to solve a linear system $Ax = b$, where $A = WZ$, we solve the two systems $Zx = y$ and $Wy = b$. In the other hand, by expressing A in the form $A = WZW^{-1}$we compute the eigenvalues of A using p step factorization : At step k, $1 \leq k \leq p$, we compute an elementary matrix W_k such that $W_1 \cdots W_k$ and $Z_k = W_k^{-1} Z_{k-1} W_k = (W_k \cdots W_1)^{-1} A (W_k \cdots W_1)$ have the form (2). Application of W_k introduces zeros below and before diagonal of A in columns k and $n - k$.
The W matrix is recovered by formatting the product $W_p \cdots W_1$ and $Z = Z_p$. So, at step 1, we calculate

$$w_{.,1}^T = (0, w_{2,1}, \ldots, w_{n-1,1}, 0) \ , \ w_{.,n}^T = (0, w_{2,n}, \ldots, w_{n-1,n}, 0)$$

Then, $Z_1 = W_1^{-1} A W_1$ imply that $\forall 2 \leq i \leq n - 1$.

$$\begin{cases} w_{i,1}(a_{1,1} + < Aw_{.,1}, e_1 >) + w_{i,n}(a_{n,1} + < Aw_{.,1}, e_{1,n} >) = a_{i,1} + < Aw_{.,1}, e_1 >) \\ w_{i,1}(a_{1,n} + < Aw_{.,n}, e_1 >) + w_{i,n}(a_{n,n} + < Aw_{.,n}, e_n >) = a_{i,n} + < Aw_{.,n}, e_1 >) \end{cases}$$

At step 2, we compute

$$w_{.,2}^T = (0, 0, w_{3,2}, \ldots, w_{n-2,2}, 0, 0) \ , \ w_{.,n-1}^T = (0, 0, w_{3,n-1}, \ldots, w_{n-2,n-1}, 0, 0)$$

to form the W_2matrix. Then we compute $Z_2 = W_2^{-1} Z_1 W_2$.

Note that the W_2 transformation let the first and last columns and the first and last rows of Z_1 unchanged i.e

$$z_{1,i}^{(2)} = z_{1,i}^{(1)} \quad , \quad z_{n,i}^{(2)} = z_{n,i}^{(1)} \quad , \quad z_{i,1}^{(2)} = z_{i,1}^{(1)} \quad , \quad z_{i,n}^{(2)} = z_{i,n}^{(1)} \quad , \quad \forall 2 \leq i \leq n - 1$$

Hence, we obtain the same result if we let

$$\tilde{w}_{.,2}^T = (0, w_{3,2}, \ldots, w_{n-2,2}, 0) \ , \ \tilde{w}_{.,n-1}^T = (0, w_{3,n-1}, \ldots, w_{n-2,n-1}, 0)$$

and

$$\tilde{W}_2 = I + \tilde{w}_{.,2} e_2^T + \tilde{w}_{.,n-1} e_{n-1}^T$$

here e_2, e_{n-1} are the first and last column of I_{n-1}. Next, we compute the $n-1$ by $n-1$ matrix $\tilde{Z}_2 = \tilde{W}_2^{-1}\tilde{Z}_1\tilde{W}_2$ where $\tilde{Z}_1 = (z_{i,j}^{(1)})_{2\le i,j\le n-1}$.

$$z_{l,i}^{(k)} = z_{l,i}^{(k-1)}, z_{n-l+1,i}^{(k)} = z_{n-l+1,i}^{(k-1)}, z_{i,l}^{(k)} = z_{i,l}^{(k-1)}, z_{i,n-l+1}^{(k)} = z_{i,n-l+1}^{(k-1)},$$

$$\forall 1 \le i \le n,\ 1 \le l \le k-1$$

So, at step k, it suffice to compute $(n-2k+2)x(n-2k+2)$ block of the matrix Z by taken

$$\tilde{w}_{.,k}^T = (0, w_{k+1,k}, \ldots, w_{n-k,k}, 0)\ ,$$

$$\tilde{w}_{.,n-k+1}^T = (0, w_{k+1,n-k+1}, \ldots, w_{n-k,n-k+1}, 0) \text{ in } \mathbb{R}^{n-2k+2}$$

The method can be summarized as follow.

$Z_0 = A$

For k=1 to p

Solve the following systems for $w_{.,k}$and $w_{.,k'}$ where $k' = n-k+1$ and $z_{i,j}$ are the elements of Z_{k-1}

$$\begin{cases} k+1 \le i \le k'-1 \\ w_{i,k}(z_{k,k}+ < Z_{k-1}w_{.,k}, e_k >) + w_{i,k'}(z_{k',k} \\ + < Z_{k-1}w_{.,k}, e_{k'} >) = z_{i,k}+ < Z_{k-1}w_{.,k}, e_i > \\ w_{i,k}(z_{k,k'}+ < Z_{k-1}w_{.,k'}, e_k >) + w_{i,k'}(z_{k',k'} \\ + < Z_{k-1}w_{.,k'}, e_{k'} >) = z_{i,k'}+ < Z_{k-1}w_{.,k'}, e_i > \end{cases}$$

$W_k = I_k + w_{.,k}e_k^T + w_{.,k'}e_{k'}^T$; $Z_k = W_k^{-1}Z_{k-1}W_k$

EndFor

$W = I_n + \sum_{k=1}^{p}(w_{.,k}e_k^T + w_{.,k'}e_{k'}^T)$; $Z = Z_p$

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