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NEW TRICORNS AND MULTICORNS ANTIFRACTALS IN JUNGCK MANN ORBIT

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Abstract: The aim of this paper is to study the visualization of tricorns and multicorns antifractals and the pattern among them in Jungck Mann orbit.

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1. Introduction

The fractal geometry in mathematics has presented some attractive complex graphs and objects to computer graphics. Fractal is a Latin word, derived from

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the word “fractus” which means “broken”. Julia [4] introduced the concept of iterative function system and by using it, he derived the Julia set in 1918. After that Mandelbrot [6] extended the work of Julia and introduced the Mandelbrot set; a set of all connected Julia sets. He expanded the ideas of Julia and introduced the Mandelbrot set by using the complex function $z^2 + c$ with using z as a complex function and c as a complex parameter. The fractal structure of Mandelbrot and Julia sets have been demonstrated for quadratic, cubic and higher degree polynomials, by using Picard orbit which is an application of one-step feedback process [3].

The dynamics of antiholomorphic complex polynomials $\bar{z}^n + c$, for $n \geq 2$, leads to interesting tricorns and multicorns antifractals with respect to one-step feedback process [3], two step-feedback process [13, 14] and three-step feedback process [1, 7].

In 2003, Nakane and Schleicher [9] has presented the various properties of multicorns and tricorns along with beautiful figures. They have quoted the multicorns as the generalized tricorns or the tricorns of higher order. The dynamics of antipolynomial $\bar{z}^n + c$ of complex polynomial $z^n + c$, where $n \geq 2$ leads to interesting tricorns and multicorns antifractals with respect to function iteration (see [3, 8, 9]). Tricorns are being used for commercial purpose, e.g. Tricorn mugs and Tricorn shirts. Multicorns are symmetrical objects. Their symmetry has been studied by Lau and Schleicher [5].

The study of connectedness locus for antiholomorphic polynomials $\bar{z}^2 + c$ defined as tricorns, coined by Milnor [7], plays intermediate role between quadratic and cubic polynomials. In 1989, Crowe et al. [2] considered it as an formal analogy with Mandelbrot sets and named it as Mandelbar set and also brought their bifurcation features along arcs rather than at points. Milnor [7] found it as a real slice of cubic connected locus. Winter [17] showed that the boundary of the tricorns contains arc. The symmetries of tricorns and multicorns have been analyzed by Lau and Schleicher [5], and Nakane and Schleicher [8, 9] presented their various properties along with beautiful figures and quoted that multicorns are the generalized tricorns or the tricorns of higher order. Superior tricorns and uperior multicorns using the Mann iterates rather than function iterates is studied and explored by Negi [10].

In this paper we present a new class of tricorns and multicorns using Jungck Mann orbit and analyze them.

2. Preliminaries

Definition 2.1. (Mandelbrot set [6]). The Mandelbrot set M for the quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of the point 0 is bounded, that is,

$$M = \{c \in \mathbb{C} : \{Q_c^n(0) : n = 0, 1, 2, ...\} \text{ is bounded}\}.$$

An equivalent formulation is

$$M = \{c \in \mathbb{C} : Q_c^n(0) \text{ doesn't tends to } \infty \text{ as } n \to \infty\}.$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

Definition 2.2. (Julia set [4]). The attractor basin of infinity is never all of $\mathbb{C}$, since f_c has fixed points $z_f = 1/2 \pm \sqrt{1/4 + c}$ (and also points of period n, that satisfy a polynomial equation of degree 2^n, namely $f^n(z) = z$). The nonempty, compact boundary of the attractor basin of infinity is called the Julia set of f_c,

$$J_c = \partial A_\infty(c).$$

Definition 2.3. (Filled Julia set [4]). The filled in Julia set of the function f is defined as

$$K(f) = \{z \in \mathbb{C} : f^k(z) \to \infty\}.$$

Definition 2.4. ([4]) The Julia set of the function f is defined to be the boundary of $K(f)$, that is,

$$J(f) = \partial K(f).$$

Definition 2.5. ([3]) Let $\{z_n : n = 1, 2, 3, 4...\}$, denoted by $\{z_n\}$ be a sequence of complex numbers. Then, we say $\underset{n\to\infty}{Lim}\ z_n = \infty$ if for given $M > 0$, there exists $N > 0$, such that for all $n > N$, we must have $|z_n| > M$. Thus all the values of z_n lies outside a circle of radius M for sufficiently large values of n. Let

$$Q(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z^1 + a_n z^0, \quad a_0 \neq 0$$

be a polynomial of degree n, where $n \geq 2$. The coefficients are allowed to be complex numbers. In other words, it follows that $Q_c(z) = z^2 + c$.

Definition 2.6. Following Milnor [7], Nakane and Schleicher [8] have defined the tricorns, as the connectedness locus for antiholomorphic polynomials $\bar{z}^n + c$ for $n \geq 2$.

Definition 2.7. The multicorns S_c for the quadratic function $S_c z = \bar{z}^n + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$S_c z = \{c \in C : S_c^n(0) \text{ doesn't tends to } \infty \text{ as } n \to \infty\},$$

where C is a complex space, S_c^n is the nth iterate of the function $S_c z$. An equivalent formulation is that the connectedness of loci for higher degree anti-holomorphic polynomials $S_c z = \bar{z}^n + c$ are called multicorns [14].

Notice that at $n = 2$, multicorns reduce to tricorns. Moreover, the tricorns naturally lives in the real slice $d = \bar{c}$ in the two dimensional parameter space of maps $z \to (z^2 + d)^2 + c$. They have $(n+1)$-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns [14].

Definition 2.8. (Picard orbit [3]) Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, the Picard's orbit is defined as the set of iterates of a point x_0, that is,

$$O(f, x_0) = \{x_n : x_n = f(x_{n-1}), n = 1, 2, 3, ...\}.$$

Definition 2.9. (Jungck Mann orbit) Let us consider the sequence $\{x_n\}$ of iterates for any initial point $x_0 \in X$ such that

$$Sx_{n+1} = (1 - \alpha)Sx_n + \alpha T x_n,$$

where $\alpha \in (0, 1)$ for $n = 0, 1, 2, ...$ The above sequence of iterates is called Jungck Mann orbit, denoted by JMO, which is a function of five tuple (T, x_0, α).

In nonlinear dynamics, we have two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded, that is, there exists M, such that $|Q^n(z)| \leq M$ for all n is called as a prisoner set, while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Mandelbrot set for Q.

To visualize antifractals in JMO for $\bar{z}^n + c$, we shall require escape criterion with respect to JMO. Escape criterion for $\bar{z}^n + c$ in JMO is

$$|z_k| > \max\left\{|c|, \left(\frac{2(1+|a|)}{\alpha}\right)^{n-1}\right\}.$$

3. Generation of Tricorns and Multicorns in Jungck Mann Orbit

By using the Mathematica, we have generated tricorns and multicorns for $f(x) = \bar{z}^n + az + c$, where $a, c \in C$ with $Tz = \bar{z}^n + c$ and $Sz = az$ in JMO. A few selected tricorns and multicorns for the complex function $f(x) = \bar{z}^n + az + c$, where $a, c \in C$ with $Tz = \bar{z}^n + c$ and $Sz = az$ are generated at different values of n and a in JMO (see Figures 1-13). Following are the observations made from generated multicorns:

- Here we notice that the number of ovoids in the tricorns and Multicorns is $n+1$, where n is the power of $\bar{z}$. Also, few branches have n subbranches.
- Multicorns exhibit $(n+1)$-fold rotational symmetries.
- For an n, there exist many multicorns.
- Higher degree multicorns become circular saw.

Figure 1. Tricorn for $n = 2$, $a = 3$ and $\alpha = 0.7$

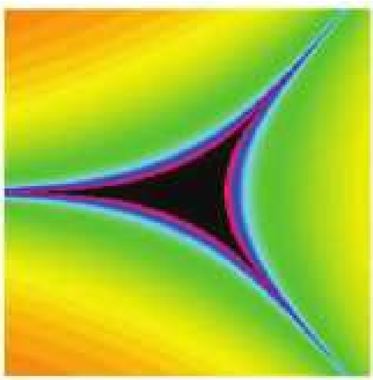

Figure 2. Tricorn for $n = 2$, $a = \frac{1}{2}$ and $\alpha = 0.1$

Figure 3. Tricorn for $n = 2$, $a = 9$ and $\alpha = 0.9$

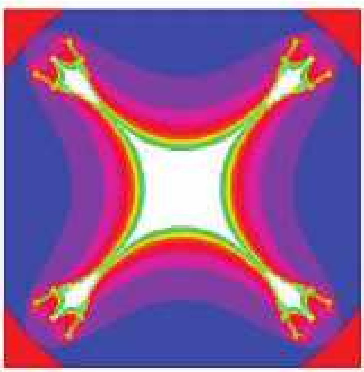

Figure 4. Multicorn for $n = 3$, $a = 7$ and $\alpha = 0.7$

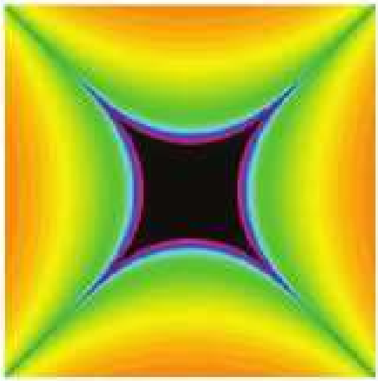

Figure 5. Multicorn for $n = 3$, $a = \frac{1}{2}$ and $\alpha = 0.1$

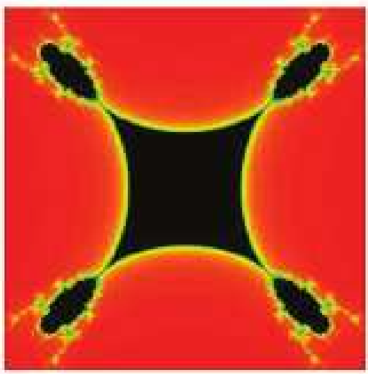

Figure 6. Multicorn for $n = 3$, $a = 9$ and $\alpha = 0.9$

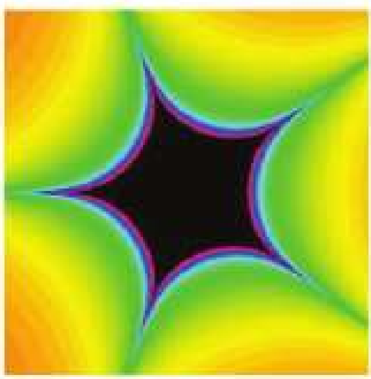

Figure 7. Multicorn for $n = 4$, $a = \frac{1}{2}$ and $\alpha = 0.2$

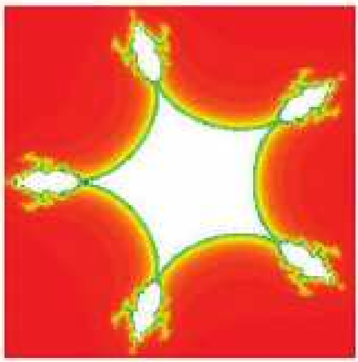

Figure 8. Multicorn for $n = 4$, $a = 9$ and $\alpha = 0.87$

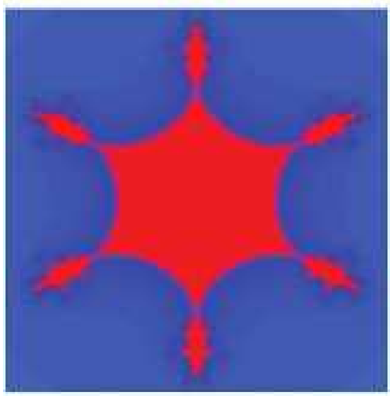

Figure 9. Multicorn for $n = 5$, $a = 3$ and $\alpha = 0.9$

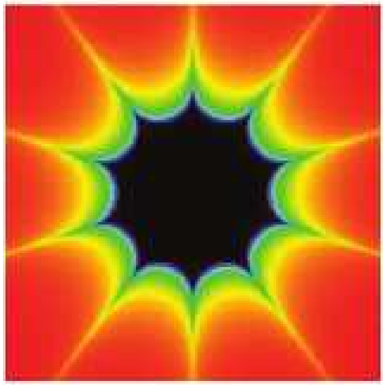

Figure 10. Multicorn for $n = 9$, $a = \frac{1}{2}$ and $\alpha = 0.099$

Figure 11. Multicorn for $n = 15$, $a = 3$ and $\alpha = 0.9$

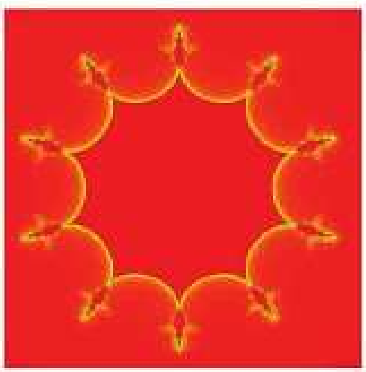

Figure 12. Multicorn for $n = 9$, $a = 9$ and $\alpha = 0.9$

Figure 13. Multicorn for $n = 19$, $a = \frac{1}{2}$ and $\alpha = 0.8$

4. Generation of Anti Julia Sets in Jungck Mann Orbit

Anti Julia sets have been generated for the complex function $f(x) = \bar{z}^n + az + c$, where $a, c \in C$ with $Tz = \bar{z}^n + c$ and $Sz = az$ are generated at different values of n, a and c in JMO (see Figures 14-24). Figures 14-16 show that at $n = 2$, the anti Julia sets take the shape of tricorns. Further, it has been observed that the higher degree anti Julia sets become circular saw (Figures 23 and 24).

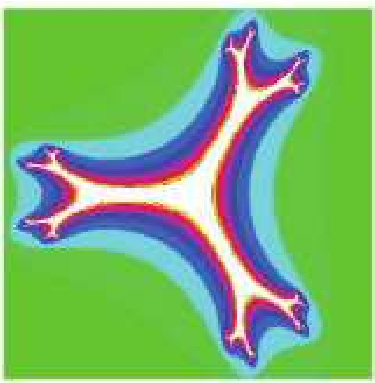

Figure 14. Anti Julia set for $n = 2$, $a = 3$, $c = 3i$ and $\alpha = 0.1$

Figure 15. Anti Julia set for $n = 2$, $a = 9$, $c = 0.2 - 3i$ and $\alpha = 0.1$

Figure 16. Anti Julia set for $n = 2, a = 9, c = 35 + 60i$ and $\alpha = 0.9$

Figure 17. Anti Julia set for $n = 3$, $a = 3$, $c = 0.5i$ and $\alpha = 0.3$

Figure 18. Anti Julia set for $n = 3$, $a = 3$, $c = -2i$ and $\alpha = 0.1$

Figure 19. Anti Julia set for $n = 3$, $a = 5$, $c = -1$ and $\alpha = 0.01$

Figure 20. Anti Julia set for $n = 4$, $a = 9$, $c = 0.2 - 3i$ and $\alpha = 0.1$

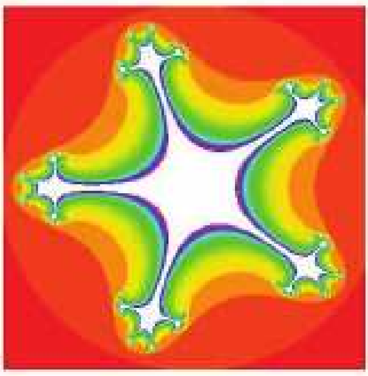

Figure 21. Anti Julia set for $n = 4$, $a = 5$, $c = -1$ and $\alpha = 0.01$

Figure 22. Anti Julia set for $n = 9$, $a = 9$, $c = -10$ and $\alpha = 0.9$

Figure 23. Anti Julia set for $n = 15$, $a = 9$, $c = 2.2 - 3i$ and $\alpha = 0.1$

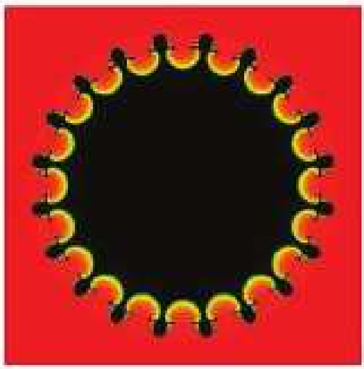

Figure 24. Anti Julia set for $n = 19$, $a = 5$, $c = -1$ and $\alpha = 0.01$

5. Conclusions

In the dynamics of antipolynomial of complex polynomial function $f(x) = \bar{z}^n + az + c$ with $Tz = \bar{z}^n + c$ and $Sz = az$, where $a, c \in C$ and $n \geq 2$, there exist many multicorns for the same value of n in Jungck Mann orbit. Anti Julia sets have also been generated in Jungck Mann orbit. Further, for the odd values of n, all the multicorns are symmetrical objects, and for even values of n, all the multicorns (including tricorns) are symmetrical about x-axis. It was found that for higher degrees of the polynomial, all the antifractals become circular saw.

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