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GENERALIZED NOWHERE DENSE SETS IN CLUSTER TOPOLOGICAL SETTING

M. Matejdes
Department of Mathematics and Computer Science
Faculty of Education
Trnava University in Trnava
Priemyselná 4, 918 43 Trnava, SLOVAKIA

Abstract: The aim of the article is to generalize the notion of nowhere dense set with respect to a cluster topological space which is defined as a triplet $(X, \tau, \mathcal{E})$ where (X, τ) is a topological space and $\mathcal{E}$ is a nonempty family of nonempty subsets of X. The notions of $\mathcal{E}$-nowhere dense and locally $\mathcal{E}$-scattered sets are introduced and the necessary and sufficient conditions under which the family of all $\mathcal{E}$-nowhere dense sets is an ideal are given.

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1. Introduction and Basic Definitions

Cluster topological spaces provide a general framework with the involvement of ideal topological spaces [1], [2], [3], [9]. They have a wider application and its progress can find in [6], [7], [10]. The paper can be considered as a continuation of [5] where some cluster topological notions were introduced and it corresponds with the efforts to generalize the Baire classification of sets and the Baire category theorem [4], [11].

In [5] one can find an open problem to discover a necessary and sufficient condition under which the family of all $\mathcal{E}$-nowhere dense sets forms an ideal. In the first part we recall the basic notions and results of [5], a few counter examples are given and Section 3 is devoted to our main goal.

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In the sequel, (X, τ) is a nonempty topological space. By $\overline{A}$, A°, we denote the closure, the interior of A in X, respectively. By $\overline{A}^\circ$ we denote the interior of $\overline{A}$.

Definition 1.1. (see [5]) Any nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ will be called a cluster system in X. If G is a nonempty open set and any nonempty open subset of G contains a set from $\mathcal{E}$, then $\mathcal{E}$ is called a π-network in G. For a cluster system $\mathcal{E}$ and a subset A of X, we define the set $\mathcal{E}(A)$ of all points $x \in X$ such that for any neighborhood U of x, the intersection $U \cap A$ contains a set form $\mathcal{E}$. A triplet $(X, \tau, \mathcal{E})$ is called a cluster topological space. If $\emptyset \neq Y \subset X$ and $\mathcal{E}_Y := \{E \cap Y : E \in \mathcal{E} \text{ and } E \cap Y \neq \emptyset\}$, then $(Y, \tau_Y, \mathcal{E}_Y)$ where τ_Y is the subspace topology is called a cluster topological subspace of $(X, \tau, \mathcal{E})$, provided $\mathcal{E}_Y \neq \emptyset$.

Remark 1.1. (see [1], [2], [3], [9]) Specially, if $\mathcal{I}$ is a proper ideal on X, then a cluster system $\mathcal{E}_\mathcal{I} = \{E \subset X : E \notin \mathcal{I}\}$ leads to a local function of A, i.e., $\mathcal{E}_\mathcal{I}(A) = \{x \in X :$ for any open set U containing x there is $E \in \mathcal{E}_\mathcal{I}$ such that $E \subset U \cap A\} = \{x \in X : U \cap A \notin \mathcal{I},$ for any open set U containing $x\} =:$ $A^*(\mathcal{I}, \tau)$ what is called the local function of A with respect to $\mathcal{I}$ and τ. Note, if $\mathcal{E} = 2^X \setminus \{\emptyset\}$, then $\mathcal{E}(A) = \overline{A}$.

The next definition introduces some basic notions derived from $\mathcal{E}$-operator reminding the properties of local function which are known from an ideal topological space.

Definition 1.2. A set A is called $\mathcal{E}$-scattered if A contains no set from $\mathcal{E}$. A set A is locally $\mathcal{E}$-scattered at a point $x \in X$ if there is an open set U containing x such that $U \cap A$ is $\mathcal{E}$-scattered (i.e., $x \notin \mathcal{E}(A)$). A is locally $\mathcal{E}$-scattered if A is locally $\mathcal{E}$-scattered at any point from A (i.e., $A \cap \mathcal{E}(A) = \emptyset$) and A is $\mathcal{E}$-dense in itself if $A \subset \mathcal{E}(A)$. A set A is $\mathcal{E}$-nowhere dense if for any nonempty open set U there is a nonempty open subset H of U such that $H \cap A$ is $\mathcal{E}$-scattered. The family of all $\mathcal{E}$-nowhere dense sets, locally $\mathcal{E}$-scattered sets, nowhere dense sets, is denoted by $\mathcal{N}_\mathcal{E}$, $\mathcal{S}_\mathcal{E}$, $\mathcal{N}$, respectively.

Remark 1.2. By Definition 1.1, $\mathcal{E}$ is a nonempty system of nonempty subsets of X. A trivial case $\mathcal{E} = \emptyset$ ($\emptyset \in \mathcal{E}$) leads to the trivial results, since $\mathcal{E}(A) = \emptyset$ and $\mathcal{N}_\mathcal{E} = 2^X$ ($\mathcal{E}(A) = X$ and $\mathcal{N}_\mathcal{E} = \emptyset$) for any $A \subset X$.

Definition 1.3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two cluster systems. $\mathcal{E}_1 < \mathcal{E}_2$ if for any $E_1 \in \mathcal{E}_1$ there is $E_2 \in \mathcal{E}_2$ such that $E_2 \subset E_1$. $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent, $\mathcal{E}_1 \sim \mathcal{E}_2$, if $\mathcal{E}_1 < \mathcal{E}_2$ and $\mathcal{E}_2 < \mathcal{E}_1$.

2. Preliminary Results

The next properties of $\mathcal{E}$-operator are clear and the proof of the following lemma is omitted.

Lemma 2.1. (see [5])

(1) $\mathcal{E}(\emptyset) = \emptyset$,

(2) $\mathcal{E}(A)$ *is closed,*

(3) $\mathcal{E}(A) \subset \overline{A}$,

(4) $\mathcal{E}(\mathcal{E}(A)) \subset \mathcal{E}(A)$,

(5) $\mathcal{E}$ *is a* π*-network in an open set* $G \neq \emptyset$ *if and only if* $\mathcal{E}(G) = \mathcal{E}(\overline{G}) = \overline{G}$.

Lemma 2.2.

(1) If $\mathcal{E}_1 \subset \mathcal{E}_2$*, then* $\mathcal{E}_1 < \mathcal{E}_2$,

(2) if $\mathcal{E}_1 < \mathcal{E}_2$*, then* $\mathcal{E}_1(A) \subset \mathcal{E}_2(A)$ *and* $\mathcal{N}_{\mathcal{E}_2} \subset \mathcal{N}_{\mathcal{E}_1}$,

(3) if $\mathcal{E}_1 \sim \mathcal{E}_2$*, then* $\mathcal{E}_1(A) = \mathcal{E}_2(A)$ *and* $\mathcal{N}_{\mathcal{E}_2} = \mathcal{N}_{\mathcal{E}_1}$,

(4) $\mathcal{N}_{\mathcal{E}_1 \cup \mathcal{E}_2} \subset \mathcal{N}_{\mathcal{E}_i}$, $i = 1, 2$,

(5) $\mathcal{N}_{\mathcal{E}_i} \subset \mathcal{N}_{\mathcal{E}_1 \cap \mathcal{E}_2}$, $i = 1, 2$,

(6) if $A_1 \subset A_2$*, then* $\mathcal{E}(A_1) \subset \mathcal{E}(A_2)$,

(7) let $A \subset Y \subset X$*. If* A *is* $\mathcal{E}_Y$*-nowhere dense, then* $A \in \mathcal{N}_{\mathcal{E}}$,

(8) if A_t *is* $\mathcal{E}$*-dense in itself for any* $t \in T$*, then* $\cup_{t \in T} A_t$ *is so,*

(9) if A *is* $\mathcal{E}$*-dense in itself, then* $\overline{A}$ *is so.*

Proof. We will show only (7), (8) and (9). Other items are easy to prove.

(7) Let G be a nonempty open subset of X. If $G \cap Y = \emptyset$, there is noting to prove. Suppose $G \cap Y \neq \emptyset$. Then $G \cap Y \in \tau_Y$, so there is a nonempty open set $H \in \tau$, such that $\emptyset \neq H \cap Y \subset G \cap Y$ and $H \cap Y \cap A$ contains no set from $\mathcal{E}_Y$, hence $H \cap Y \cap A = H \cap A$ contains no set from $\mathcal{E}$. Since $H \cap G$ is a nonempty open subset of G and $H \cap G \cap A$ contains no set from $\mathcal{E}$, $A \in \mathcal{N}_{\mathcal{E}}$.

(8) It follows from $\cup_{t \in T} A_t \subset \cup_{t \in T} \mathcal{E}(A_t) \subset \mathcal{E}(\cup_{t \in T} A_t)$.

(9) Since $A \subset \mathcal{E}(A)$, $\overline{A} \subset \overline{\mathcal{E}(A)} = \mathcal{E}(A) \subset \mathcal{E}(\overline{A})$ by Lemma 2.1 (2) and Lemma 2.2 (6).

Lemma 2.3. (see [5]) *The next conditions are equivalent:*

(1) $A \in \mathcal{N}_{\mathcal{E}}$,

(2) $\mathcal{E}(A) \in \mathcal{N}$,

(3) $(\mathcal{E}(A))^\circ = \emptyset$.

The following theorem summarizes the basic properties of $\mathcal{N}_{\mathcal{E}}$. For completeness, we will prove it because some items are new or slightly different.

Theorem 2.1. (see [5])

(1) $\mathcal{N} \subset \mathcal{N}_{\mathcal{E}}$. Consequently, if $A \in \mathcal{N}$, then $\overline{A} \in \mathcal{N}_{\mathcal{E}}$,

(2) if $A \in \mathcal{N}_{\mathcal{E}}$ and $B \subset A$, then $B \in \mathcal{N}_{\mathcal{E}}$,

(3) $A \setminus \mathcal{E}(A) \in \mathcal{N}_{\mathcal{E}}$ and $A \setminus \mathcal{E}(A)$ is locally $\mathcal{E}$-scattered,

(4) any $\mathcal{E}$-scattered set is locally $\mathcal{E}$-scattered and any locally $\mathcal{E}$-scattered set is from $\mathcal{N}_{\mathcal{E}}$,

(5) if $A \in \mathcal{N}$ and $B \in \mathcal{N}_{\mathcal{E}}$, then $A \cup B \in \mathcal{N}_{\mathcal{E}}$,

(6) if $G_t \in \mathcal{N}_{\mathcal{E}}$ and G_t is open for any $t \in T$, then $\cup_{t\in T} G_t \in \mathcal{N}_{\mathcal{E}}$. Consequently, if $A_t \subset Y \subset X$ is τ_Y-open and $A_t \in \mathcal{N}_{\mathcal{E}_Y}$, then $\cup_{t\in T} A_t \in \mathcal{N}_{\mathcal{E}}$,

(7) $A \in \mathcal{N}_{\mathcal{E}}$ if and only if A is a sum of a locally $\mathcal{E}$-scattered set and a set from $\mathcal{N}$,

(8) if $A, B \in \mathcal{N}_{\mathcal{E}}$ and one of them is closed, then $A \cup B \in \mathcal{N}_{\mathcal{E}}$.

Proof. (1): Let $A \in \mathcal{N}$. By Lemma 2.1 (3), $\mathcal{E}(A) \subset \overline{A}$, so $(\mathcal{E}(A))^\circ \subset \overline{A}^\circ = \emptyset$ and by Lemma 2.3, $A \in \mathcal{N}_{\mathcal{E}}$.

(2): It follows from the implications: $B \subset A \Rightarrow \mathcal{E}(B) \subset \mathcal{E}(A) \Rightarrow (\mathcal{E}(B))^\circ \subset (\mathcal{E}(A))^\circ = \emptyset$ and Lemma 2.3.

(3): Let G be nonempty open. If $G \cap (A \setminus \mathcal{E}(A))$ is empty, there is nothing to prove. Let $x \in G \cap (A \setminus \mathcal{E}(A))$. Then there is an open subset H of G containing x, such that $H \cap A$ contains no set from $\mathcal{E}$. Then $H \cap (A \setminus \mathcal{E}(A))$ contains no set from $\mathcal{E}$, so $H \cap (A \setminus \mathcal{E}(A))$ is $\mathcal{E}$-scattered. That means $A \setminus \mathcal{E}(A) \in \mathcal{N}_{\mathcal{E}}$. The second part is clear.

(4): The first part is clear. Let A be locally $\mathcal{E}$-scattered. Since $A \cap \mathcal{E}(A) = \emptyset$, $A = A \setminus \mathcal{E}(A)$ is $\mathcal{E}$-nowhere dense by (3).

(5): Let G be nonempty open. Since A is nowhere dense and B is $\mathcal{E}$-nowhere dense, there are two nonempty open sets $G_0 \subset G$ and $H \subset G_0$ such that $A \cap G_0 = \emptyset$ and $B \cap H$ is $\mathcal{E}$-scattered. Hence $(A \cup B) \cap H = B \cap H$ is $\mathcal{E}$-scattered, so $A \cup B$ is $\mathcal{E}$-nowhere dense.

(6): Let $\{H_s\}_{s\in S}$ be a maximal family of pairwise disjoint open sets such that any H_s is a subset of some set from $\{G_t\}_{t\in T}$ and $A := \cup_{t\in T} G_t \setminus \cup_{s\in S} H_s$ is nowhere dense. It is clear that $B := \cup_{s\in S} H_s$ is $\mathcal{E}$-nowhere dense. By item (5), $\cup_{t\in T} G_t = A \cup B$ is $\mathcal{E}$-nowhere dense. The consequence follows from Lemma 2.2 (7).

(7): " $\Rightarrow$ " It follows from equation $A = (A \setminus \mathcal{E}(A)) \cup (A \cap \mathcal{E}(A))$, item (3) and Lemma 2.3. The opposite implication follows from the items (4) and (5).

(8): Suppose A is closed. If $A^\circ = \emptyset$, then A is nowhere dense and by item (5), $A \cup B \in \mathcal{N}_\mathcal{E}$.

Let $A^\circ \neq \emptyset$ and U be a nonempty open set. Suppose $U \cap A^\circ = \emptyset$. Since $A \setminus A^\circ$ is nowhere dense, so there is a nonempty open set $H \subset U$ such that $H \cap (A \setminus A^\circ) = H \cap A = \emptyset$. Since $B \in \mathcal{N}_\mathcal{E}$, there is a nonempty open set $H_0 \subset H$, such that $H_0 \cap B = (H_0 \cap A) \cup (H_0 \cap B) = H_0 \cap (A \cup B)$ contains no set from $\mathcal{E}$, hence $A \cup B \in \mathcal{N}_\mathcal{E}$. Finally, suppose $U \cap A^\circ \neq \emptyset$. Since $A \in \mathcal{N}_\mathcal{E}$, there is a nonempty open set $H_1 \subset U \cap A^\circ \subset A$, such that $H_1 \cap A = H_1$ contains no set from $\mathcal{E}$, consequently $(A \cup B) \cap H_1$ contains no set from $\mathcal{E}$. So $A \cup B \in \mathcal{N}_\mathcal{E}$. □

If $\mathcal{E}_{II} = \{E : E \text{ is of second category in } (X, \tau)\}$, then $\mathcal{N}_{\mathcal{E}_{II}}$ is the family of all sets of first category. So, item (6) of Theorem 2.1 is a generalization of the Banach category theorem.

3. Main Results

Next theorem deals with a relationship between an $\mathcal{E}$-nowhere dense set and a nowhere dense one and we will find some conditions under which $\mathcal{N}_\mathcal{E}$ forms an ideal.

Theorem 3.1. *Let $\mathcal{E}$ be a π-network in an open set G_0. If $A \subset \overline{G_0}$ is closed and $\mathcal{E}$-nowhere dense, then A is nowhere dense. Consequently, if $\mathcal{E}(X) = X$ and A is a closed subset of X, then A is nowhere dense if and only if A is $\mathcal{E}$-nowhere dense.*

Proof. Let G_1 be nonempty open. If $A \cap G_1 = \emptyset$, there is nothing to prove. Let $A \cap G_1 \neq \emptyset$ and $G := G_1 \cap G_0$. Since $\mathcal{E}$ is a π-network in G_0, $H := G \cap (X \setminus A) \neq \emptyset$ (if $G \subset A$, then there is a nonempty open subset H_0 of G such that $H_0 \cap A = H_0$ contains no set from $\mathcal{E}$, contradiction with assumption that $\mathcal{E}$ is a π-network in G_0). So, H is a nonempty open subset of G and disjoint from A. □

Remark 3.1. No assumption in Theorem 3.1 can be omitted. Let $X = \{a, b\}$, $\tau = \{X, \emptyset\}$, $\mathcal{E} = \{X\}$, $A = \{a\}$. Then $\mathcal{E}$ is a π-network in X. The set A is $\mathcal{E}$-nowhere dense, A is not closed and A is not nowhere dense.

The assumption that $\mathcal{E}$ is a π-network can not be omitted. Consider $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ with the usual topology, $\mathcal{E} = \{E : E \text{ is infinite}\}$. It is clear that $\mathcal{E}$ is not a π-network in X. Put $A = \{1, \frac{1}{2}\}$. The set A is closed and $\mathcal{E}$-nowhere dense. But A is not nowhere dense.

It is known that A is nowhere dense iff $\overline{A}$ is so. An analogous equivalence for the $\mathcal{E}$-nowhere dense sets leads to the fact that $\mathcal{N}_{\mathcal{E}}$ is an ideal.

Theorem 3.2. (see [5]) *If $\overline{A}$ is $\mathcal{E}$-nowhere dense whenever A is $\mathcal{E}$-nowhere dense, then $\mathcal{N}_{\mathcal{E}}$ is an ideal.*

An obvious question is whether the assumption of Theorem 3.2 implies the equality $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$ and if the opposite implication holds. Next examples will give the negative answers.

Example 3.1. *Let $X = \{a, b\}$, $\tau = 2^X$, $\mathcal{E} = \{X\}$. Then $\mathcal{N}_{\mathcal{E}} = 2^X$ and $\mathcal{N} = \{\emptyset\} \neq \mathcal{N}_{\mathcal{E}}$.*

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$, $\mathcal{E} = \{\{a\}\}$. Then $\mathcal{N}_{\mathcal{E}} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ is an ideal. The set $\{b\} \in \mathcal{N}_{\mathcal{E}}$, but $\overline{\{b\}} = \{a, b, c\} \notin \mathcal{N}_{\mathcal{E}}$.*

In the case if $\mathcal{E}$ is a π-network in X, the opposite implication is valid but the assumption that $\overline{A}$ is $\mathcal{E}$-nowhere dense whenever A is $\mathcal{E}$-nowhere dense seems to be too strong and it leads to the equation $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$.

Theorem 3.3. (see [5]) *Let $\mathcal{E}$ be a π-network in X. Then the next conditions are equivalent:*

(1) $\overline{A}$ is $\mathcal{E}$-nowhere dense if and only is A is $\mathcal{E}$-nowhere dense,

(2) $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$.

In [5] it is recommended to investigate a condition under which $\mathcal{N}_{\mathcal{E}}$ is an ideal. In this section we introduce a notion of additive cluster system.

Theorem 3.4. *$\mathcal{N}_{\mathcal{E}}$ is an ideal if and only if any sum of two locally $\mathcal{E}$-scattered sets is from $\mathcal{N}_{\mathcal{E}}$.*

Proof. "$\Rightarrow$" By Theorem 2.1 (4), any locally $\mathcal{E}$-scattered set is from $\mathcal{N}_{\mathcal{E}}$, so the sum of two $\mathcal{E}$-scattered sets is from $\mathcal{N}_{\mathcal{E}}$.

"$\Leftarrow$" Let $A, B \in \mathcal{N}_{\mathcal{E}}$. By Theorem 2.1 (7), $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where A_1, B_1 are locally $\mathcal{E}$-scattered and $A_2, B_2 \in \mathcal{N}$. Then $A \cup B = (A_1 \cup B_1) \cup (A_2 \cup B_2)$ is a sum of a locally $\mathcal{E}$-scattered and a nowhere dense set, so $A \cup B \in \mathcal{N}_{\mathcal{E}}$, by Theorem 2.1 (7). □

Corollary 3.1. *If $\mathcal{S}_{\mathcal{E}}$ is an ideal, then $\mathcal{N}_{\mathcal{E}}$ is so.*

The opposite implication does not hold, as the next example shows.

Example 3.3. *Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ with the usual topology and $\mathcal{E} = \{E \subset X : X \setminus E$ is finite$\}$. Then $X_1 = \{\frac{1}{n} : n = 1, 2, 3, \dots\}$ and $X_2 = \{0\}$ are locally $\mathcal{E}$-scattered, but $X_1 \cup X_2 = X$ is not so. It is clear $\mathcal{N}_{\mathcal{E}} = 2^X$ is an ideal.*

Definition 3.1. A cluster system $\mathcal{E}$ is $\mathcal{N}$-additive if for any $A, B \subset X$ there is a nowhere dense set R, such that $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B) \cup R$.

Theorem 3.5. *$\mathcal{N}_{\mathcal{E}}$ is an ideal if and only if $\mathcal{E}$ is $\mathcal{N}$-additive.*

Proof. "$\Rightarrow$" Let $A, B \in \mathcal{N}_{\mathcal{E}}$. First, we will show $(\mathcal{E}(A \cup B))^\circ \subset \mathcal{E}(A) \cup \mathcal{E}(B)$. Let $x \in (\mathcal{E}(A \cup B))^\circ$ and $x \notin \mathcal{E}(A) \cup \mathcal{E}(B)$. Then there is an open subset H of $\big(\mathcal{E}(A \cup B)\big)^\circ$ containing x and $H \cap A$ and $H \cap B$ contain no set from $\mathcal{E}$. So, $H \cap A$ and $H \cap B$ are $\mathcal{E}$-nowhere dense set. Since $\mathcal{N}_{\mathcal{E}}$ is an ideal, there is a nonempty open subset G of H, such that $G \cap (A \cup B)$ contains no set from $\mathcal{E}$. On the other hand, $x \in (\mathcal{E}(A \cup B))^\circ$, hence $G \cap (A \cup B)$ contains a set from $\mathcal{E}$, a contradiction.

Since $\mathcal{E}(A \cup B) \setminus (\mathcal{E}(A \cup B))^\circ$ in nowhere dense and $\mathcal{E}(A \cup B) = [\mathcal{E}(A \cup B) \setminus (\mathcal{E}(A \cup B))^\circ] \cup \mathcal{E}(A) \cup \mathcal{E}(B)$, $\mathcal{E}$ is $\mathcal{N}$-additive.

"$\Leftarrow$" Let $A, B \in \mathcal{N}_{\mathcal{E}}$. Then $\mathcal{E}(A \cup B) = R \cup \mathcal{E}(A) \cup \mathcal{E}(B)$, where R is a nowhere dense set. Since $\mathcal{E}(A), \mathcal{E}(B)$ are nowhere dense, $\mathcal{E}(A \cup B)$ is a nowhere dense set, so $A \cup B \in \mathcal{N}_{\mathcal{E}}$, by Lemma 2.3. □

4. Derived Cluster Systems

It is well known that a set A is of first category if and only if $D(A) = \emptyset$ where $D(A)$ is the set of all points in which A is of first category, i.e., for any $x \in A$ there is an open set U containing x such that $A \cap U$ does not contain a set of second category. Question is if there is a similar characterization of $\mathcal{E}$-nowhere dense sets, namely $A \in \mathcal{N}_{\mathcal{E}}$ iff $\mathcal{E}(A) = \emptyset$. Next example shows that similar characterization exists for the ideal $\mathcal{N}$ of all nowhere dense sets.

Example 4.1. *Let* $\mathcal{E}_{\mathcal{N}} = \{E : E \notin \mathcal{N}\}$*. It is clear that* $\mathcal{E}_{\mathcal{N}}(A) = \overline{\overline{A}^{\circ}}$*. By Lemma 2.3,* $A \in \mathcal{N}_{\mathcal{E}_{\mathcal{N}}}$ *iff* $\mathcal{E}_{\mathcal{N}}(A)$ *is nowhere dense iff* $\overline{A}^{\circ}$ *is nowhere dense iff* $\overline{A}^{\circ} = \emptyset$ *iff* A *is nowhere dense. Consequently,* $\mathcal{N}_{\mathcal{E}_{\mathcal{N}}} = \mathcal{N}$*. So,* A *is a nowhere dense set iff* $A \in \mathcal{N}_{\mathcal{E}_{\mathcal{N}}}$ *iff* $\mathcal{E}_{\mathcal{N}}(A) = \overline{\overline{A}^{\circ}} = \emptyset$*.*

The next example shows that the equivalence $A \in \mathcal{N}_{\mathcal{E}}$ if and only if $\mathcal{E}(A) = \emptyset$ does not hold in general.

Example 4.2. *Let* $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ *with the usual topology and* $\mathcal{E} = \{\{1\}\} \cup \{E : E \text{ is infinite}\}$*. Then* $\mathcal{E}(A) = \emptyset$ *iff* A *is finite and* $1 \notin A$*. It is clear* $\mathcal{N}_{\mathcal{E}} = \{A : 1 \notin A\}$*.*

Definition 4.1. Let $\mathcal{E}$ be a cluster system. Put $\mathcal{E}^* = \{E : E \notin \mathcal{N}_{\mathcal{E}}\} = \{E : (\mathcal{E}(E))^{\circ} \neq \emptyset\}$ (see Lemma 2.3). A set A is called $\mathcal{E}$-preopen if $A \subset (\mathcal{E}(A))^{\circ}$. A cluster system of all nonempty $\mathcal{E}$-preopen sets is denoted by $\mathcal{E}^{\mathsf{po}}$.

Now we will study a connection among $\mathcal{N}_{\mathcal{E}}$, $\mathcal{N}_{\mathcal{E}^*}$ and $\mathcal{N}_{\mathcal{E}^{\mathsf{po}}}$.

Lemma 4.1. *Let* $(\mathcal{E}(A))^{\circ} \neq \emptyset$*. If* H *is a nonempty open subset of* $(\mathcal{E}(A))^{\circ}$*, then* $A \cap H \in \mathcal{E}^*$ *and* $A \cap (\mathcal{E}(A))^{\circ}$ *is* $\mathcal{E}$*-preopen.*

Proof. Denote $G := (\mathcal{E}(A))^{\circ}$. First we prove $H \subset \mathcal{E}(A \cap H)$. Let $x \in H$ and $U \subset H \subset G$ be an open set containing x. Since $x \in U \subset G \subset \mathcal{E}(A)$, $A \cap U = A \cap H \cap U$ contains a set from $\mathcal{E}$, so $x \in \mathcal{E}(A \cap H)$. Since $H \subset \mathcal{E}(A \cap H)$, $(\mathcal{E}(A \cap H))^{\circ}$ is nonempty, so $A \cap H \in \mathcal{E}^*$.

Let $x \in (\mathcal{E}(A))^{\circ}$. Then for any open set U containing x and $U \subset (\mathcal{E}(A))^{\circ} \subset \mathcal{E}(A)$ there is $E \in \mathcal{E}$ such that $E \subset U \cap A \subset U \cap A \cap (\mathcal{E}(A))^{\circ}$, hence $x \in \mathcal{E}(A \cap (\mathcal{E}(A))^{\circ})$. We have proved $(\mathcal{E}(A))^{\circ} \subset \mathcal{E}(A \cap (\mathcal{E}(A))^{\circ})$. That means $A \cap (\mathcal{E}(A))^{\circ} \subset (\mathcal{E}(A))^{\circ} \subset [\mathcal{E}(A \cap (\mathcal{E}(A))^{\circ})]^{\circ}$, so $A \cap (\mathcal{E}(A))^{\circ}$ is $\mathcal{E}$-preopen. □

Theorem 4.1. *Let* $\mathcal{E}$ *be a cluster system. Then* $\mathcal{E}^{\mathsf{po}} \sim \mathcal{E}^* < \mathcal{E}$ *and* $\mathcal{N}_{\mathcal{E}} = \mathcal{N}_{\mathcal{E}^*} = \mathcal{N}_{\mathcal{E}^{\mathsf{po}}}$*.*

Proof. Since $\mathcal{E}^{\mathsf{po}} \subset \mathcal{E}^*$, $\mathcal{E}^{\mathsf{po}} < \mathcal{E}^*$. Let $A \in \mathcal{E}^*$. Then $(\mathcal{E}(A))^{\circ} \neq \emptyset$ and by Lemma 4.1, $A \cap (\mathcal{E}(A))^{\circ}$ is $\mathcal{E}$-preopen subset of A, so $\mathcal{E}^* < \mathcal{E}^{\mathsf{po}}$. That means $\mathcal{E}^* \sim \mathcal{E}^{\mathsf{po}}$ and by Lemma 2.2 (3), $\mathcal{N}_{\mathcal{E}^*} = \mathcal{N}_{\mathcal{E}^{\mathsf{po}}}$.

The relation $\mathcal{E}^* < \mathcal{E}$ is clear, so by Lemma 2.2 (2), $\mathcal{N}_{\mathcal{E}} \subset \mathcal{N}_{\mathcal{E}^*}$. Let $A \in \mathcal{N}_{\mathcal{E}^*}$ and $A \notin \mathcal{N}_{\mathcal{E}}$. Then $G := (\mathcal{E}(A))^{\circ} \neq \emptyset$. Since $A \in \mathcal{N}_{\mathcal{E}^*}$, there is a nonempty open set $H \subset G$, such that $A \cap H$ contains no set from $\mathcal{E}^*$. By Lemma 4.1, $A \cap H \in \mathcal{E}^*$, a contradiction. □

The following theorem gives a characterization of sets from $\mathcal{N}_{\mathcal{E}}$ by the $\mathcal{E}^*$-operator.

Theorem 4.2. *$A \in \mathcal{N}_{\mathcal{E}}$ if and only if $\mathcal{E}^*(A) = \emptyset$.*

Proof. If $\mathcal{E}^*(A) = \emptyset$, then $A \in \mathcal{N}_{\mathcal{E}^*}$ and by Theorem 4.1, $A \in \mathcal{N}_{\mathcal{E}}$.

Let $A \in \mathcal{N}_{\mathcal{E}}$ and suppose $\mathcal{E}^*(A) \neq \emptyset$. Let $x \in \mathcal{E}^*(A)$. Then for any open set G containing x the intersection $G \cap A$ contains a set $B \in \mathcal{E}^*$ and by Definition 4.1, $(\mathcal{E}(B))^\circ \neq \emptyset$. Since $A \in \mathcal{N}_{\mathcal{E}}$, for $(\mathcal{E}(B))^\circ$ there is a nonempty open set $H \subset (\mathcal{E}(B))^\circ$ such that $A \cap H$ does not contain a set from $\mathcal{E}$. By Lemma 4.1 and Theorem 4.1, $A \cap H \in \mathcal{E}^* < \mathcal{E}$, so $A \cap H$ contains a set from $\mathcal{E}$, a contradiction. □

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