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LOWER BOUND FOR THE REGULARITY INDEX OF FAT POINTS

Phan Van Thien
Department of Mathematics
Hue Normal University
VIETNAM

Abstract: The problem to find an upper bound for the regularity index of fat points has been dealt with by many authors. In this paper we give a lower bound for the regularity index of fat points. It shall be an useful tool for determining the regularity index.

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1. Introduction

Let $P_1, \ldots, P_s$ be distinct points in the projective space $\mathbb{P}^n := \mathbb{P}^n(k)$, k an algebraically closed field. Denote by $\wp_1, \ldots, \wp_s$ the prime ideals in the polynomial ring $R := k[X_0, \ldots, X_n]$ corresponding to the points $P_1, \ldots, P_s$. Let $m_1, \ldots, m_s$ be positive integers. We will denote by Z the zero-scheme defined by the ideal $I := \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ and call Z a set of *fat points* in $\mathbb{P}^n$.

The homogeneous coordinate ring of Z is R/I. This ring is a one-dimensional Cohen-Macaulay graded ring, $R/I = \bigoplus_{t \geq 0} (R/I)_t$, whose multiplicity is

$$e(R/I) = \sum_{i=1}^{s} \binom{m_i + n - 1}{n}.$$

The function $H_{R/I}(t) := \dim_k (R/I)_t$ strictly increases until it reaches the mul-

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tiplicity $e(R/I)$, at which it stabilizes. The *regularity index* of Z, denote by $\operatorname{reg}(Z)$, is defined to be the least integer t such that $H_{R/I}(t) = e(R/I)$. It is well known that $\operatorname{reg}(Z) = \operatorname{reg}(R/I)$, the Castelnuovo-Mumford regularity of R/I. Hence we will also denote $\operatorname{reg}(Z)$ by $\operatorname{reg}(R/I)$.

The problem to exactly determine the regularity index $\operatorname{reg}(Z)$ is fairly difficult. So, instead of determining $\operatorname{reg}(Z)$, one tries to find an upper bound for it. The problem to find an upper bound for $\operatorname{reg}(Z)$ has been dealt with by many authors (see [1]-[14]). In this paper we will give a lower bound for the regularity index of fat points. The lower bound and upper bound are useful tools for determining the regularity index.

The algebraic method used in this paper as well as in [6], [12], [13], [14].

2. Preliminaries

From now on, we say a j-plane, i.e. a linear j-space. We will identify a hyperplane as the linear form defining it.

We will use the following lemmas which have been proved in [6].

Lemma 1. *[6, Lemma 1] Let $P_1, \ldots, P_r, P$ be distinct points in $\mathbb{P}^n$ and let $\wp$ be the defining ideal of P. If $m_1, \ldots, m_r$ and a are positive integers, $J := \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$, and $I = J \cap \wp^a$, then*

$$\operatorname{reg}(R/I) = \max\{a-1, \operatorname{reg}(R/J), \operatorname{reg}(R/(J+\wp^a))\}.$$

Lemma 2. *[6, Lemma 3] Let $P_1, \ldots, P_r, P$ be distinct points in $\mathbb{P}^n$ and let $\wp$ be the defining ideal of P. Let $a, m_1, \ldots, m_r$ positive integers. Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ and $\wp = (X_1, \ldots, X_n)$. Then*

$$\operatorname{reg}(R/(J+\wp^a)) \leq b$$

if and only if $X_0^{b-i} M \in J + \wp^{i+1}$ for every monomial M of degree i in $X_1, \ldots, X_n$, $i = 0, \ldots, a-1$.

Suppose that we can find t hyperplanes $H_1, \ldots, H_t$ avoiding P such that $H_1 \cdots H_t M \in J$ for every monomial M of degree i in $X_1, \ldots, X_n$, $i = 0, \ldots, a-1$. Since we can write $H_j = X_0 + G_j$ for some linear form $G_j \in \wp$ for $j = 1, \ldots, t$, we get $X_0^t M \in J + \wp^{i+1}$. Therefore, we have the following lemma:

Lemma 3. *Assume that $H_1, \ldots, H_t$ are hyperplanes avoiding P such that $H_1 \cdots H_t M \in J$ for every monomial M of degree i in $X_1, \ldots, X_n$, $i = 0, \ldots, a-1$. If*

$$\delta \geq \max\{t + i | 0 \leq i \leq a-1\}$$

then

$$\operatorname{reg}(R/(J+\wp^a)) \le \delta.$$

The following lemma has been proved in [14].

Lemma 4. *[14, Lemma 3.3] Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$, and $m_1, \dots, m_s$ be positive integers. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. If $Y = \{P_{i_1}, \dots, P_{i_r}\}$ is a subset of X and $J = \wp_{i_1}^{m_{i_1}} \cap \cdots \cap \wp_{i_r}^{m_{i_r}}$, then*

$$\operatorname{reg}(R/I) \ge \operatorname{reg}(R/J).$$

3. Lower Bound for the Regularity Index of Fat Points

Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ and $m_1, \dots, m_s$ be positive integers. Let $n_1, \dots, n_s$ be non-negative integers with $(n_1, \dots, n_s) \ne (0, \dots, 0)$ and $m_i \ge n_i$ for $i = 1, \dots, s$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, $N = \wp_1^{n_1} \cap \cdots \cap \wp_s^{n_s}$ ($\wp_i^{n_i} = R$ if $n_i = 0$). Then we have $e(R/I) \ge e(R/N)$ and $H_{R/I}(t) \ge H_{R/N}(t)$. So, we can not compare $\operatorname{reg}(R/I)$ with $\operatorname{reg}(R/N)$ by definition of the regularity index. In Proposition 6 we will prove that $\operatorname{reg}(R/I) \ge \operatorname{reg}(R/N)$.

The first, we get the following result.

Lemma 5. *Let $X = \{P_1, \dots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ and $m_1, \dots, m_s$, $n_1, \dots, n_s$ be positive integers with $m_i \ge n_i$ for $i = 1, \dots, s$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ and $N = \wp_1^{n_1} \cap \cdots \cap \wp_s^{n_s}$, then*

$$\operatorname{reg}(R/I) \ge \operatorname{reg}(R/N).$$

Proof. In case $m_i = n_i$ for $i = 1, \dots, s$, we have the equality. In case there exists j such that $m_j > n_j$, we may assume that $m_s > n_s$. Put $I_1 = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}} \cap \wp_s^{m_s - 1}$. We will prove $\operatorname{reg}(R/I) \ge \operatorname{reg}(R/I_1)$.

Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$. By Lemma 2 we have

$$\begin{aligned}
&\operatorname{reg}(R/(J+\wp_s^{m_s})) \le b\\
&\Leftrightarrow X_0^{b-i} M \in J + \wp_s^{i+1} \text{ for every } M = X_1^{c_1} \cdots X_n^{c_n}, c_1 + \cdots + c_n = i,\\
&\qquad i = 0, \dots, m_s - 1\\
&\Rightarrow X_0^{b-i} M \in J + \wp_s^{i+1} \text{ for every } M = X_1^{c_1} \cdots X_n^{c_n}, c_1 + \cdots + c_n = i,\\
&\qquad i = 0, \dots, m_s - 2\\
&\Leftrightarrow \operatorname{reg}(R/(J+\wp_s^{m_s - 1})) \le b.
\end{aligned}$$

This implies $\operatorname{reg}(R/(J+\wp_s^{m_s})) \geq \operatorname{reg}(R/(J+\wp_s^{m_s-1}))$. By Lemma 1 we have

$$\begin{aligned} \operatorname{reg}(R/I) &= \max\{m_s - 1, \operatorname{reg}(R/J), \operatorname{reg}(R/(J+\wp_s^{m_s}))\}. \\ \operatorname{reg}(R/I_1) &= \max\left\{m_s - 2, \operatorname{reg}(R/J), \operatorname{reg}(R/(J+\wp_s^{m_s-1}))\right\}. \end{aligned}$$

Therefore, we get

$$\operatorname{reg}(R/I) \geq \operatorname{reg}(R/I_1).$$

By inductive argue on m_s we get

$$\operatorname{reg}(R/I) \geq \operatorname{reg}(R/(\wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}} \cap \wp_s^{n_s})).$$

By induction on number of points we get

$$\operatorname{reg}(R/I) \geq \operatorname{reg}(R/N).$$

□

From the above lemma and Lemma 4 we get the following proposition.

Proposition 6. *Let $X = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ and $m_1, \ldots, m_s$ be positive integers. Let $n_1, \ldots, n_s$ be non-negative integers with $(n_1, \ldots, n_s) \neq (0, \ldots, 0)$ and $m_i \geq n_i$ for $i = 1, \ldots, s$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$, $N = \wp_1^{n_1} \cap \cdots \cap \wp_s^{n_s}$ ($\wp_i^{n_i} = R$ if $n_i = 0$). We have*

$$\operatorname{reg}(R/I) \geq \operatorname{reg}(R/N).$$

A rational normal curve in $\mathbb{P}^j$ to be a curve of degree j that may be given parametrically as the image of the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^j \\ (s,t) &\mapsto (s^n, s^{j-1}t, s^{j-2}t^2, \ldots, t^j). \end{aligned}$$

Let $Q_1, \ldots, Q_r$ be distinct points on a linear j-space, say α, in $\mathbb{P}^n$. If there exist a rational normal curve, say $\mathcal{C}$, in $\mathbb{P}^j$ and an isomorphism of a linear change of coordinates

$$\varphi : \mathbb{P}^j \to \alpha$$

such that $Q_1, \ldots, Q_r \in \varphi(\mathcal{C})$, then we said that $Q_1, \ldots, Q_r$ are in Rnc-j. In $\mathbb{P}^n$ the points whose coordinators satisfying parametric equations

$$X_0 = t^j, X_1 = t^{j-1}u, \ldots, X_{j-1} = tu^{j-1}, X_j = u^j, X_{j+1} = \cdots = X_n = 0$$

lie on a image of a rational normal curve in $\mathbb{P}^j$ by an isomorphism of a linear change of coordinates. So, if $Q_1,\dots,Q_r$ are in Rnc-j in $\mathbb{P}^n$, then we may assume that their coordinators satisfying the above parametric equations.

Let $Z=m_1P_1+\cdots+m_sP_s$ be a set of fat points in $\mathbb{P}^n$. Then the set

$$\{P_{i_1},\dots,P_{i_q}\in\{P_1,\dots,P_s\}|P_{i_1},\dots,P_{i_q} \text{ are in } Rnc-1\}$$

is non-empty.

The following theorem shows a lower bound for the regularity index of fat points.

Theorem 7. *Let $Z=m_1P_1+\cdots+m_sP_s$ be a set of fat points in $\mathbb{P}^n$. Then,*

$$\operatorname{reg}(Z)\geq\max\{D_j|\ j=1,\dots,n\},$$

where

$$D_j=\max\left\{\left[\frac{\sum_{l=1}^{q}m_{i_l}+j-2}{j}\right]\mid P_{i_1},\dots,P_{i_q} \text{ are in Rnc-}j\right\}.$$

Proof. Suppose that points $P_{i_1},\dots,P_{i_q}$ of $\{P_1,\dots,P_n\}$ are Rnc-j in $\mathbb{P}^n$. We may assume that $m_{i_1}\geq\cdots\geq m_{i_q}$ (after relabeling the points, if necessary). Let $\wp_{i_1},\dots,\wp_{i_q}$ be the homogeneous prime ideals of R corresponding to the points $P_{i_1},\dots,P_{i_q}$. Put

$$J=\wp_1^{m_{i_1}}\cap\cdots\cap\wp_{i_q}^{m_{i_q}}.$$

By Lemma 4 we have

$$\operatorname{reg}(Z)\geq\operatorname{reg}(R/J).$$

We will prove that

$$\operatorname{reg}(R/J)\geq\left[\frac{\sum_{l=1}^{q}m_{i_l}+j-2}{j}\right].$$

Since the points $P_{i_1},\dots,P_{i_q}$ are in Rnc-j in $\mathbb{P}^n$, we may assume that their coordinators satisfying parametric equations:

$$X_0=t^j,X_1=t^{j-1}u,\dots,X_{j-1}=tu^{j-1},X_j=u^j,X_{j+1}=\cdots=X_n=0$$

and the points $P_{i_q}=(1,0,\dots,0)$. Then $\wp_{i_q}=(X_1,\dots,X_n)$. Put

$$J_1=\wp_{i_1}^{m_{i_1}}\cap\cdots\cap\wp_{i_{q-1}}^{m_{i_{q-1}}}.$$

The first, we will prove that

$$\operatorname{reg}(R/(J_1+\wp_{i_q}^{m_{i_q}})) \geq \left[\frac{\sum_{l=1}^{q} m_{i_l}+j-2}{j}\right].$$

Put $T=\left[\frac{\sum_{l=1}^{q} m_{i_l}+j-2}{j}\right]$. Consider the monomial $X_1^{m_{i_q}-1}$. If

$$X_0^{T-m_{i_q}}X_1^{m_{i_q}-1} \in J_1+\wp_{i_q}^{m_{i_q}},$$

then there exists a form $h \in \wp_{i_q}^{m_{i_q}}$ of degree $T-1$ such that

$$X_0^{T-m_{i_q}}X_1^{m_{i_q}-1}+h \in J_1.$$

Since $X_0^{T-m_{i_q}}X_1^{m_{i_q}-1} \in \wp_{i_q}^{m_{i_q}-1}$ and $h \in \wp_{i_q}^{m_{i_q}} \subset \wp_{i_q}^{m_{i_q}-1}$, we have

$$X_0^{T-m_{i_q}}X_1^{m_{i_q}-1}+h \in J_1 \cap \wp_{i_q}^{m_{i_q}-1}.$$

Moreover, $m_{i_1}+\cdots+m_{i_q}-1 > j(T-1)$, hence by Bezout's theorem

$$X_0^{T-m_{i_q}}X_1^{m_{i_q}-1}+h$$

vanishing on the points $(1,\lambda,\ldots,\lambda^j,0,\ldots,0) \in \mathbb{P}^n$, for every λ in the field k. This implies

$$\lambda^{m_{i_q}-1}+h(1,\lambda,\ldots,\lambda^j,0,\ldots,0)=0$$

for every $\lambda \in k$. Since $h \in \wp_{i_q}^{m_{i_q}}=(X_1,\ldots,X_n)^{m_{i_q}}$, we have $h(1,\lambda,\ldots,\lambda^j,0,\ldots,0)=0$ or $h(1,\lambda,\ldots,\lambda^j,0,\ldots,0)=\lambda^{m_{i_q}}g(\lambda)$, for some non-zero polynomial $g \in k[x]$. Hence, $\lambda^{m_{i_q}-1}=0$ or $\lambda^{m_{i_q}-1}+\lambda^{m_{i_q}}g(\lambda)=0$ for every $\lambda \in k$, a contradiction. Thus, we get

$$X_0^{T-m_{i_q}}X_1^{m_{i_q}-1} \notin J_1+\wp_{i_q}^{m_{i_q}}.$$

By Lemma 2 we have

$$\operatorname{reg}(R/(J_1+\wp_{i_q}^{m_{i_q}})) \geq T.$$

Next, by Lemma 1 we get

$$\begin{aligned}\operatorname{reg}(R/J) &= \max\{m_{i_q}-1, \operatorname{reg}(R/J_1), \operatorname{reg}(R/(J_1+\wp_{i_q}^{m_{i_q}}))\}\\ &\geq \operatorname{reg}(R/(J_1+\wp_{i_q}^{m_{i_q}})) \geq T.\end{aligned}$$

The proof of Theorem 7 is now completed. □

4. Application of Lower Bound

The first, by using the lower bound we can compute the regularity index of fat points whose support on a line. This formula was showed by E.D. Davis and A.V. Geramita in [7] by using another method.

Proposition 8. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$. If $P_1, \dots, P_s$ lie on a line, then*

$$\operatorname{reg}(Z) = m_1 + \cdots + m_s - 1.$$

Proof. We may assume that $m_1 \geq \cdots \geq m_s$. If $P_1, \dots, P_s$ lie on a line, then $D_1 = m_1 + \cdots + m_s - 1$. Put $I = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$. We will prove that

$$\operatorname{reg}(R/I) = D_1.$$

By Theorem 7 we have

$$\operatorname{reg}(R/I) \geq \max\{D_j |\ j = 1, \dots, n\}.$$

So, it suffices to prove by induction on s that

$$\operatorname{reg}(R/I) \leq D_1.$$

Put $J = \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}$, by the inductive assumption, we get

$$\operatorname{reg}(R/J) \leq m_1 + \cdots + m_{s-1} - 1 \leq D_1. \qquad (1)$$

Choose $P_s = (1, 0, \dots, 0)$, then $\wp_s = (X_1, \dots, X_n)$. Since $P_1, \dots, P_s$ lie on a line, there exists hyperplane, say H_j, passing throught P_j and avoiding P_s for $j = 1, \dots, s-1$. This implies

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} \in J.$$

Therefore, for every monomial $M = X_1^{c_1} X_2^{c_2} \cdots X_n^{c_n}$ of degree i, $i = 0, \dots, m_s - 1$, we have

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} M \in J.$$

By Lemma 3 we get

$$\operatorname{reg}(R/(J + \wp_s^{m_s})) \leq D_1. \qquad (2)$$

From (1), (2) and Lemma 1 we get

$$\operatorname{reg}(R/I) \leq D_1.$$

□

Now we consider a set of fat points whose support is in Rnc-j.

Lemma 9. *Let $Z = m_1P_1 + \cdots + m_sP_s$ be a set of fat points in $\mathbb{P}^n$. Suppose that j is the least integer such that $P_1, \ldots, P_s$ are in Rnc-j. If t is an integer such that*

$$t \geq \max\left\{m_l, \left[\frac{m_1 + \cdots + m_{s-1} + j - 1}{j}\right] \mid l = 1, \ldots, s-1\right\},$$

then we can find t hyperplanes, say $H_1, \ldots, H_t$, avoiding P_s sucth that

$$H_1 \cdots H_t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

Proof. Since the points $P_1, \ldots, P_s$ are in Rnc-j in $\mathbb{P}^n$, we may assume that their coordinators satisfying parametric equations:

$$X_0 = v^j, X_1 = v^{j-1}u, \ldots, X_{j-1} = vu^{j-1}, X_j = u^j, X_{j+1} = \cdots = X_n = 0.$$

If $t = 1$, then $P_1, \ldots, P_s$ lie on a line. For $j = 1, \ldots, s-1$, there exists a hyperplane, say H_j, passing throught P_j and avoiding P_s. Then we have $t = m_1 + \cdots + m_{s-1}$ hyperplanes $\underbrace{H_1, \ldots, H_1}_{m_1}, \ldots, \underbrace{H_{s-1}, \ldots, H_{s-1}}_{m_{s-1}}$ avoiding P_s such that

$$H_1^{m_1} \cdots H_{s-1}^{m_{s-1}} \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

If $t \geq 2$, then no $l+2$ points of $\{P_1, \ldots, P_s\}$ are on a l-plane for $l < j$. This implies that there does not exist any $(j-1)$-plane containing $j+1$ points of $\{P_1, \ldots, P_s\}$. We will prove the lemma by induction on $\sum_{i=1}^{s-1} m_i$.

We may assume that $m_1 \geq \cdots \geq m_{s-1}$. Since j is the least integer such that $P_1, \ldots, P_s$ are in Rnc-j, we have $j \leq s-1$. Let σ_1 be the $(j-1)$-plane passing throught $P_1, \ldots, P_j$. Then σ_1 avoids P_s. Therefore, there is a hyperplane, say L_1, containing σ_1 and avoiding P_s.

Case $s-1 = j$: Then

$$L_1^t \in \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_1} \subset \wp_1^{m_1} \cap \cdots \cap \wp_{s-1}^{m_{s-1}}.$$

Case $s-1 \geq j+1$: Since $t \geq \left[\frac{m_1+\cdots+m_{s-1}+j-1}{j}\right]$ and $m_1 \geq \cdots \geq m_{s-1}$, we have

$$\begin{aligned} t-1 &\geq \left[\frac{m_1 + \cdots + m_{s-1} + j - 1}{j}\right] - 1 \geq \left[\frac{(j+1)m_{j+1} - 1}{j}\right] \\ &\geq m_{j+1}. \end{aligned}$$

On the other hand, since $t \geq \left[\frac{m_1+\cdots+m_{s-1}+j-1}{j}\right]$, we get

$$t-1 \geq \left[\frac{(m_1-1)+\cdots+(m_j-1)+m_{j+1}+\cdots+m_{s-1}+j-1}{j}\right].$$

Consider

$$Z_1 = (m_1-1)P_1+\cdots+(m_j-1)P_j+m_{j+1}P_{j+1}+\cdots+m_{s-1}P_{s-1}+m_sP_s.$$

By the inductive assumption we can find $(t-1)$ hyperplanes, say $L_2,\ldots,L_t$, avoiding P_s such that

$$L_2\cdots L_t \in \wp_1^{m_1-1}\cap\cdots\cap\wp_j^{m_j-1}\cap\wp_{j+1}^{m_{j+1}}\cap\cdots\cap\wp_{s-1}^{m_{s-1}}.$$

Moreover, since $L_1 \in \wp_1\cap\cdots\cap\wp_j$, we get

$$L_1L_2\cdots L_t \in \wp_1^{m_1}\cap\cdots\cap\wp_{s-1}^{m_{s-1}}.$$

□

We can compute the regularity index of fat points whose support is in Rnc-j.

Proposition 10. *Let $Z = m_1P_1+\cdots+m_sP_s$ be a set of fat points in $\mathbb{P}^n$. If $P_1,\ldots,P_s$ are in Rnc-t, then*

$$\operatorname{reg}(Z) = \max\{D_j | j = 1,\ldots,t\},$$

where

$$D_j = \max\left\{\left[\frac{\sum_{l=1}^{q} m_{i_l}+j-2}{j}\right] \mid P_{i_1},\ldots,P_{i_q} \text{ are in Rnc-}j\right\}.$$

Proof. We may assume that $m_1 \geq \cdots \geq m_s$. We will argue by induction on s. If $s=1$, then $\operatorname{reg}(Z) = m_1-1 = D_1$. If $s \geq 2$, we consider two following cases:
Case $t=1$: Then $P_1,\ldots,P_s$ lie on a line and $D_1 = m_1+\cdots+m_s-1 = \max\{D_j|\ j=1,\ldots,n\}$. By Proposition 8 we have $\operatorname{reg}(Z) = D_1$.
Case $t \geq 2$: Since $P_1,\ldots,P_s$ are in Rnc-t, there is the least integer $p \leq t$ such that $P_1,\ldots,P_s$ are in Rnc-p. Then

$$D_1 = m_1+m_2-1 \geq D_2 \geq \cdots \geq D_{p-1},$$
$$D_p = \left[\frac{m_1+\cdots+m_s+p-2}{p}\right] \geq D_{p+1} \geq \cdots \geq D_n.$$

So, $\max\{D_j|\ j=1,\dots,n\}=\max\{D_j|\ j=1,\dots,t\}=\max\{D_1,D_p\}$. Hence, by Theorem 7 we get

$$\operatorname{reg}(Z)\geq\max\{D_1,D_p\}.$$

It sufficies to prove that

$$\operatorname{reg}(Z)\leq\max\{D_1,D_p\}.$$

Put $Z_1=m_1P_1+\cdots+m_{s-1}P_{s-1}$, $J=\wp_1^{m_1}\cap\cdots\cap\wp_{s-1}^{m_{s-1}}$ and $Y=\{P_1,\dots,P_{s-1}\}$. We have $\operatorname{reg}(Z_1)=\operatorname{reg}(R/J)$. By inductive hypothesis we have

$$\operatorname{reg}(Z_1)=\max\{D'_j|j=1,\dots,t\},$$

where

$$D'_j=\max\left\{\left[\frac{\sum_{l=1}^{q}m_{i_l}+j-2}{j}\right]|Y\ni P_{i_1},\dots,P_{i_q}\text{ are in Rnc-j}\right\}.$$

Since $\{P_1,\dots,P_{s-1}\}\subset\{P_1,\dots,P_{s-1},P_s\}$, we have $D'_j\leq D_j$ for $j=1,\dots,t$. Therefore, we get

$$\operatorname{reg}(R/J)\leq\max\{D_1,D_p\}. \qquad (3)$$

Consider $R/(J+\wp_s^{m_s})$. We may assume that

$$P_s=(1,0,\dots,0),\ P_1=(0,\underbrace{1}_{2},0,\dots,0),\dots,P_p=(0,\dots,0,\underbrace{1}_{p+1},0,\dots,0).$$

For every monomial $M=X_1^{c_1}\cdots X_n^{c_n}$, $c_1+\cdots+c_n=i$, $i=0,\dots,m_s-1$. Put

$$m'_l=\begin{cases}m_l-i+c_l & \text{for } l=1,\dots,p,\\ m_l & \text{for } l=p+1,\dots,s-1.\end{cases}$$

Put $J'=\wp_1^{m'_1}\cap\cdots\cap\wp_{s-1}^{m'_{s-1}}$. By Proposition 4 we can find

$$t=\max\left\{m'_l,\left[\frac{m'_1+\cdots+m'_{s-1}+p-1}{p}\right]|l=1,\dots,s-1\right\}$$

hyperplanes, say $H_1,\dots,H_t$, avoiding P_s such that

$$H_1\cdots H_t\in J'.$$

Since $M\in\wp_1^{i-c_1}\cap\cdots\cap\wp_p^{i-c_p}$ and $J'=\wp_1^{m_1-i+c_1}\cap\cdots\cap\wp_p^{m_p-i+c_p}\cap\wp_{p+1}^{m_{p+1}}\cdots\cap\wp_{s-1}^{m_{s-1}}$, we get

$$H_1\cdots H_tM\in J.$$

By Lemma 3 we get

$$\operatorname{reg}(R/(J+\wp_s^{m_s})) \leq \max\{t+i|i=1,\ldots,m_s-1\} \leq \max\{D_1, D_p\}. \quad (4)$$

Put $I = J \cap \wp_s^{m_s}$. We have $\operatorname{reg}(Z) = \operatorname{reg}(R/I)$. From (3), (4) and Lemma 1 we have

$$\operatorname{reg}(Z) \leq \max\{D_1, D_p\}.$$

□

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