

**DERIVATIONS AND CENTROIDS OF
FOUR-DIMENSIONAL ASSOCIATIVE ALGEBRAS**

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Abstract: In the paper derivations and centroids of four-dimensional associative algebras are described. We also identify the class of algebras called characteristically nilpotent among four-dimensional associative algebras.

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1. Introduction

The interest in the study of derivations of algebras goes back to a paper by Jacobson [1]. There, Jacobson proved that any Lie algebra over a field of characteristic zero which has non degenerate derivations is nilpotent. In the same paper, he asked for the converse. Dixmier and Lister [2] have given a negative answer to the converse of Jacobson's hypothesis by constructing an example of a nilpotent Lie algebra all of whose derivations are nilpotent (hence degenerate). Lie algebras whose derivations are nilpotent have been called characteristically nilpotent. The result of Dixmier and Lister in [2] is assumed to be the origin of the theory of characteristically nilpotent Lie algebras. A few years later in 1959, Leger and Togo [3] published a paper showing the importance of the characteristically nilpotent Lie algebras. The theory of characteristically nilpotent Lie algebras constitutes an independent research object since 1955. Until then, most studies about Lie algebras were oriented to the classical aspects of the theory, such as semi-simple and reductive Lie algebras (see [4]).

The structural theory of finite dimensional associative algebras have been treated by [5]. Many interesting results related to the problem have appeared since then. Further works in this field can be found in [6], [7], [8], [9] and [10]. Further development of the theory of associative algebras was in 80-s of the last century when many open problems, remaining on unsolved since 30-s, have been solved

Centroids of algebras play an important role in the classification problems and in different areas of structure theory of algebras. The centroid of some classes of a Lie algebra is known to be a field and this fact plays an important role in the classification problem of the class considered (see [11]). Benkart and Neher studied extended affine and root graded Lie algebras in [11]. Melville in [12] studied the centroids of nilpotent Lie algebra. Centroids and derivations of associative algebras in dimension less than four has been treated in [13] and provided an impetus to further the study in higher dimension.

In this study, we concentrate on the derivations and centroids of associative algebras in dimension four. Using the classification result of associative algebras, we give a description of derivations and centroids of associative algebras in dimension 4.

2. Preliminaries

In this section we review some elementary facts on algebras, derivations and centroids.

Definition 1. An algebra A over a field \mathbb{K} is a vector space over \mathbb{K} equipped with a bilinear map

$$\lambda : A \times A \rightarrow A.$$

Definition 2. An associative algebra A is an algebra with bilinear map $\lambda : A \times A \rightarrow A$ satisfying the associative law:

$$\lambda(\lambda(x, y), z) = \lambda(x, \lambda(y, z)) \text{ for all } x, y, z \in A.$$

Definition 3. A Lie algebra L over a field \mathbb{K} is an algebra satisfying the following conditions:

$$\begin{aligned} \lambda(x, x) &= 0, \\ \lambda(\lambda(x, y), z) + \lambda(\lambda(y, z), x) + \lambda(\lambda(z, x), y) &= 0, \end{aligned}$$

for all $x, y, \in L$.

It is customary to denote the binary operation λ in Lie algebras case as the bracket $[\cdot, \cdot]$.

Definition 4. Let (A_1, \cdot) and (A_2, \bullet) be two associative algebras over a field \mathbb{K} . A homomorphism from A_1 to A_2 is a \mathbb{K} -linear mapping $f : A_1 \rightarrow A_2$ such that

$$f(x \cdot y) = f(x) \bullet f(y), \text{ for all } x, y \in A_1.$$

The set of all homomorphisms from A_1 to A_2 is denoted by $Hom_{\mathbb{K}}(A_1, A_2)$. A bijective homomorphism is called isomorphism and the corresponding algebras are said to be isomorphic. If $A_1 = A_2 = A$, then $Hom_{\mathbb{K}}(A)$ is an associative algebra with respect to the composition operation. The algebra $Hom_{\mathbb{K}}(A)$ is called the algebra of endomorphisms of A .

Example 5. Let (A, \cdot) be an associative algebra. Define the product of two elements in A as follows:

$$[a, b] = a \cdot b - b \cdot a.$$

Then $(A, [\cdot, \cdot])$ is a Lie algebra. Particularly, the associative algebra $Hom_{\mathbb{K}}(A)$ above is a Lie algebra under $[f, g] = f \circ g - g \circ f$.

Definition 6. A linear transformation d of an associative algebra (A, \cdot) is called a derivation if for any $x, y \in A$ one has

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y).$$

The set of all derivations of an associative algebra A is denoted by $Der(A)$.

In Lie algebras case the property to be characteristically nilpotent is an important property. For associative algebras case this property is defined similarly to that of Lie algebras case.

The following facts can easily be proven.

Proposition 7. Let A be an algebra. Then $Der(A)$ is a Lie algebra with respect to the bracket $[f, g] = f \circ g - g \circ f$.

For $a \in A$, let L_a and R_a be the elements of $Hom_{\mathbb{K}}(A)$ defined by

$$L_a(x) = a \cdot x, \quad R_a(x) = x \cdot a, \text{ for } x \in A.$$

Theorem 8. Let $d \in Hom_{\mathbb{K}}(A)$. Then the following conditions are equivalent :

1. $d \in Der(A)$,
2. $[d, L_x] = L_{d(x)}, \quad x \in A$,
3. $[d, R_y] = R_{d(y)}, \quad y \in A$.

Proof. The proof is straightforward. □

Definition 9. Let A be an arbitrary associative algebra over a field \mathbb{K} : The centroid of $\Gamma(A)$, is defined by

$$\Gamma(A) = \{\phi \in End(A) : \phi(x \cdot y) = \phi(x) \cdot y = x \cdot \phi(y)\}.$$

Definition 10. Let H be a non empty subset of A . The set

$$Z_A(H) = \{x \in A : x \cdot H = H \cdot x = 0\}.$$

is said to be centralizer of H in A .

It must however be noted that $Z_A(A) = Z(A)$, the center of A .

Definition 11. Let $\phi \in End_{\mathbb{K}}(A)$. If $\phi(A) \subseteq Z(A)$ and $\phi(A^2) = 0$, then ϕ is called a central derivation.

The set of all central derivations of A is denoted by $C(A)$.

In the foregoing, we give a few earlier results on some properties of centroids of associative algebras which show the relationship between derivation and centroid. The proof of some of the facts given below can be found in [13].

Proposition 12. *Let A be an associative algebra over a field \mathbb{K} . Then*

1. *If $\phi \in \Gamma(A)$ and $d \in \text{Der}(A)$ then $\phi \circ d$ is a derivation of A .*
2. *$C(A) = \Gamma(A) \cap \text{Der}(A)$.*
3. *For any $d \in \text{Der}(A)$ and $\phi \in \Gamma(A)$ one has*
 - *The composition $d \circ \phi$ is in $\Gamma(A)$ if and only if $\phi \circ d$ is a central derivation of A ;*
 - *The composition $d \circ \phi$ is a derivation of A if and only if $[d \circ \phi]$ is a central derivation of A .*

3. Procedure for finding derivations

Let A be an n -dimensional associative algebra and d be its derivation. Fix a basis $\{e_1, e_2, \dots, e_n\}$ of A . Then,

$$d \circ L_{e_i}(y) = L_{d(e_i)}(y) + L_{e_i} \circ d(y),$$

for basis vectors $\{e_1, e_2, \dots, e_n\}$, i.e.,

$$(d \circ L_{e_i} - L_{e_i} \circ d)(y) = L_{d(e_i)}(y).$$

The derivation d being a linear transformation of the vector space A is represented in a matrix form $(d_{ij})_{i,j=1,2,\dots,n}$ i.e., $d(e_i) = \sum_{j=1}^n d_{ji}e_j$, $i = 1, 2, \dots, n$.

Hence,

$$\begin{aligned} (d \circ L_{e_i} - L_{e_i} \circ d)(y) &= \sum_{j=1}^n d_{ji}L_{e_j}(y) \\ d \circ L_{e_i} - L_{e_i} \circ d &= \sum_{j=1}^n d_{ji}L_{e_j} \end{aligned} \tag{1}$$

for all $i = 1, 2, \dots, n$.

The last system of equations along with structure of A gives constraints for elements of the matrix d . Solving the system of equations, we find the matrix

$$d = (d_{ij}) = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix}.$$

As mention earlier, we provide classification results of four-dimensional associative algebras from [10] which we make use in our study. Recall that m^{th} isomorphism class of associative algebras in dimension n is denoted by As_n^m

3.1. Four-dimensional Associative algebras

Theorem 13. *Any four-dimensional complex associative algebra can be included in one of the following isomorphism classes of algebras:*

$$\begin{aligned}
 As_4^1 &: e_1e_1 = e_3, & e_2e_2 = e_4; \\
 As_4^2 &: e_1e_2 = e_3, & e_2e_1 = e_4; \\
 As_4^3 &: e_1e_2 = e_4, & e_3e_1 = e_4; \\
 As_4^4 &: e_1e_1 = e_4, & e_2e_2 = e_2, & e_2e_3 = e_3 \\
 As_4^5 &: e_1e_1 = e_4, & e_2e_2 = e_2, & e_3e_2 = e_3; \\
 As_4^6 &: e_1e_2 = e_3, & e_2e_1 = e_4, & e_2e_2 = -e_3; \\
 As_4^7 &: e_1e_2 = e_3, & e_2e_1 = -e_3, & e_2e_2 = e_4; \\
 As_4^8 &: e_1e_2 = e_4, & e_2e_1 = -e_4, & e_3e_3 = e_4; \\
 As_4^9(\alpha) &: e_1e_2 = e_4, & e_2e_1 = \frac{1+\alpha}{1-\alpha}e_4, & e_2e_2 = e_3, & \alpha \in \mathbb{C} \setminus \{1\}; \\
 As_4^{10} &: e_1e_1 = e_1, & e_1e_2 = e_2, & e_1e_3 = e_3, & e_1e_4 = e_4; \\
 As_4^{11} &: e_1e_1 = e_1, & e_1e_2 = e_2, & e_1e_3 = e_3, & e_4e_1 = e_4; \\
 As_4^{12} &: e_1e_1 = e_1, & e_1e_2 = e_2, & e_1e_4 = e_4, & e_2e_3 = e_4; \\
 As_4^{13} &: e_1e_1 = e_1, & e_1e_2 = e_2, & e_3e_3 = e_3, & e_3e_4 = e_4; \\
 As_4^{14} &: e_1e_1 = e_1, & e_1e_4 = e_4, & e_2e_1 = e_2, & e_2e_4 = e_3; \\
 As_4^{15} &: e_1e_1 = e_1, & e_1e_4 = e_4, & e_2e_1 = e_2, & e_3e_1 = e_3; \\
 As_4^{16} &: e_1e_1 = e_1, & e_2e_1 = e_2, & e_3e_1 = e_3, & e_4e_1 = e_4; \\
 As_4^{17} &: e_1e_1 = e_1, & e_2e_1 = e_2, & e_3e_2 = e_4, & e_4e_1 = e_4; \\
 As_4^{18} &: e_1e_1 = e_1, & e_2e_1 = e_2, & e_3e_3 = e_3, & e_4e_3 = e_4; \\
 As_4^{19} &: e_1e_1 = e_1, & e_2e_2 = e_2, & e_2e_4 = e_4, & e_3e_1 = e_3; \\
 As_4^{20} &: e_1e_1 = e_1, & e_2e_2 = e_2, & e_3e_3 = e_3, & e_4e_4 = e_4; \\
 As_4^{21} &: e_1e_1 = e_3, & e_1e_3 = e_4, & e_2e_2 = -e_4, & e_3e_1 = e_4;
 \end{aligned}$$

$$As_4^{22} : e_1e_1 = e_4, \quad e_1e_2 = e_3, \quad e_2e_1 = -e_3, \quad e_2e_2 = -2e_3 + e_4;$$

- $As_4^{23}(\alpha) : e_1e_1 = e_4, \quad e_1e_2 = e_3, \quad e_2e_1 = -\alpha e_4, \quad e_2e_2 = -e_3, \quad \alpha \in \mathbb{C}$
- $As_4^{24} : e_1e_1 = e_4, \quad e_1e_2 = e_4, \quad e_2e_1 = -e_4, \quad e_3e_3 = e_4;$
- $As_4^{25} : e_1e_1 = e_4, \quad e_1e_4 = -e_3, \quad e_2e_1 = e_3, \quad e_4e_1 = -e_3;$
- $As_4^{26} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2;$
- $As_4^{27} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_4 = e_4, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{28} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_2e_1 = e_2, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{29} : e_1e_1 = e_1, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_2 = e_2, \quad e_3e_2 = e_3;$
- $As_4^{30} : e_1e_1 = e_1, \quad e_1e_3 = e_3, \quad e_2e_2 = e_2, \quad e_2e_4 = e_4, \quad e_4e_1 = e_4;$
- $As_4^{31} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{32} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_3e_1 = e_3, \quad e_4e_2 = e_4;$
- $As_4^{33} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_3e_2 = e_3, \quad e_4e_3 = e_3, \quad e_4e_4 = e_4;$
- $As_4^{34} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_3e_3 = e_3, \quad e_3e_4 = e_4, \quad e_4e_3 = e_4;$
- $As_4^{35}(\alpha) : e_1e_1 = e_4, \quad e_1e_2 = \alpha e_4, \quad e_2e_1 = -\alpha e_4, \quad e_2e_2 = e_4, \quad e_3e_3 = e_4, \quad \alpha \in \mathbb{C};$
- $As_4^{36} : e_1e_1 = e_4, \quad e_1e_4 = -e_3, \quad e_2e_1 = e_3, \quad e_2e_2 = e_3, \quad e_4e_1 = -e_3;$
- $As_4^{37} : e_1e_2 = e_4, \quad e_1e_3 = e_4, \quad e_2e_1 = -e_4, \quad e_2e_2 = e_4, \quad e_3e_1 = e_4;$
- $As_4^{38} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_3e_1 = e_3;$
- $As_4^{39} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_3e_1 = e_3, \quad e_3e_2 = e_4;$
- $As_4^{40} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_2e_1 = e_2, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{41} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_2e_1 = e_2, \quad e_3e_3 = e_3, \quad e_3e_4 = e_4, \quad e_4e_3 = e_4;$
- $As_4^{42} : e_1e_1 = e_1, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_2 = e_2, \quad e_3e_1 = e_3, \quad e_4e_2 = e_4;$
- $As_4^{43} : e_1e_1 = e_1, \quad e_1e_3 = e_3, \quad e_2e_1 = e_2, \quad e_2e_3 = e_4, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{44} : e_1e_1 = e_1, \quad e_1e_3 = e_3, \quad e_2e_2 = e_2, \quad e_2e_4 = e_4, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{45} : e_1e_1 = e_1, \quad e_1e_4 = e_4, \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_3e_1 = e_3, \quad e_4e_2 = e_4;$
- $As_4^{46} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_2e_4 = e_4, \quad e_3e_1 = e_3, \quad e_4e_1 = e_4;$
- $As_4^{47} : e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_2e_4 = e_4, \quad e_3e_2 = e_3, \quad e_4e_2 = e_4;$
- $As_4^{48} : e_1e_1 = e_2, \quad e_1e_2 = e_3, \quad e_1e_3 = e_4, \quad e_2e_1 = e_3, \quad e_2e_2 = e_4, \quad e_3e_1 = e_4;$
- $As_4^{49} : e_1e_4 = e_2, \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3, \quad e_4e_3 = e_4;$
- $As_4^{50} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_2 = e_4,$
 $e_4e_1 = e_4;$
- $As_4^{51} : e_1e_1 = e_1, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_3e_1 = e_3,$

$$\begin{aligned}
 & e_4e_1 = e_4; \\
 As_4^{52} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_2 = e_4, \quad e_3e_1 = e_3, \\
 & e_4e_1 = e_4; \\
 \\
 As_4^{53} : e_1e_1 = e_1, & \quad e_2e_2 = e_2, \quad e_2e_3 = e_3, \quad e_2e_4 = e_4, \quad e_3e_2 = e_3, \quad e_3e_3 = e_4, \\
 & e_4e_2 = e_4; \\
 As_4^{54} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_2e_3 = e_1, \quad e_2e_4 = e_2, \quad e_3e_1 = e_3, \quad e_3e_2 = e_4, \\
 & e_4e_3 = e_3, \quad e_4e_4 = e_4; \\
 As_4^{55} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_2 = e_4, \\
 & e_3e_1 = e_3, \quad e_4e_1 = e_4; \\
 As_4^{56} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_3 = e_4, \\
 & e_3e_1 = e_3, \quad e_3e_2 = -e_4, \quad e_4e_1 = e_4; \\
 As_4^{57} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_2 = e_3, \\
 & e_2e_3 = e_4, \quad e_3e_1 = e_3, \quad e_3e_2 = e_4, \quad e_4e_1 = e_4; \\
 As_4^{58} : e_1e_1 = e_1, & \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_1e_4 = e_4, \quad e_2e_1 = e_2, \quad e_2e_2 = -e_4, \\
 & e_2e_3 = -e_4, \quad e_3e_1 = e_3, \quad e_3e_2 = e_4, \quad e_4e_1 = e_4.
 \end{aligned}$$

In what follows, the following notations are used:

1. IC: isomorphism classes of algebras.
2. Dim: Dimensions of the algebra of derivations.
3. $m_2 = d_{11} + d_{21}$,
4. $m_3 = d_{33} - d_{11}$,
5. $m_4 = \frac{2}{1-\alpha}d_{12}, \quad \alpha \neq 1$
6. $m_5 = d_{22} + d_{33}$,
7. $m_6 = d_{33} - d_{22}$.

The derivations of four-dimensional associative algebra are given as follows.

Theorem 14. *The derivation of four-dimensional associative algebras has the following form:*

Table 1: Derivations of four-dimensional associative algebras

IC	Derivation	Dim	IC	Derivation	Dim
As_4^1	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{22} \end{pmatrix}$	6	As_4^2	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & m_2 & 0 \\ d_{41} & d_{42} & 0 & m_2 \end{pmatrix}$	6
As_4^3	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ -d_{21} & 0 & d_{22} & 0 \\ d_{41} & d_{42} & d_{43} & m_2 \end{pmatrix}$	6	As_4^4	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & 2d_{11} \end{pmatrix}$	4
As_4^5	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & 2d_{11} \end{pmatrix}$	4	As_4^6	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix}$	5
As_4^7	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & m_2 & 0 \\ d_{41} & d_{42} & 0 & 2d_{22} \end{pmatrix}$	7	As_4^8	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ d_{21} & m_3 & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{33} \end{pmatrix}$	7
$As_4^9(\alpha)$	$\begin{pmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{22} & 0 \\ d_{41} & d_{42} & m_4 & m_2 \end{pmatrix}$	7	As_4^{10}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}$	12
As_4^{11}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & 0 \\ d_{31} & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	8	As_4^{12}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & -d_{21} & m_5 \end{pmatrix}$	5
As_4^{13}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{43} & d_{44} \end{pmatrix}$	4	As_4^{14}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & d_{32} & d_{33} & d_{21} \\ d_{32} & 0 & 0 & m_6 \end{pmatrix}$	4

Continued to the next page

IC	Derivation	Dim	IC	Derivation	Dim
As_4^{15}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & 0 \\ d_{31} & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	8	As_4^{16}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}$	12
As_4^{17}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & -d_{21} & m_5 \end{pmatrix}$	5	As_4^{18}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{43} & d_{44} \end{pmatrix}$	4
As_4^{19}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & 0 & d_{33} & 0 \\ 0 & d_{42} & 0 & d_{44} \end{pmatrix}$	4	As_4^{20}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	0
As_4^{21}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & \frac{3}{2}d_{11} & 0 & 0 \\ d_{31} & d_{21} & 2d_{11} & 0 \\ d_{41} & d_{42} & 2d_{31} & 3d_{11} \end{pmatrix}$	5	As_4^{22}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix}$	5
$As_4^{23}(\mu)$	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{11} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{11} \end{pmatrix}$	5	As_4^{24}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & 0 & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix}$	5
As_4^{25}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 2d_{11} & 0 & 0 \\ d_{31} & d_{32} & 3d_{11} & m_7 \\ d_{41} & 0 & 0 & 2d_{11} \end{pmatrix}$	5	As_4^{26}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & 0 & d_{33} & d_{34} \\ d_{41} & 0 & d_{43} & d_{44} \end{pmatrix}$	7
As_4^{27}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ d_{31} & 0 & d_{33} & d_{34} \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	5	As_4^{28}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & 0 & d_{33} & d_{34} \\ d_{41} & 0 & d_{43} & d_{44} \end{pmatrix}$	7
As_4^{29}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & -d_{31} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	4	As_4^{30}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & 0 & d_{33} & 0 \\ d_{41} & -d_{41} & 0 & d_{44} \end{pmatrix}$	4
As_4^{31}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & -d_{31} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	4	As_4^{32}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & -d_{31} & d_{33} & 0 \\ 0 & d_{42} & 0 & d_{44} \end{pmatrix}$	4
As_4^{33}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & d_{32} & d_{33} & -d_{32} \\ 0 & 0 & 0 & 0 \end{pmatrix}$	2	As_4^{34}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	1

Continued to the next page

IC	Derivation	Dim	IC	Derivation	Dim
As_4^{35}	$\begin{pmatrix} d_{11} & -d_{21} & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix}$	5	As_4^{36}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ d_{31} & d_{32} & 0 & m_7 \\ d_{41} & d_{21} & 0 & 0 \end{pmatrix}$	4
As_4^{37}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & d_{11} & 0 & 0 \\ 0 & -d_{21} & d_{11} & 0 \\ d_{41} & d_{42} & d_{43} & 2d_{11} \end{pmatrix}$	5	As_4^{38}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	6
As_4^{39}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & d_{21} & m_5 \end{pmatrix}$	5	As_4^{40}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & d_{32} & d_{33} & 0 \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix}$	6
As_4^{41}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	2	As_4^{42}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & -d_{41} & 0 & d_{44} \end{pmatrix}$	3
As_4^{43}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & d_{42} & d_{21} & m_5 \end{pmatrix}$	5	As_4^{44}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & -d_{41} & 0 & d_{44} \end{pmatrix}$	3
As_4^{45}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & -d_{31} & d_{33} & 0 \\ d_{41} & -d_{41} & 0 & d_{44} \end{pmatrix}$	4	As_4^{46}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & -d_{31} & d_{33} & d_{34} \\ d_{41} & -d_{41} & d_{43} & d_{44} \end{pmatrix}$	6
As_4^{47}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & d_{34} \\ 0 & 0 & d_{43} & d_{44} \end{pmatrix}$	4	As_4^{48}	$\begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 2d_{11} & 0 & 0 \\ d_{31} & 2d_{21} & 3d_{11} & 0 \\ d_{41} & 2d_{31} & 3d_{21} & 4d_{11} \end{pmatrix}$	4
As_4^{49}	$\begin{pmatrix} d_{11} & 0 & d_{13} & 0 \\ d_{21} & d_{22} & 0 & -d_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -d_{21} & m_1 \end{pmatrix}$	4	As_4^{50}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & 0 & d_{33} & 0 \\ 0 & d_{42} & 0 & 2d_{22} \end{pmatrix}$	4
As_4^{51}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & d_{32} & d_{33} & d_{34} \\ 0 & d_{42} & d_{43} & d_{44} \end{pmatrix}$	9	As_4^{52}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & 0 & d_{33} & 0 \\ 0 & d_{42} & 0 & 2d_{22} \end{pmatrix}$	4
As_4^{53}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & d_{43} & 2d_{33} \end{pmatrix}$	2	As_4^{54}	$\begin{pmatrix} 0 & d_{12} & -d_{21} & 0 \\ d_{21} & d_{22} & 0 & -d_{21} \\ -d_{12} & 0 & -d_{22} & -d_{12} \\ 0 & -d_{12} & d_{13} & 0 \end{pmatrix}$	3
As_4^{55}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & d_{32} & d_{33} & 0 \\ 0 & d_{42} & d_{43} & 2d_{22} \end{pmatrix}$	5	As_4^{56}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & d_{23} & 0 \\ 0 & d_{32} & d_{33} & 0 \\ 0 & d_{42} & d_{43} & m_5 \end{pmatrix}$	6
As_4^{57}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & d_{32} & 2d_{22} & 0 \\ 0 & d_{42} & 2d_{32} & 3d_{22} \end{pmatrix}$	3	As_4^{58}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & d_{32} & d_{22} & 0 \\ 0 & d_{42} & d_{43} & 2d_{22} \end{pmatrix}$	4

Proof. We provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly with small modification(s).

Let us consider As_4^1 . Applying the system of equation (1), we get $d_{12} = d_{13} = d_{14} = d_{21} = d_{23} = d_{24} = d_{34} = d_{43} = 0, d_{33} = 2d_{11}, d_{44} = 2d_{22}$. Therefore, the derivations for As_4^1 are given by the matrices

$$d = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{22} \end{pmatrix}.$$

The dimension of the algebra $Der(As_4^1)$ of derivation of the class As_4^1 is 6. \square

Remark 15.

1. There is one class of characteristically nilpotent associative algebras in the list of isomorphism classes of four-dimensional associative algebras.
2. The dimensions of the derivation algebras in this case vary between zero and nine.

4. Description of centroids of four dimensional associative algebras

In this section, we give the description of centroids of four-dimensional complex associative algebras. Let $\{e_1, \dots, e_n\}$ be a basis of a n -dimensional associative algebra A , then $e_i e_j = \sum_{k=1}^n \gamma_{ij}^k e_k$. The coefficients, $\{\gamma_{ij}^k\} \in \mathbb{C}^{n^3}$, of the above linear combinations are called the structure constants of A on the basis $\{e_1, \dots, e_n\}$. An element ϕ of the centroid $\Gamma(A)$, being a linear transformation of the vector space A , is represented in a matrix form $(a_{ij})_{i,j=1,2,\dots,n}$, i.e., $\phi(e_i) = \sum_{j=1}^n a_{ji} e_j, i = 1, 2, \dots, n$. must satisfy the following systems of equations:

$$\sum_{k=1}^n (\gamma_{ij}^t a_{kt} - a_{ti} \gamma_{tj}^k) = 0; \tag{2}$$

and

$$\sum_{k=1}^n (\gamma_{ij}^k a_{kt} - a_{tj} \gamma_{it}^k) = 0 \tag{3}$$

It is observed that if the structure constants $\{\gamma_{ij}^k\}$ of the associative algebra A are given, then, in order to describe its centroid one has to solve the system of equations above with respect to the a_{ij} , $i, j = 1, 2, \dots, n$. Again, we make use classification results of four-dimensional complex associative algebras from [10] to compute the centroids of four-dimensional complex associative algebras.

Theorem 16. *The centroids of four dimensional complex associative algebras are given as follows:*

Table 2: Centroids of four-dimensional associative algebras

IC	Centroid $\Gamma(A)$	Dim	IC	Centroid $\Gamma(A)$	Dim
As_4^1	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	2	As_4^2	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	5
As_4^3	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	4	As_4^4	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	3
As_4^5	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	3	As_4^6	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	5
As_4^7	$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & a_{12} \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	6	As_4^8	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	4
$As_4^9(\alpha)$	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	5	As_4^{10}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	4
As_4^{11}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{12}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1
As_4^{13}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	2	As_4^{14}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1

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IC	Centroid $\Gamma(A)$	Dim	IC	Centroid $\Gamma(A)$	Dim
As_4^{15}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{16}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1
As_4^{17}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{18}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	2
As_4^{19}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix}$	2	As_4^{20}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$	4
As_4^{21}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{31} & a_{11} \end{pmatrix}$	4	As_4^{22}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	5
$As_4^{23}(\mu)$	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & a_{42} & 0 & a_{11} \end{pmatrix}$	5	As_4^{24}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	4
As_4^{25}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	7	As_4^{26}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	4
As_4^{27}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	2	As_4^{28}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	2
As_4^{29}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{30}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	2
As_4^{31}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{32}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1
As_4^{33}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix}$	2	As_4^{34}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$	1

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IC	Centroid $\Gamma(A)$	Dim	IC	Centroid $\Gamma(A)$	Dim
As_4^{35}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	4	As_4^{36}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & a_{32} & a_{11} & -a_{41} \\ a_{41} & a_{42} & a_{43} & a_{11} \end{pmatrix}$	4
As_4^{37}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{38}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	3
As_4^{39}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ 0 & a_{31} & 0 & a_{11} \end{pmatrix}$	2	As_4^{40}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & a_{24} \\ a_{31} & 0 & a_{11} & a_{34} \\ 0 & a_{31} & 0 & a_{11} \end{pmatrix}$	5
As_4^{41}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & a_{24} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{33} \end{pmatrix}$	4	As_4^{42}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	2
As_4^{43}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ 0 & a_{31} & 0 & a_{11} \end{pmatrix}$	2	As_4^{44}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	2
As_4^{45}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1	As_4^{46}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1
As_4^{47}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{22} & 0 \\ 0 & a_{42} & 0 & a_{22} \end{pmatrix}$	4	As_4^{48}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & a_{21} & a_{11} & 0 \\ 0 & a_{31} & a_{21} & a_{11} \end{pmatrix}$	3
As_4^{49}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & a_{23} & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	3	As_4^{50}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	4
As_4^{51}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{21} & 0 & a_{11} \end{pmatrix}$	3	As_4^{52}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & a_{21} & 0 & a_{11} \end{pmatrix}$	3
As_4^{53}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{22} & 0 \\ 0 & a_{42} & a_{32} & a_{22} \end{pmatrix}$	4	As_4^{54}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$	1

Continued to the next page

IC	Centroid $\Gamma(A)$	Dim	IC	Centroid $\Gamma(A)$	Dim
As_4^{55}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & a_{21} & 0 & a_{11} \end{pmatrix}$	4	As_4^{56}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ a_{31} & 0 & a_{11} & 0 \\ a_{41} & -a_{31} & -a_{21} & a_{11} \end{pmatrix}$	4
As_4^{57}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	2	As_4^{58}	$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ a_{41} & 0 & 0 & a_{11} \end{pmatrix}$	2

Proof. Let us consider As_4^1 , the structure constants are given as follows $\gamma_{11}^3 = 1, \gamma_{22}^4 = 1$ others being zero. From the systems of equations (2) and (3), we have

$a_{12} = a_{13} = a_{14} = a_{21} = a_{23} = a_{24} = a_{31} = a_{32} = a_{34} = a_{41} = a_{42} = a_{43} = 0$ and $a_{33} = a_{11} = a_{44}$. Therefore, we obtain the centroids of As_4^1 in matrix form as follows:

$$\Gamma(As_4^1) = \left\{ \left(\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix} \mid a_{11}, a_{22} \in \mathbb{C} \right) \right\}.$$

The computations for the other classes are carried out similarly. □

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