

A RBSDES APPROACH FOR THE NUMERICAL EVALUATION OF THE AMERICAN OPTION PRICING PROBLEM

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Abstract: The problem of pricing American type options is a typical example of a non linear problem characterized by the absence of closed expressions for its evaluation. Therefore, during recent years, such an issue has been approached , both deterministically and randomly, from the algorithmic point of view, trying to derive suitable numerical approximations. In this paper, starting from the aforementioned solutions, we review some computational, stochastic inspired, methods, mainly based on the the existing link between the above recalled pricing task, and the theory of Reflected Backward Stochastic Differential Equations (BSDEs). In particular we show how suitable numerical schemes can be developed within the SBDEs framework by mean of *quantization techniques* as well as considering Monte Carlo methods.

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1. Introduction

There is a vast literature about Backward Stochastic Differential Equations (BSDEs) and their applications. The fundamental definitions and first results concerning BSDEs have been introduced by Bismut, see [6], for the linear case and by Pardoux and Peng, see [31], in the general case. Duffie and Epstein [17, 18], have shown that the BSDEs point of view gives a simple formulation

of recursive utilities and their properties. In fact, recursive utility corresponds to the solution of a particular BSDE associated with a generator which does not depend on z -component, see below for more details. Moreover, the BSDEs theory has been extensively used in stochastic control, see, e.g., [3] and references therein, as they appear as adjoint equations in the stochastic version of Pontryagin maximum principle, and in Mathematical Finance, since any pricing problem by replication can be written in terms of linear BSDEs, or non-linear BSDEs when portfolios constraints are taken into account, see [19, 11, 12, 13, 14].

More recently, El Karoui, Kapoudjian, Pardoux, Peng and Quenez generalized these results to BSDEs with reflection, see [20], that is, to a setting with an additional continuous, increasing process added in the equation; the function of this additional process is to keep the solution above a certain prescribed lower-boundary process and to do so in a minimal fashion. Moreover, these authors make the crucial observation that the solution is the value function of an optimal stopping problem. Those RBSDEs have been intensively investigated due to their connections with financial mathematics, optimal control and stochastic game. In [1] showed a new construction of the solution of RBSDEs using a maximum principle. This appears as an alternative to the classical penalization method used in [19]. Penalization method produces the solution of a RBSDE as an increasing limit of a sequence of solution of standard (non-reflected) BSDEs. In [23] the author gives an application linked to the stochastic mixed control problems, while in [21] it has been solved the problem of existence and uniqueness of solution to RBSDEs with jumps. Note that the latter problem can be also studied by the point of view of semilinear stochastic evolution equations, also in connection with approximating procedures, as it is done, e.g., in [27]

Concerning the connection with the Mathematical Finance framework, we recall what follows: the holder of a stock option has the right either sell or buy a certain quantity of a company's stock by a certain date at a given price. The price in the contract is known as the strike price and the date is known as the expiration date, or maturity.

A put, resp. a call, option gives the holder the right to sell, resp. to buy, the underlying asset. There exists several types of stock option, spanning from *plain Vanilla*, or European, ones, to highly structured derivatives. Between latter plethora of possibilities, one of the most traded type is represented by American option. Unlike the European stock option, which can be exercised at maturity only, American options can be exercised at any time up to maturity, which implies that the problem of American option valuation does not have a closed-form solution which, instead, can be given exploiting numerical meth-

ods. In one-dimension, the most popular approach to American option pricing and hedging is represented by the implementation of the dynamic programming formula on a binomial tree, originally initiated by Cox-Ross & Rubinstein as an elementary alternative to continuous time Black & Scholes model. These methods are quite efficient to handle vanilla American options on a single asset but they quickly become intractable as the number of underlying assets increases. In order to increase the efficiency and speed of the previous method, different proposals have been studied, leading to the specification of three main families of computational approaches: *Binomial Trees*, *Finite Difference methods* and *Monte Carlo methods*. Note also that previous approaches can be partially applied to the study of financial time series in order to better understand the comovements of assets dynamics and related (stochastic) volatilities of the markets they are traded in. In particular, latter type of problem can be effectively analysed by means of techniques as the Gibbs sampling, or the regime switching one, see, e.g., [15, 16], and references therein.

Concerning the Monte Carlo methods, one can use them when aiming to solve a problem by simulating the underlying process, e.g., a stock, and then calculate the average of some function of the simulated process. A great advantage of the Monte Carlo approach is its flexibility, since it can treat a large set of fundamental problems in statistics, biology, numerical analysis, etc. Moreover, Monte Carlo methods can also handle high dimensional problems effectively. The ideas behind the Monte Carlo method were developed in the 40's by Stanislaw Ulam with together with the help of John von Neumann. Their ideas were later used within the Manhattan project, namely the research program that led to the development of the first atomic bombs during the second World War.

In the 60's the first financial applications of Monte Carlo methods were suggested by Hertz, see [24], and Boyle, see [10]. Then, Bossaerts [9] and Tilley [33], suggested that Monte Carlo methods could be also used to price American style derivatives. Nevertheless, such type of numerical approaches are outperformed by *ad hoc* solutions at least in some particular frameworks as those when *once for all* computations can be realized. The latter is the case, just to give concrete examples, of the evaluation of expectation of known random variables by means of *polynomial chaos decomposition*, as happens for relevant financial models like, e.g., the Vasicek model, or the Cox-Ingersoll-Ross (CIR) model, see [8]. Concerning the *quantization method*, see, e.g., [7] and references therein, it is worth to mention that it is well-adapted for the pricing and hedging of American options. Its purpose is to compute a large number of conditional expectations by projection of the diffusion on optimal

grid designed to minimize the (square mean) projection error.

Quantization methods have been also extensively investigated in *Information Theory* and signal processing [22], while, since the early 2000's, the associated infinite dimensional setting has been investigated from both theoretical and numerical point of view, with a special attention paid to functional quantization, see [26, 29]. Other types of quantization method, as *Marginal quantization* and *Markovian quantization*, were proposed in [28], the first one, relies on an optimal approximation of the marginal distributions of the process, while the second one, looks for an optimal approximation of transition probabilities of the Markov process at some points. And more recently, quantization was introduced in numerical probability to devise numerical integration method [30] and to solve multidimensional stochastic control problems such as swing options, see [4].

In the present paper we shall exploit the approach mainly developed in [2, 34], to the links between the RBSDEs theory, Monte Carlo and quantization method, to price American options.

In particular, the paper is organized as follows: in Section 2 we give some basic notions on BSDEs, as, e.g., existence and uniqueness of solutions and the comparison principle; in Section 3 we report almost the same basic notions, but in the RBSDEs setting, in Section 4 we will show some possible applications of the theoretical techniques developed for BSDEs and RBSDEs in Finance, in particular in Subsection 4.1, resp. in Subsection 4.2, we present the link between RBSDEs and American option, resp. the link between quantization method and RBSDE, while the application of the Monte Carlo approach to the same problem will be treated in Subsection 4.3.

Notations

Throughout the work we will make use of the following notations

- $C_b^\infty(\mathbb{R}^d)$ denotes the set of functions infinitely differentiable with bounded differentials (so that they have at most linear growth).
- The letters C and K denote positive real constants that may vary from line to line.
- $|\cdot|$ will denote the Euclidean norm and " \cdot " the inner product on $\mathbb{R}^d = \mathbb{R}^{1 \otimes d}$. $\|M\| := \sup_{|x| \leq 1} |Mx|$ will denote the operator norm of the matrix $M \in \mathbb{R}^{d \otimes q}$ (d rows, q columns) and M^* its transpose (sometimes also denoted by M^T). In particular $x \cdot y = x^*y$ (or $x \cdot y = x^T y$).

2. Backward Stochastic Differential Equations

Roughly speaking, BSDEs are used to analyse what happens if instead of imposing an initial condition for a Stochastic Differential Equation (SDE), one considers a terminal condition.

Let W be a 1-dimensional Brownian motion defined on a probability space (Ω, F, \mathbb{P}) , and let us consider the filtration:

$$F_t^W := \sigma(W_s \mid s \in [0, t]) , F_t := F_t^W \vee N_p , t \geq 0 .$$

We fix a finite horizon $0 < T < +\infty$, and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ to be Lipschitz functions. The SDE:

$$dY_t = f(Y_t)dt + g(Y_t)dW , t \in [0, T] , \tag{1}$$

has a unique solution $Y \in S^2([0, T])$, provided we set an initial condition $Y_0 = x$. Recall that $Y_T \in L^2(\Omega, F_T, \mathbb{P})$. We use the martingale representation theorem (MRT), since $\xi \in L^2(\Omega, F_T, \mathbb{P})$ and $(Y_t)_{t \geq 0} = \{\mathbb{E}[\xi \mid F_t]\}_{t \geq 0}$ is a square-integrable martingale; by MRT, there exists a process $Z \in L^2_{ad}([0, T] \times \Omega)$ such that:

$$Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dW_s , t \in [0, T] ,$$

namely

$$\begin{cases} dY_t = Z_t dW_t , t \in [0, T] \\ Y_T = \xi \end{cases} .$$

In general, instead of (1), we consider equations of the following form:

$$\begin{cases} dY_t = f(Y_t)dt + g(Y_t)dW_t + Z_t dW_t , t \in [0, T] \\ Y_T = \xi \in L^2(\Omega, F_T, \mathbb{P}) \end{cases} ,$$

where the pair $(Y, Z) \in S^2([0, T]) \times L^2_{ad}([0, T] \times \Omega)$ is the unknown. Substituting $\tilde{Z}_t := g(Y_t) + Z_t$, we see that it is enough to look for the solutions of the following equation:

$$\begin{cases} dY_t = f(Y_t)dt + \tilde{Z}_t dW_t , t \in [0, T] \\ Y_T = \xi \in L^2(\Omega, F_T, \mathbb{P}) \end{cases} ,$$

defined to be the BSDE we are interested in.

Let us consider the following general form for a BSDE

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t , t \in [0, T] \\ Y_t = \xi \in L^p(\Omega, F_T, \mathbb{P}) \end{cases} , \tag{2}$$

under suitable assumptions on the driver f and final data ξ , which were described below, and for $p > 2$.

In what follows we report a general result about existence and uniqueness of the solution of a BSDE when the coefficient are Lipschitz-continuous. Let $(\Omega, F_T, \mathbb{P})$ be a complete probability space, W a d -dimensional Brownian motion defined on $(\Omega, F_T, \mathbb{P})$, $F_t = F_t^W \vee N_p$, $0 \leq t \leq T$, $\mathbb{F} = (F_t)_{t \in [0, T]}$. We impose the following conditions on the coefficients:

(C1): The driver $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$ is measurable and:

1. $f(\cdot, \cdot, y, z)$ is progressively measurable for every $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$;
2. $\mathbb{E}[\int_0^T |f(t, 0, 0)|^2 dt] < +\infty$;
3. $\exists L > 0 : |f(\omega, t, y, z) - f(\omega, t, y', z')| \leq L(|y - y'| + |z - z'|)$
 $\forall y, y' \in \mathbb{R}^k, \forall z, z' \in \mathbb{R}^{k \times d}, P(d\omega)dt - a.e. \text{ on } \Omega \times [0, T]$;

(C2): $\xi \in L^2(\Omega, F_T, \mathbb{P}, \mathbb{R}^k)$.

Theorem. *Under the conditions C1 and C2, the equation (2) has unique solution $(Y, Z) \in B^2 = L^2_{ad}(\Omega \times [0, T], \mathbb{R}^k) \times L^2_{ad}(\Omega \times [0, T], \mathbb{R}^{k \times d})$.*

Proof. see Section 3 in [31]. Or for a shorter direct proof using useful a priori estimates, see Theorem 2.1 in [19].

A second, particularly relevant, result is the so called *Comparison Principle*, which is of great interest when one has to compare the solutions of two BSDEs.

Comparison Principle. *Let $f^i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $i=1,2$ two functions satisfying (C1) and $\xi^1, \xi^2 \in L^2(\Omega, F_T, \mathbb{P})$. For $i=1,2$, let $(Y^i, Z^i) \in B^2$ be the unique solution of BSDE(f^i, ξ^i). If:*

- *it holds $f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1)$, $dtdP$ -a.e. on $[0, T] \times \Omega$ or $f^1(t, Y_t^2, Z_t^2) \leq f^2(t, Y_t^2, Z_t^2)$, $dtdP$ -a.e. on $[0, T] \times \Omega$;*
- $\xi^1 \leq \xi^2$;

then $Y_t^1 \leq Y_t^2 \forall t \in [0, T]$, P -a.s.

Proof. see Section 2.2 in [19].

3. Reflected Backward Stochastic Differential Equations

The notion of Reflected Backward Stochastic Differential Equation (RBSDE) has been introduced by El Karoui in [20], defining BSDEs characterized by an increasing process K introduced in order to force the solution Y of the (original) BSDE to remain larger than a given process S representing an *obstacle* or *barrier*.

In order to provide the basics notions and results for RBSDEs, let us start by fixing a d -dimensional standard Brownian motion, W_t , $0 \leq t \leq T$, defined on a complete probability space $(\Omega, F_T, \mathbb{P})$. Moreover, let F_t , $0 \leq t \leq T$ be the natural filtration of W_t and let P be the σ -algebra of predictable subsets of $\Omega \times [0, T]$.

Further, consider the following notations:

$$\begin{aligned} \mathbb{L}^2 &= \{ \xi \text{ an } F_t\text{-measurable random variable such that } \mathbb{E}[|\xi|^2] < +\infty \}; \\ \mathbb{H}^2 &= \{ \phi_t, 0 \leq t \leq T \text{ a predictable process such that } \mathbb{E}[\int_0^T |\phi_t|^2 dt] < +\infty \}; \\ \varrho^2 &= \{ \phi_t, 0 \leq t \leq T \text{ a predictable process such that } \mathbb{E}[\sup_{0 \leq t \leq T} |\phi_t|^2] < +\infty \}. \end{aligned}$$

Previous definitions can be re-read in terms of the following three *objects*:

1. the first is a terminal value ξ such that $\xi \in \mathbb{L}^2$;
2. the second is a coefficient f , which is a map: $f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, such that:
 - (a) $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, f(\cdot, y, z) \in \mathbb{H}^2$,
 - (b) for some $K_0 > 0$ and all $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$, a.s.

$$|f(t, y, z) - f(t, y', z')| \leq K_0(|y - y'| + |z - z'|),$$
3. the third is an *obstacle* or *barrier*, $S_t, 0 \leq t \leq T$, which is a continuous progressively measurable real-valued process satisfying $\mathbb{E}[\sup_{0 \leq t \leq T} (S_t^+)^2] < +\infty$.

Then, a solution of a given RBSDE is a triple of process (Y, Z, K) , taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$, and satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dW_s, \forall t \in [0, T].$$

The additional process K is continuous and non-decreasing, moreover its role is to push upwards the process Y in order to keep it above the *barrier* S , provided that:

- i) $(Y, Z) \in B^2 = L^2_{ad}([0, T] \times \Omega) \times L^2_{ad}([0, T] \times \Omega; \mathbb{R}^d)$ and $K \in S^2([0, T])$ is a non-decreasing process with $K_0 = 0$;
- ii) $Y_t \geq S_t \forall t \in [0, T]$ and $\int_0^T (Y_t - S_t) dK_t = 0$.

The latter implies that the process K acts only when Y reaches the *barrier* S , but in a minimal way.

Theorem. *Under the above conditions, if $\xi \geq S_T$, then the RBSDE has a unique solution.*

Proof. See Section 2 in [20].

To what concerns the RBSDE counterpart of the *BSDE-comparison principle*, we have :

Comparison Principle for RBSDEs. *Let (ξ, f, S) and (ξ', f', S') be two sets of data taken according to the object specifications given at points [1], [2], [3], above, but with the exception that the Lipschitz condition [2] could be satisfied by either f or f' only. Suppose in addition that the following inequalities hold*

- $\xi \leq \xi'$ a.s.
- $f(t, y, z) \leq f'(t, y, z) dP \times dt$ a.e. $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$
- $S_t \leq S'_t$ a.s. $0 \leq t \leq T$.

Let (Y, Z, K) be a solution of the RBSDE with data (ξ, f, S) and (Y', Z', K') a solution of the RBSDE with data (ξ', f', S') . Then:

$$Y_t \leq Y'_t, \quad 0 \leq t \leq T \text{ a.s.}$$

Proof. see Theorem 4.1 in [20].

4. Application in Finance

4.1. RBSDEs and American Option

We start this section giving a short introduction to the *Snell envelope* technique, which plays an important role in evaluating American type options, indeed the valuation of American options is a special case of the *Snell envelope*, see [32].

Definition. If $X = (X_n)_{n=0}^N$ is a sequence of random variables, adapted to a filtration F_n with $\mathbb{E}[|X_n|] < \infty$, the sequence $Z = (Z_n)_{n=0}^N$ defined by:

$$\begin{cases} Z_N = X_N \\ Z_n = \max(X_n, \mathbb{E}[Z_{n+1} | F_n]) , n \leq N - 1 \end{cases} , \tag{3}$$

is called the Snell envelope of X.

The sequence Z can be considered to be an *optimal value process*:

$$Z_n = \text{ess sup}_{n \leq \tau \leq T} \mathbb{E}[X_\tau | F_n] . \tag{4}$$

The recursion (3) defines the optimal stopping strategy to find the optimal value Z_n , and it is usual to refer to it, as the method of *dynamic programming*. The time that maximizes the essential supremum (4), is called an *optimal stopping time* and is denoted $\hat{\tau}_n$. We also recall that:

Definition. Let (Ω, F, \mathbb{P}) be a probability space and N_t be an increasing family of σ -algebras of subset of Ω . A function $\tau : \Omega \rightarrow [0, \infty]$ is called a stopping time with respect to N_t if:

$$\{\omega | \tau(\omega) \leq t\} \in N_t , \forall t \geq 0 .$$

Concerning the financial framework, let us recall that the *holder* of an American option has the option to either buy (call) or sell (put) the underlying asset S at a certain, fixed *a priori*, strike price K , at an *optional time* $\tau \leq T$, T being the terminal, or maturity, time of the investment. The related payoff function is given by:

$$\Phi(S) = \begin{cases} (S_\tau - K)^+ & \text{for call} \\ (K - S_\tau)^+ & \text{for put} \end{cases} .$$

Latter definition implies that the value of an American option is the value achieved by exercising optimally. Finding the optimal exercise rule is an *optimal stopping problem*. Indeed, for each value of S_t , $t = T, \dots, 0$, we have to determine the price of the option and check if it is favorable to exercise. This problem is known as a *free boundary problem*. At each time t we have two possibilities: hold on to the option or exercise it. The price, which we can denote S_τ , is called the *optimal exercise price*. We do not know S_τ a priori and thus we do not know how to apply the boundary conditions, why this problem is referred

to as a free boundary problem. Hence, the problem of pricing an American option is to find the optimal expected discounted payoff:

$$\sup_{\tau \in \mathcal{T}} e^{r(T-\tau)} \mathbb{E}[\Phi(S_\tau)] ,$$

where \mathcal{T} is the class of admissible stopping times, taking values in $[0, T]$.

So we restate that, finding a fair value is not an easy task due to the complexity of the derivative, that is, in the case of simple derivatives, we can derive the value in a closed-form solution, for the *American option* we requires numerical procedures.

Therefore, in what follows, we state the model linking the American options pricing task to the theory of RBSDEs.

In order to achieve such a goal, let us start considering a stock market model constituted by a non-risky asset (bond) S^0 , and a finite family of risky assets S^i $i=1, \dots, m$. The price of one bond unit is given by:

$$dS_t^0 = S_t^0 r_t dt , \quad t \geq 0 ,$$

where r is the (risk free) interest rate, while the price processes of the risky assets are modelled by:

$$dS_t^i = S_t^i (\mu_t^i dt + \sigma_t^i dW_t) , \quad t \geq 0 ,$$

where W is a d -dimensional Brownian motion. We assume that $S_0^0 > 0$ and $S_0^i > 0$, $i=1, \dots, m$. The processes r and μ^i are supposed progressively measurable and bounded. Here, σ^i are known as the volatilities of the prices, and we suppose that the matrix $\sigma = (\sigma^i)_{1 \leq i \leq m}$ is invertible, hence, in particular $m = d$., with its inverse bounded, which corresponds to the fact that the market is *arbitrage-free* and *complete*.

Let $\bar{\pi} = (\pi_t^0, \pi_t^1, \dots, \pi_t^m)_{0 \leq t \leq T}$ be a *portfolio*, where the processes π^0 and π^i , $i=1, \dots, m$; are the amounts invested in the bond and the risky asset i , respectively. The value of the portfolio $\bar{\pi}$ is given by the process:

$$V_t = V_y^{\bar{\pi}} := \pi_t^0 + \pi_t^1 + \dots + \pi_t^m , \quad t \in [0, T] .$$

We recall that a portfolio is called *admissible* if $\bar{\pi}$ is progressively measurable and if:

$$\int_0^T |\pi_t^0| dt < +\infty , \quad \int_0^T |\pi_t^i \sigma_t^i|^2 dt < +\infty$$

$$\forall i = 1, \dots, m \text{ } P - a.s. , \quad V_t^{\bar{\pi}} \geq 0 , \quad \forall t \in [0, T] .$$

An admissible portfolio $\bar{\pi}$ is called *self-financing* if, taking $\sigma^0 = 0$, its value $V_t^{\bar{\pi}}$ satisfies:

$$dV_t = \pi_t^0 \frac{dS_t^0}{S_t^0} + \pi_t^1 \frac{dS_t^1}{S_t^1} + \dots + \pi_t^m \frac{dS_t^m}{S_t^m} = r_t V_t dt + \sum_{i=1}^m (\pi_t^i (\mu_t^i - r_t) dt + \pi_t^i \sigma_t^i dW_t) = r_t V_t dt + \pi_t (\mu_t - r_t) dt + \pi_t \sigma_t dW_t = (r_t V_t + \pi_t \sigma_t \theta_t) dt + \pi_t \sigma_t dW_t ,$$

where $\pi_t = (\pi_t^1, \dots, \pi_t^m)$ is the *risky portfolio*, while we indicate by $\theta_t = (\sigma)^{-1}(\mu_t - r_t)$ is the *risk premium vector*. The set of admissible self-financing portfolios will be denoted by A .

It is customary to write $(V, \pi) \in A$, meaning $(\pi^0, \pi) \in A$ and $\pi_t^0 = V_t - \sum_{i=1}^m \pi_t^i, t \in [0, T]$.

For our purpose, the wealth-portfolio (V, π) has to satisfy:

$$-dV_t = b(t, V_t, \pi_t) dt - \pi_t \sigma_t dW_t , t \geq 0 ,$$

where b is convex and Lipschitz, with respect to (x, π) . Indeed, in the standard case, we have:

$$b(t, x, \pi) = -r_t x - \pi \sigma_t \theta_t .$$

When the market is constrained, namely when the interest rates are not the same whether we borrow or invest money, then the function b takes the alternative form:

$$b(t, x, \pi) = -r_t x - \pi \sigma_t \theta_t + (R_t - r_t)(x - \langle \pi, 1 \rangle)^- ,$$

where R_t and r_t are the spot rates to borrow, respectively to invest. Additionally, we assume that:

$$\mathbb{E} \left[\int_0^T |b(t, 0, 0)|^2 dt \right] < +\infty ,$$

where $(\sigma_t)_{0 \leq t \leq T}$, which represents the *volatility matrix* of the n risky assets, is invertible and its inverse $(\sigma_t)^{-1}$ is bounded. Without loss of generality, we can even suppose that $\sigma_t = I$.

It follows that we are concerned with the problem of pricing an *American option* at time $t = 0$. When it is exercised at a stopping time $\tau \leq T$, the yield is given by:

$$\tilde{P}_\tau := P_\tau 1_{(\tau < T)} + \xi 1_{(\tau = T)} ,$$

Recall that, in case of American put options, we have $P_t = (K - S_t^1)^+, \xi P_T$, so $\tilde{P}_\tau = P_\tau$.

Then there exists a unique pair $(V_p(\tau, \tilde{P}_\tau), \pi_p(\tau, \tilde{P}_\tau))_{p \in [0, \tau]} = (V_p^\tau, \pi_p^\tau)_{p \in [0, \tau]}$, which replicates \tilde{P}_τ , i.e:

$$\begin{cases} -dV_p^\tau = b(p, V_p^\tau, \pi_p^\tau)dt - \pi_p^\tau dW_s, & 0 \leq p \leq \tau \\ V_\tau^\tau = \tilde{P}_\tau \end{cases} .$$

Therefore, the price is given by the *Snell Envelope*:

$$X_0 = \operatorname{ess\,sup}_\tau V_0(\tau, \tilde{P}_\tau),$$

Theorem. *The value X_0 is equal to Y_0 , where (Y, π, K) is the solution of the following reflected BSDE:*

$$\begin{cases} -dY_t = b(t, Y_t, \pi_t)dt + dK_t - \pi_t dW_t, & t \in [0, T] \\ Y_T = \xi \\ T_t \geq P_t, & \forall t \in [0, T] \\ \int_0^T (Y_t - P_t) dK_t = 0 \end{cases} .$$

Furthermore, the stopping time:

$$v := \inf \{s \geq 0 \mid Y_s = P_s\} \wedge T,$$

is the optimal exercise time, namely $X_0 = V_0(v, \tilde{P}_v)$.

Proof. The existence of a solution $(Y, \pi, K) \in L^2_{ad}(\Omega \times [0, T]) \times L^2_{ad}(\Omega \times [0, T]; \mathbb{R}^d) \times S^2([0, T])$ of the RBSDE is guaranteed by the existence and uniqueness result. Moreover, if τ is a stopping time such that $\tau \leq T$, then we have:

$$\begin{cases} -dV_p^\tau = b(p, V_p^\tau, \pi_p^\tau)dt - \pi_p^\tau dW_p, & 0 \leq p \leq \tau \\ V_\tau^\tau = \tilde{P}_\tau \end{cases} . \tag{5}$$

Since $-dY_s = b(s, Y_s, \pi_s)ds + dK_s - \pi_s dW_s$, $s \in [0, \tau]$, and $Y_\tau \geq \tilde{P}_\tau$, then, by the comparison principle, we have:

$$Y_s \geq V_s^\tau, \forall s \in [0, \tau], P - a.s .$$

We can view equation (5) as a reflected BSDE, with barrier $V_t^\tau \wedge \tilde{P}_t$; $t \in [0, \tau]$, hence:

$$Y_0 \geq \operatorname{ess\,sup}_\tau V_0(\tau, \tilde{P}_\tau) .$$

On the other hand we have:

$$-dY_t = b(t, Y_t, \pi_t)dt - \pi_t dW_t ; \forall t \in [0, v],$$

with $Y_v = \tilde{P}_v$, so that $Y_0 = V_0(v, \tilde{P}_v) \leq \operatorname{ess\,sup}_\tau V_0(\tau, \tilde{P}_\tau)$. □

4.2. RBSDEs-Quantization Approach to American Option Pricing

In what follows, mainly exploiting the results given in [2], see also [7] and references therein, we describe a probabilistic method for pricing American style options on multidimensional baskets of traded assets and we see the connection between this method and RBSDEs. First of all the asset dynamics follow a d -dimensional, $d \geq 1$, diffusion model in the time interval $[0, T]$, where T is the *maturity time*. Taking into account a continuous time Markov process $(S_t)_{t \in [0, T]}$ for the underlying, the pricing procedure consists in solving the continuous optimal stopping time problem related to an *obstacle process*, typically of the form $h_t = h(t, S_t)$.

In order to apply a quantization method based on grids, we first discretize the asset price process at times $t_k := kT/n$, where $k = 0, \dots, n$, as explained in [2], then we simulate large samples of $(S_{t_k})_{0 \leq k \leq n}$ to produce at each time t_k a grid Γ_k of size N_k , which is *optimally fitted* to S_{t_k} , among all grids with size N_k , namely the *closest neighbor rule projection* $q_{\Gamma_k}(S_{t_k})$ of S_{t_k} onto the grid Γ_k is the best least square approximation of S_{t_k} among *all random vectors* Z s.t. $|Z(\Omega)| \leq N_k$, so that

$$\|S_{t_k} - q_{\Gamma_k}(S_{t_k})\|_2 = \min\{\|S_{t_k} - Z\|_2, Z : \Omega \rightarrow \mathbb{R}^d, |Z(\Omega)| \leq N_k\}.$$

Following such an approach, like in [2], we will work with the best possible grid of size N_k , at each time step t_k , to approximate the d -dimensional random vector S_{t_k} . Both the function q_{Γ_k} and the set $q_{\Gamma_k}(\Omega)$ are called *optimal quantizer* of S_{t_k} with resulting error bound represented by $\|S_{t_k} - q_{\Gamma_k}(S_{t_k})\|_2$, the latter being called the lowest quantization error, in the *quadratic mean sense*.

Hence the model is constructed in the following way: we consider a market on which are traded $d \geq 1$ risky assets S^1, \dots, S^d , and a deterministic riskless asset $S_t^0 := e^{rt}, r \in \mathbb{R}$, in a finite time interval $[0, T], T > 0$. The price process $S := (S^1, \dots, S^d)$ for the risky assets is taken to have the following structure:

$$dS_t^i = S_t^i(rdt + \sum_{1 \leq j \leq q} \sigma_{ij}(e^{-rt} S_t) dW_t^j), S_0^i := s_0^i > 0, 1 \leq i \leq d, \quad (6)$$

where $W := (W^1, \dots, W^q)$ is a q -dimensional Brownian Motion defined on a probability space (Ω, F, \mathbb{P}) and $\sigma : \mathbb{R}^d \rightarrow M(d \times q) := \mathbb{R}^{d \otimes q}$ is bounded and Lipschitz continuous. The associated filtration is denoted by $F := (F_t^s)_{t \in [0, T]}$, and it is the one generated by S , which coincides with that of W when $\sigma \sigma^T(\xi) > 0$ for every $\xi \in \mathbb{R}^d$. Moreover we introduce:

$$c(\xi) := \text{Diag}(\xi)\sigma(\xi), \xi := (\xi^1, \dots, \xi^d) \in \mathbb{R}^d,$$

where $Diag(\xi)$ denotes the diagonal matrix with diagonal entry ξ^i at row i . Assuming that the functions $c(\xi)$, and the drift $b(\xi) := r\xi$, are Lipschitz continuous, we have the existence and uniqueness of the solution for (6) on (Ω, F, \mathbb{P}) . The discounted price process $\tilde{S}_t := e^{-rt}S_t$ is a positive \mathbb{P} -martingale satisfying:

$$d\tilde{S}_t = c(\tilde{S}_t)dW_t, \quad \tilde{S}_0 := s_0. \tag{7}$$

In what follows we assume that \mathbb{P} represents the *risk neutral probability* of the model we are considering. When $q \neq d$, the completeness of the market fails, nevertheless, this does not influence the implementation of the quantization method to compute the price of the derivatives. When $q = d$, and $\sigma(x) \equiv \sigma \in M(d \times d)$, eq. (6) represents the usual d -dimensional Black & Scholes model, namely the risky assets follow a geometric Brownian motions whose dynamics are given by:

$$S_t^i = s_0^i \exp \left(\left(r - \frac{1}{2} \|\sigma_i\|^2 \right) t + \sum_{1 \leq j \leq d} \sigma_{ij} W_t^j \right), \quad 1 \leq i \leq d.$$

Let us now take into consideration an American option related to a payoff process $(h_t)_{t \in [0, T]}$, which is a F -adapted nonnegative process. We will always consider the sub-class of payoffs h_t that only depend on (t, S_t) , i.e. they satisfy:

$$h_t := h(t, S_t), \quad t \in [0, T], \quad h : [0, T] \rightarrow \mathbb{R}_+ \text{ being Lipschitz continuous.} \tag{8}$$

Under the assumptions (6) and (8), one has:

$$\mathbb{E} \left(\sup_{t \in [0, T]} |h_t|^p \right) < +\infty, \quad \forall p \geq 1.$$

In a complete market, the fair price ν_t at time t is:

$$\nu_t := e^{rt} \text{ess sup} \{ \mathbb{E}(e^{-r\tau} h_\tau | F), \tau \in \mathfrak{T}_t \}, \tag{9}$$

where $\mathfrak{T}_t := \{ \tau : \Omega \rightarrow [t, T], F\text{-stopping time} \}$. The latter simply means that the discounted price of the option represented by $\tilde{\nu}_t := e^{-rt}\nu_t$, is nothing but the *Snell envelope* of the discounted American payoff:

$$\tilde{h}_t := \tilde{h}(t, \tilde{S}_t), \text{ with } \tilde{h}(t, x) := e^{-rt}h(t, e^{rt}x). \tag{10}$$

Using the Markov property of the diffusion process $(S_t)_{t \in [0, T]}$, it is possible to show that, see [2], $\nu_t := v(t, S_t)$, where v solves the variational inequality:

$$\max \left(\frac{\partial v}{\partial t} + L_{r, \sigma} v, v - h \right) = 0, \quad v(T, \cdot) = h(T, \cdot),$$

$L_{r,\sigma}$ being the infinitesimal generator associated to eq. (6). Therefore the approximation problem for ν_t appears as a special case of the Snell envelope of a d -dimensional diffusion with Lipschitz coefficients.

Following [2], we assume without loss of generality, that the interest rate r in eq. (6) is 0, hence are in the so called *discounted world* setting, with \tilde{S} given by (7), and \tilde{h} given by (10), instead of S_t and h , respectively.

The peculiarity of the martingale diffusion dynamics proposed for the risky assets by eq. (7), and with $r = 0$, can be shown to have little influence on the results, hence we can consider a general drifted Brownian diffusion of the following type

$$S_t = S_0 + \int_0^t b(S_s) ds + \int_0^t c(S_s) dW_s ,$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and $c : \mathbb{R}^d \rightarrow M(d \times q)$ are both Lipschitz continuous vector fields, and $(W_t)_{t \in [0, T]}$ is q -dimensional Brownian motion. When σ is not constant, for example, in the model described by eq. (6), the exact simulation of a diffusion at time t is not feasible, and it is customary to implement a Markovian discretization scheme, as, e.g., the Euler scheme. The latter consists in setting $t_k = kT/n$, so that

$$\bar{S}_{t_{k+1}} = \bar{S}_{t_k} + b(\bar{S}_{t_k}) \frac{T}{n} + c(\bar{S}_{t_k}) \cdot (W_{t_k} - W_{t_{k-1}}) , \quad \bar{S}_0 = s_0 \mathbf{1} ; .$$

Sometimes, the diffusion can be simulated in a rather straightforward manner, essentially because it can be expressed in a closed form, e.g. $S_t := \phi(t, W_t)$. This is the case of the regular multi-dimensional Black & Scholes model, taking $\sigma(x) := \sigma$ in (6), which allows to consider directly the *Snell envelope of the homogeneous Markov chain* $(S_{t_k})_{0 \leq k \leq n}$, for our quantization aims.

Latter procedure corresponds, from the derivatives point of view, to approximate the original continuous time American option by a Bermuda option, characterized by a finite set of exercise times, either on \tilde{S} , or on S itself. Moreover the Snell envelope can be quantized considering (S_{t_k}) or (\bar{S}_{t_k}) , or of any family of homogeneous discrete time F_{t_k} -Markov chains $(X_k^{(n)})_{0 \leq k \leq n}$ whose transition, denoted $P^{(n)}(x, dy)$, preserves the Lipschitz continuity property, at least in the following sens: for every Lipschitz continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we require that

$$\left[P^{(n)} f \right]_{\text{Lip}} \leq (1 + C_{b,\sigma,T} T/n) [f]_{\text{Lip}} , \quad [f]_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} .$$

To simplify notations, we drop the dependency in n , keeping just $(X_k)_{0 \leq k \leq n}$.

The (F_{t_k}) -Snell envelope of $h(t_k, X_k)$, denoted by $(V_k)_{0 \leq k \leq n}$, is defined by:

$$V_k := \text{ess sup}\{\mathbb{E}(h(\theta, X_\theta) \mid F_{t_k}), \theta \in \Theta_k\},$$

where Θ_k denotes the set of $\{t_k, \dots, t_n\}$ -valued (F_{t_l}) -stopping times; it satisfies the *backward dynamic programming formula*:

$$\begin{cases} V_n := h(t_n, X_n) \\ V_k := \max(h(t_k, X_k), \mathbb{E}(V_{k+1} \mid F_{t_k})) \quad 0 \leq k \leq n - 1 \end{cases} \quad (11)$$

Exploiting the Markov property, it is possible to derive a dynamic programming formula in distribution: $V_k = v_k(X_k), k \in \{0, \dots, n\}$, where the functions v_k are recursively defined as follow

$$\begin{cases} v_n := h(t_n, \cdot) \\ v_k := \max(h(t_k, \cdot), P^{(n)}(v_{k+1})) \quad 0 \leq k \leq n - 1 \end{cases} ,$$

nevertheless, from numerical point of view, such type of formula are intractable, since they require to compute at each time step a conditional expectation.

Hence we are in position to describe the quantization method core. In particular, we first discretize the random variables X_k by some $\sigma(X_k)$ -random variables \hat{X}_k taking finite values in \mathbb{R}^d , so that it is natural to define \hat{X}_k as the *quantized* counterpart of X_k . Then, being $N = N_0 + N_1 + \dots + N_n$ the total number of elementary quantizers used to quantize the whole Markov chain $(X_k)_{0 \leq k \leq n}$. Moreover we assume that, $\forall k \in \{0, 1, \dots, n\}$, we have a sequence of quantizations $\hat{X}_k = q_k(X_k), k = 0, \dots, n$ of the Markov chain $(X_k)_k$, and we also denote by $\{x_1^k, \dots, x_{N_k}^k\} = q_k(\mathbb{R}^d)$, the grid of N_k points used to quantize X_k , and by $x^k = (x_1^k, \dots, x_{N_k}^k)$ the induced N_k -tuple.

Then we can state the quantized dynamic programming formula below, in tight analogy with eq. (11). In particular, one simply replaces X_k by its quantized random vector \hat{X}_k :

$$\begin{cases} \hat{V}_n := h(t_n, \hat{X}_n) \\ \hat{V}_k := \max(h(t_k, \hat{X}_k), \mathbb{E}(\hat{V}_{k+1} \mid \hat{X}_k)) \quad 0 \leq k \leq n - 1 \end{cases} \quad (12)$$

In what follows, for the sake of simplicity, we will use the following notation $\hat{\mathbb{E}}_k(\cdot) := \mathbb{E}(\cdot \mid \hat{X}_k)$. Since $(\hat{X}_k)_{k \in \mathbb{N}}$ is not Markovian, we consider the conditional expectation with respect to \hat{X}_k . On the other hand, even if the N_k -tuple x^k has been set up a priori for every X_k , the numerical processing of this algorithm is not possible. Indeed, one needs to know the joint distributions of $(\hat{X}_k, \hat{X}_{k+1}), k = 0, \dots, n - 1$, as shown by the the following proposition, see [2]:

Proposition (Quantization tree algorithm) *For every $k \in \{0, \dots, n\}$, let $x^k := (x_1^k, \dots, x_{N_k}^k)$, $q_k : \mathbb{R}^d \rightarrow \{x_1^k, \dots, x_{N_k}^k\}$ and $\hat{X}_k = q_k(X_k)$ be a quantization of X_k . Set, for every $k \in \{0, \dots, n\}$ and every $i \in \{1, \dots, N_k\}$:*

$$p_i^k := \mathbb{P}(\hat{X}_k = x_i^k) = \mathbb{P}(X_k \in C_i(x^k)) \tag{13}$$

and, for every $k \in \{0, \dots, n - 1\}$, $i \in \{1, \dots, N_k\}$, $j \in \{1, \dots, N_{k+1}\}$:

$$\begin{aligned} \pi_{i,j}^k &:= \mathbb{P}(\hat{X}_{k+1} = x_j^{k+1} \mid \hat{X}_k = x_i^k) = \mathbb{P}(X_{k+1} \in C_j(x^{k+1}) \mid X_k \in C_i(x^k)) = \\ &= \frac{p_{ij}^k}{p_i^k} \text{ with } p_{ij}^k := \mathbb{P}(X_{k+1} \in C_j(x^{k+1}), X_k \in C_i(x^k)), \end{aligned} \tag{14}$$

then, by backward induction, one defines the function \hat{v}_k by:

$$\begin{aligned} \hat{v}_n(x_i^n) &:= h_n(x_i^n), \quad i \in \{1, \dots, N_n\}, \\ \hat{v}_k(x_i^k) &:= \max \left(h(t_k, x_i^k), \sum_{j=1}^{N_{k+1}} \pi_{ij}^k \hat{v}_{k+1}(x_j^{k+1}) \right), \quad 1 \leq i \leq N_k, \quad 0 \leq k \leq n - 1. \end{aligned} \tag{15}$$

Therefore $\hat{V}_k = \hat{v}_k(\hat{X}_k)$ satisfies the above dynamic programming (12) of the pseudo-Snell envelope.

Remark. If $X_k = S_{t_k}$ or \bar{S}_{t_k} , then $X_0 = \hat{X}_0 = s_0$ and $\hat{v}_0(\hat{X}_0) = \hat{v}_0(s_0)$ is deterministic. In more general settings one approximates $\mathbb{E}[v_0(X_0)]$ by:

$$\mathbb{E}[\hat{v}_0(\hat{X}_0)] = \sum_{i=1}^{N_0} p_i^0 \hat{v}_0(x_i^0).$$

The RBSDEs theory comes into play when we want to evaluate the global residual risk on time interval $[0, T']$, $T' < T$, which is induced by the use of the time discretization of the diffusion with step T/n , namely:

$$\mathbb{E} \left[\int_0^{T-\delta} |c^*(S_s)(Z_s - \zeta_s)|^2 ds \right], \tag{16}$$

where $Z_t := [c(S_t)^*]^{-1}H_t$, H_t being an adapted process, while $\zeta_t := proj_n(Z)$ is the projection on the set P_n of elementary predictable strategies. Our basic assumption, let us denote it by (Σ) , reads as follows:

- (i) $c \in C_b^\infty((0, +\infty)^d)$,
- (ii) $\sigma\sigma^*(x) \geq \epsilon_0 I_d$.

Assumption (i) is fulfilled when $\sigma \in C_b((0, +\infty)^d) \cap C_b^\infty((0, +\infty)^d)$ and:

$$\partial^k \sigma_i(x) = O(1/|x^i|) \text{ as } |x| \rightarrow +\infty \text{ } k \geq 1, i = 1, \dots, d.$$

The difficulties arise when we want evaluate the term in (16) directly. So, like in [2], we reduce the problem to a simpler one, according to the following two steps:

STEP 1. Lemma. *Set $H_s := c^*(S_s) \cdot Z_s$ and $\eta_s := \frac{n}{T} \mathbb{E}_{t_k} \left[\int_{t_k}^{t_{k+1}} H_u du \right]$, $s \in [t_k, t_{k+1}]$. Then, under the assumptions of the previous theorem, we have*

$$\mathbb{E} \left[\int_0^T |c^*(S_s)(Z_s - \zeta_s)|^2 ds \right] \leq \frac{C}{n} + 2\mathbb{E} \left[\int_0^T |H_s - \eta_s|^2 ds \right],$$

as stated in [2, Subsec. 4.3, Lemma 1].

STEP 2. The second type of difficulty we have to take into account is due to the following two facts

- the *obstacle* $h(t, S_t)$ is not sufficiently smooth, then we do not have a nice control on the increasing process K_t ;
- the process S_t is not uniformly elliptic, since $c(0) = 0$, then we cannot count on nice evaluations for the (probability) density of S_t .

In order to overcome previous issues, we will replace S by an elliptic diffusion denoted \underline{S} and the *obstacle* h by a smoother one, denoted by \underline{h} . Let $\epsilon \in (0, 1]$ and $\lambda > 0$, then we consider:

— a function $\underline{h} \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$, with a regularization by convolution of order ϵ of the original *obstacle* h , then, since h is Lipschitz continuous, we have:

$$\|h - \underline{h}\|_\infty \leq C_h \epsilon \text{ and } \|(\partial_t + L_c)\underline{h}\|_\infty \leq C_h \epsilon^{-1},$$

where L_c is the infinitesimal generator of the diffusion S .

— a function $\phi_\lambda \in C_b^\infty(\mathbb{R}, \mathbb{R})$ satisfying:

$$\phi_\lambda(\xi) := \xi \text{ if } \xi \geq e^{-\lambda}, \phi_\lambda(\xi) := \frac{e^{-\lambda}}{2} \text{ if } \xi \leq \frac{1}{2}e^{-\lambda},$$

and

$$\|\phi_\lambda^{(m)}\|_\infty \leq C_m e^{C_m \lambda}, \forall m \geq 1,$$

where C_m is a real constant which does not depend on λ . Then the approximating diffusion coefficient c_λ , which is defined for every $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ by:

$$c_\lambda(x) := c(\phi_\lambda(x^1), \dots, \phi_\lambda(x^d)),$$

satisfies:

$$c_\lambda \in C_b^\infty(\mathbb{R}^d) \quad \text{and} \quad c_\lambda c_\lambda^*(x) \geq \frac{\epsilon_0}{4} e^{-2\lambda} I_d.$$

Taking into consideration the solution \underline{S}^x of the SDE:

$$d\underline{S}_t^x = r\underline{S}_t^x dt + c_\lambda(\underline{S}_t^x) dW_t, \quad \underline{S}_0^x = x,$$

and being $\underline{P}_t(x, dy)$ its Markov semi-group, defined by $\underline{P}_t f(x) = \mathbb{E}[f(\underline{S}_t^x)]$, we will denote by \underline{S}_t the associated solution \underline{S}^{s_0} , which starts at $s_0 \in (0, +\infty)^d$, at time $t = 0$. Then, the related Snell envelope reads as follow:

$$\underline{Y}_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E}_t[\underline{h}(\tau, \underline{S}_\tau)],$$

and it satisfies the following RBSDE:

$$\underline{Y}_t = \underline{h}(T, \underline{S}_T) + \underline{K}_T - \underline{K}_t - \int_t^T \underline{H}_s \cdot dW_s,$$

for some non decreasing process \underline{K} and some progressively, measurable $d\mathbb{P} \otimes dt$ -square integrable process \underline{H} . Finally we consider the approximation:

$$\underline{w}_s = \frac{n}{T} \mathbb{E}_{t_k} \left[\int_{t_k}^{t_{k+1}} \underline{H}_s ds \right], \quad t_k \leq s < t_{k+1},$$

and we have, see [2, Subject. 4.5, Lemma2], that the following holds by stability properties of RBSDEs:

Lemma. *Assume that (Σ) holds. Then:*

$$\mathbb{E} \left[\int_0^T |H_s - \eta_s|^2 ds \right] \leq C \left(\epsilon^2 + (1 + |s_0|)^2 e^{-\frac{c\lambda^2}{T}} \right) + \mathbb{E} \left[\int_0^T |\underline{H}_s - \underline{w}_s|^2 ds \right].$$

4.3. Pricing American Option by Monte Carlo Method

Before enter into details, let us recall some useful notions and results about the Monte Carlo (MC) method. First of all, let us underline that the main goal of the MC approach is to compute quantities of the following type

$$\tau = \mathbb{E}[\phi(X)] = \int_A \phi(x)f(x) dx ,$$

where:

- X is a random variable taking values in $A \subseteq \mathbb{R}^d, d \in \mathbb{N}$;
- $f : A \rightarrow \mathbb{R}_+$ is the probability density of X ;
- $\phi : A \rightarrow \mathbb{R}_+$ is a function such that the above expectation is finite;

Latter framework actually covers a large set of fundamental problems in statistics, numerical analysis, mechanics, engineering, etc., see, e.g., [15, 16], and references therein. As an example, if we want to compute an integral of the following type

$$\alpha = \int_0^1 f(x) dx ,$$

we can instead consider the expectation $\mathbb{E}[f(U)]$, with U uniformly distributed between 0 and 1. Then we, randomly, generate points U_1, U_2, \dots, U_n , independently and uniformly over the interval $[0, 1]$, and, by evaluating f at each U_i and by taking the associated mean, we get the following MC estimate

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^n f(U_i) ,$$

which, by the *Law of Large Numbers* (LLN), will converge to the true value when $n \rightarrow +\infty$.

In particular the LLN tells us that the expectation and the *arithmetic mean*, in the above reported sense, are equal in the limit. Moreover, by the *Central Limit Theorem* (CLT) we can deduce that the distribution of the mean is normal, regardless of the initial distribution of the random variable X .

If we want to use a MC approach to approximate the dynamic of a given stochastic process, see, e.g. [34], we easily see that such MC methods strongly depends on the possibility to perform *good* simulation of the underlying process. In particular, when we consider a 1-dimensional American option with constant

interest rate, the underlying process is the stock price S which, taking the risk-free interest rate r , and both the parameters δ and σ to be constant, is characterized by the following SDE

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t .$$

Let S_0 be the value of the stock at initial time $t = 0$, then we have

$$S_t = S_0 \exp \left((r - \delta - \frac{\sigma^2}{2})t + \sigma W_t \right) .$$

Since the increments of the standard Brownian motion W , which drives the dynamic of S , are independent and normally distributed, we can simulate paths of the process S using a recursive procedure of the following type

$$S_t = S_{t-1} \exp \left((r - \delta - \frac{\sigma^2}{2})dt + \sigma \sqrt{dt} Z \right) ,$$

where we have assumed that the time steps are equidistant, and of length dt , while $Z \sim N(0, 1)$. Latter recursive procedure allows to generate sample paths $S^i = \{S_1^i, S_2^i, \dots, S_n^i\}$, $i = 1, \dots, N$.

Most MC methods are exploited for pricing American option with respect to the *dynamic programming* setting, see [5] for its seminal definition by Bellman.

In particular, let $\Pi_i(s)$ denotes the value of the option at time t_i , given $S_i = s$, and assuming that the option has not been exercised. Finally, assume that $\Phi(s)$ is a payoff function of an American-style contract and our aim is to compute the value of $\Pi_0(S_0)$. The value of $\Pi_0(S_0)$ can be obtained recursively as follows

$$\Pi_T(s) = \Phi_T(s)$$

$$\Pi_i(s) = \max(\Phi_i(s), C_i(s)) , \quad i = 0, \dots, T - 1 ,$$

where $C_i(s)$ is the continuation value defined by

$$C_i(s) = \mathbb{E}[\Phi(S_{i+1}) \mid S_i = s] .$$

Again citing [34], we have that *The dynamic programming formulation thus boils down to at each time step $t = 1, \dots, T - 1$, compare the payoff with the expected value of the option if not exercised at this time. Nevertheless, we do not know the continuation value, which, indeed, has to be estimated. In order to do the latter, several approaches are possible, as, e.g., the *Random Tree Method*, the *Stochastic Mesh Method*, the *Duality Method*, the *Forward Backwards Method* (FBM), etc., see [34, Sec.4].*

In what follows we shall focus our attention on the FBM. Therefore the continuation value is estimated by a single value, $\hat{\Pi}(t + 1, B_{t+1}^i)$. The idea behind the FBM is to construct a grid B , by mean of backwards simulation and conditional on the initial value S_0 of the asset at time $t = 0$. Therefore, the value $\hat{\Pi}(t + 1, B_{t+1}^i)$ will be an unbiased estimator of the continuation value and the computational cost is quadratic in the number of points in the grid, while it is linear with respect to the number of exercise dates, see [34]. If we assume that $B = \{B_1^i, \dots, B_T^i\}, i = 1, \dots, n$, then the related construction proceeds as follow. First we apply the dynamic programming procedure with:

$$\hat{\Pi}(t, B_t^i) = \begin{cases} \Phi(t, B_t^i) & t = T \\ \max(\Phi(t, B_t^i), \hat{\Pi}(t + 1, B_{t+1}^i)) & t < T \end{cases} ,$$

then we define the final estimator

$$\hat{\Pi}(0, S_0) := \max \left(\Phi(0, S_0), \frac{1}{n} \sum_{i=1}^n \hat{\Pi}(1, B_1^i) \right) ,$$

which is high biased by Jensen’s inequality, see [34]. To construct the grid S we first simulate n trajectories for S , obtaining $S = \{S_1^i, \dots, S_T^i\}, i = 1, \dots, n$, then we can consider S as our *forward* grid and define $B = \{B_1^i, \dots, B_T^i\}, i = 1, \dots, n$, as the backward one. Therefore, see [34], we have

$$B_t^i = \begin{cases} S_T^i & t = T \\ S_t^{I_t^i} & t < T \end{cases} ,$$

where I_t^i is drawn from $\{1, 2, \dots, n\}$ with probability:

$$I_t^i \sim \left(\frac{p(S_{t+1} = B_{t+1}^i | S_t = S_t^\ell)}{\sum_{j=1}^n p(S_{t+1} = B_{t+1}^j | S_t = S_t^j)} \right)_\ell ,$$

p being the probability density of the underlying S , in fact we want to simulate B_t^i conditional on $S_0 = s$ and $S_{t+1}^i = B_{t+1}^i$, namely with probability:

$$\begin{aligned} p(S_t | S_{t+1}, S_0) &= \frac{p(S_t, S_{t+1}, S_0)}{p(S_{t+1}, S_0)} = \frac{p(S_t, S_{t+1}, S_0)}{\int p(S_t, S_{t+1}, S_0) ds_t} = \\ &= \frac{p(S_{t+1} | S_t, S_0)p(S_t | S_0)p(S_0)}{\int p(S_{t+1} | S_t, S_0)p(S_t | S_0)p(S_0) ds_t} , \end{aligned} \tag{17}$$

and taking into consideration the empirical measure in the denominator of (17), equation (17) simplifies to:

$$= \frac{p(S_{t+1} | S_t)}{\sum_{\ell} p(S_{t+1} | S_t^{\ell})},$$

because S is (now) a discrete Markov chain, therefore we have the same as in (17).

5. Conclusion

In this paper we have underlined how the theory behind the study of RBSDEs can be well adapted to solve the non-linear problem of pricing American option. In particular we have reported the existing link between American option and RBSDEs, namely the price of an American option, according to the notation introduced in Subsection 4.1, is given by the Snell Envelope:

$$X_0 = \operatorname{ess\,sup}_{\tau} V_0(\tau, \tilde{P}_{\tau}),$$

and X_0 equals Y_0 , the latter being the solution of the RBSDE:

$$\begin{cases} -dY_t = b(t, Y_t, \pi_t)dt + dK_t - \pi_t dW_t & t \in [0, T] \\ Y_T = \xi \\ T_t \geq P_t \\ \int_0^T (Y_t - P_t) dK_t = 0 \end{cases} \quad \forall t \in [0, T].$$

Then, we have shown how the *quantization method* can be used to numerically study the solutions of a suitable class of RBSDEs, namely those related to the task of finding the price v of an American option. In particular, according to the notation introduced in Subsection 4.2, we have that $\hat{V}_k = \hat{v}_k(\hat{X}_k)$ satisfies the dynamic programming principle:

$$\begin{cases} \hat{V}_n := h(t_n, \hat{X}_n) \\ \hat{V}_k := \max(h(t_k, \hat{X}_k), \mathbb{E}(\hat{V}_{k+1} | \hat{X}_k)) \quad 0 \leq k \leq n - 1 \end{cases},$$

which is quantized by quantization tree algorithm, where we recall that \hat{V} indicates an approximation of V .

Finally the connection between RBSDEs and Monte Carlo method have been retrieved exploiting the work of Wieslander, see [34], again by using the

dynamic programming approach, therefore, according to the notation introduced in Subsection 4.3, the price of an American option Π is *Monte Carlo* approximated as follows

$$\begin{cases} \Pi_T(s) = \Phi_T(s) \\ \Pi_i(s) = \max(\Phi_i(s), C_i(s)) \quad i = 0, \dots, T - 1 \end{cases} .$$

We would also like to recall that the RBSDEs theory can be fruitfully applied to different problems, not only related to the Mathematical Finance frameworks, as, e.g., concerning stochastic optimal control, stochastic game, insurance issues, stochastic approaches to neuronal dynamics, etc.

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