

**THE GENERAL EXACT SOLUTION FOR THE MANY
MOMENTS MACROSCOPIC APPROACH TO EXTENDED
THERMODYNAMICS OF POLYATOMIC GASES**

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Abstract: A new model for Polyatomic and for Dense Gases has been proposed in literature in the last five years in the framework of Extended Thermodynamics. The case with an arbitrary but fixed number of moments has been recently studied, both with the kinetic approach than with the macroscopic approach; this last one is more general and includes the results of the kinetic approach only as a particular case.

Scope of the “closure problem” is to find the expression of some arbitrary functions which appear in the balance equations. Up to now only a recurrence procedure has been published which outlines how to find the solution of this problem with the macroscopic approach; by using this procedure, a numberable set of solutions has been found and written explicitly, while we find here the most general exact solution. It is determined except for some arbitrary terms and it is interesting that these terms appear also in the 24 moments model; so we find here that they are transmitted from the model with 24 moments to those with an arbitrary number of moments, without any further arbitrary term.

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1. Introduction

Extended Thermodynamics of Polyatomic Gases (ETDG) with many moments proposes for every number N the following balance equations

$$\begin{aligned} \partial_t F^{i_1 \cdots i_n} + \partial_k F^{ki_1 \cdots i_n} &= P^{i_1 \cdots i_n} \quad \text{for } n = 0, \cdots, N + 2. \\ \partial_t G^{i_1 \cdots i_m} + \partial_k G^{ki_1 \cdots i_m} &= Q^{i_1 \cdots i_m} \quad \text{for } m = 0, \cdots, N. \end{aligned} \quad (1)$$

and the resulting model is called an $N - Model$ (See [1]-[4]).

Another possibility is that the index m in eq. (1)₂ goes from 0 to an arbitrary number M restricted by $M \leq N + 1$ and the resulting model can be called the $(N - 2, M) - Model$ as in [5] which belongs to the kinetic approach. But this apparently more general model can be obtained from the present one in the following way:

- Let us consider firstly the eqs. (1) with $N + M$ instead of N , that is the $(N + M) - Model$.
- Let us consider now the subsystem obtained from the above one by simply removing from (1)₁ the equations for $n = N + 3, \cdots, N + M + 2$ and from (1)₂ the equations for $m = M + 1, \cdots, N + M$; this can be done by simply putting equal to zero the corresponding Lagrange multipliers according to the technique outlined in [6].

In this way, the $(N - 2, M) - Model$ is recovered starting from the present one. Now, always in [5], it is shown that in the context of the kinetic approach is necessary to choose $N = M - 1$ if we want characteristic velocities depending on the degrees of freedom of a molecule. This is not a problem because we can proceed in the following way:

Let us consider the subsystem obtained by simply removing from (1)₁ the equation for $n = N + 2$; this can be done by simply putting equal to zero the corresponding Lagrange multiplier according to the technique outlined in [6]. In this way the number of the eqs. (1)₂ will be equal to that of (1)₁ decreased of an unity obtaining in this way the same set of equations of [5].

In conclusion, we may consider the $N - Model$ without loss of generality and this choice is motivated by the fact that within the macroscopic approach this model is easier to handle.

Before going on, we now recall that the present model belongs to the framework of Extended Thermodynamics (ET) which started with the articles [7], [8] and produced many other articles the majority of which are reported in the

book [9]. It used only eqs. (1)₁ obtaining the important result to have a symmetric hyperbolic system of partial differential equations, with finite speed of propagation of shock waves and other important properties. But the structure of ET implied the restriction $p = \frac{2}{3}\rho\epsilon$, where p is the equilibrium pressure and ϵ the internal energy. This problem have been overcome in ETDG by considering all the balance equations (1) of which the first one is called the "Mass-Block" of equations and the second is called the "Energy-Block" of equations.

The first article on this subject is [10] and, after that, a prolific production followed; we can read in the references of the book [11] the majority of the subsequent articles on this subject. The last significative article on this subject is [12], while the contribute of some of the present author in this framework are [13]-[17] besides those cited above. In (1) the independent variables are $F^{i_1 \dots i_n}$ and $G^{i_1 \dots i_m}$; the quantities $F^{ki_1 \dots i_n}$ and $G^{ki_1 \dots i_m}$ are their corresponding fluxes. We see that each flux is equal to the independent variable of the subsequent equation, except for the flux in the last equation of the Mass-Block and for that in the last equation of the Energy-Block; for these last fluxes we know only that they are symmetric tensors. We will refer to this property as "the symmetry conditions". The problem will be closed when we know the expressions of $F^{ki_1 \dots i_{N+2}}$ and $G^{ki_1 \dots i_N}$ as functions of the independent variables. Restrictions on their generality are obtained by imposing the entropy principle, the Galilean relativity principle and the symmetry conditions.

The Entropy Principle can be exploited through Liu's Theorem [18] and by using a bright idea conceived by Ruggeri [19]; so it becomes equivalent to assuming the existence of Lagrange Multipliers μ_A and λ_B which can be taken as independent variables and, after that, we have

$$\begin{aligned}
 F^{i_1 \dots i_n} &= \frac{\partial h'}{\partial \mu_{i_1 \dots i_n}} \quad , \quad G^{i_1 \dots i_m} = \frac{\partial h'}{\partial \lambda_{i_1 \dots i_m}} \\
 F^{ki_1 \dots i_n} &= \frac{\partial h'^k}{\partial \mu_{i_1 \dots i_n}} \quad , \quad G^{ki_1 \dots i_m} = \frac{\partial h'^k}{\partial \lambda_{i_1 \dots i_m}} .
 \end{aligned}
 \tag{2}$$

which expresses all the moments in terms of only two unknown functions, the 4-potentials h' , h'^k . A nice consequence of eqs. (2) is that the field equations assume the symmetric form. Another consequence of (2) is that the above mentioned symmetry conditions and the Galilean Relativity Principle can be expressed as

$$\frac{\partial h'^k}{\partial \mu_{i_1 \dots i_n}} = \frac{\partial h'}{\partial \mu_{ki_1 \dots i_n}} \quad \text{for } n = 0, \dots, N + 1; \quad \frac{\partial h'^{[k}}{\partial \mu_{i_1]i_2 \dots i_{N+2}}} = 0$$

$$\begin{aligned}
 \frac{\partial h'^k}{\partial \lambda_{i_1 \dots i_m}} &= \frac{\partial h'}{\partial \lambda_{k i_1 \dots i_m}} \quad \text{for } m = 0, \dots, N - 1 \quad ; \quad \frac{\partial h'^k}{\partial \lambda_{i_1] \dots i_N}} = 0. \\
 \frac{\partial h'}{\partial \mu_k} \mu_i &+ \sum_{n=1}^{N+1} \frac{\partial h'}{\partial \mu_{k j_1 \dots j_n}} [(n + 1) \mu_{i j_1 \dots j_n} + 2 \lambda_{j_1 \dots j_{n-1}} \delta_{j_n i}] + \\
 + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{k h_1 \dots h_s}} (s + 1) \lambda_{i h_1 \dots h_s} + h' \delta^{ki} &= 0. \tag{3}
 \end{aligned}$$

We don't go in the details on how the the Galilean Relativity Principle, thanks also to the other conditions, is equivalent to (3)₅; this equivalence can be already found in literature. In the next section, eqs. (4) and (5), we will exhibit a particular but significative solution of the conditions (3), which is expressed through a Taylor's expansion around equilibrium; this is defined as the state where $\mu_{i_1 \dots i_n} = 0$ for $n = 1, \dots, N + 2$, $\lambda_{i_1 \dots i_m} = 0$ for $m = 1, \dots, N$, so that the only variables which are not zero at equilibrium are μ and λ . The first of these is the chemical potential, while $\lambda = \frac{1}{2T}$ with T absolute temperature.

In sect. 3 we will report a part of the proof that eqs. (4) and (5) give a solution of (3). The second and final part of the proof is described in sect. 4.

2. An Exact Solution of the Conditions (3).

Before writing our solution we need to describe an hard but straightforward **notation**:

To do the derivatives with respect to $\mu_{i_1 \dots i_n}$ a number p_n of times, we use the compact form $A_{n,1}$ to indicate a set of n indexes, $A_{n,2}$ to indicate another set of n indexes, and so on up to A_{n,p_n} . For example,

$$\frac{\frac{\partial^{p_2} h'^k}{\partial \mu_{h_1 k_1} \partial \mu_{h_2 k_2} \dots \partial \mu_{h_{p_2} k_{p_2}}}}{\frac{\partial^{p_2} h'^k}{\partial \mu_{A_{2,1}} \partial \mu_{A_{2,2}} \dots \partial \mu_{A_{2,p_2}}}} \quad \text{in the compact notation becomes}$$

because the first index of $A_{\dots, \dots}$ indicates how many indexes has the corresponding μ_{\dots} , while the second index of $A_{\dots, \dots}$ indicates how many derivatives we are taking with respect to it.

$$\text{Similarly, } \frac{\partial^{p_3} h'^k}{\partial \mu_{h_1 k_1 j_1} \partial \mu_{h_2 k_2 j_2} \dots \partial \mu_{h_{p_3} k_{p_3} j_{p_3}}} \quad \text{becomes}$$

$$\frac{\partial^{p_3} h'^k}{\partial \mu_{A_{3,1}} \partial \mu_{A_{3,2}} \cdots \partial \mu_{A_{3,p_3}}}$$

and so on. By using this notation, we can now write our solution and it is

$$\begin{aligned}
 h' = & \sum_{\substack{0 \cdots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even}}} \frac{1}{p_0! p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \quad (4) \\
 & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i} \frac{1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i}{\left[1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \\
 & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{\left(\sum_{i=0}^{N+2} p_i - r_0, \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1 \right)} \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \right. \\
 & + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \cdot \\
 & \cdot \left. \left(\sum_{i=0}^N r_i \right)! C_{\left(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1 \right)} (\lambda)^{r_0} \right\} \\
 & \cdot \delta^{(A_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} (\mu)^{p_0} \cdot \\
 & \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \cdot \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}}, \\
 h'^k = & \sum_{\substack{0 \cdots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ odd}}} \frac{1}{p_0! p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \quad (5)
 \end{aligned}$$

$$\frac{\left[\sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{\left[\sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot \left\{ \sum_{r_0=1}^{\infty} C_{\left(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i\right)} \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot \cdot C_{\left(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i\right)} (\lambda)^{r_0} \right\} \cdot \delta^{(kA_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} (\mu)^{p_0} \cdot \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}},$$

where $C_{r,s}$ is a family of constants such that $C_{r,s} = 0$ if at least one of the indexes r, s is negative or if $r > s$. Despite the appearance, these expressions are not complicated. In fact, the factorial $\frac{1}{p_1!}$ appears also in the Taylor's expansions of functions depending on a single variable; we have here one of these for every variable, both of the mass block than of the energy block. Moreover, in these expressions appears the sum of these numbers $p_1 + p_2 + \cdots + p_{N+2}$ of the mass block and $r_1 + r_2 + \cdots + r_N$ of the energy block. Moreover, these numbers appear also through $p_1 + 2p_2 + \cdots + (N + 2)p_{N+2} + r_1 + 2r_2 + \cdots + Nr_N$ where these numbers are multiplied by the order of the Lagrange multiplier which they represent; after that, their sum is taken. The condition " $p_1 + 2p_2 + \cdots + (N + 2)p_{N+2} + r_1 + 2r_2 + \cdots + Nr_N$ odd" in the expression of h^{lk} is necessary because the following δ^{\cdots} must have an even number of indexes; similarly, for the expression of h' . Here we find only a letter δ^{\cdots} but it is understood that it is a shortened symbol denoting the product of some δ^{\cdots} each one with 2 indexes, and with a final symmetrization over all these indexes. Finally, in the last line there is the product of the variables with respect to which we have done the Taylor's expansions.

3. Proof of the Solution (4) and (5) - Part I

We prove now that (4), (5) is a solution of (3)₁₋₄, while (3)₅ will be considered in the next section.

- Let us begin with (3)₁.

To verify it we have to take into account that $\mu_{i_1 \dots i_n}$ here is denoted with $\mu_{A_n,1}$, or with $\mu_{A_n,2}$, and so on up to μ_{A_n,p_n} . Similarly, $\mu_{k_{i_1} \dots i_n}$ is denoted with $\mu_{A_{n+1},1}$, or with $\mu_{A_{n+1},2}$, and so on up to $\mu_{A_{n+1},p_{n+1}}$. Consequently, the left hand side of (3)₁ causes, with respect to the expression (5), a rise of one unity of the index p_n ; similarly, the right hand side of (3)₁ causes, with respect to the expression (4), a rise of one unity of the index p_{n+1} . More precisely, we obtain for both sides the following expression

$$\sum_{p_0, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot$$

$$n + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ odd}$$

$$\cdot \frac{\left[n + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{\left[n - 2 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot$$

$$\cdot \left\{ \sum_{r_0=1}^{\infty} C_{(1+\sum_{i=0}^{N+2} p_i - r_0, \quad n-2+\sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \cdot \right.$$

$$\cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)!$$

$$\cdot C_{(1+\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \quad n-2+\sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} (\lambda)^{r_0} \left. \right\} \cdot$$

$$\cdot \delta^{(k_{i_1} \dots i_n A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} (\mu)^{p_0} \cdot$$

$$\cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot$$

- It is easy to verify $(3)_{2,4}$.
- Let us continue verifying $(3)_3$ in the case $m \geq 1$.

To verify it we have to take into account that $\lambda_{i_1 \dots i_m}$ here is denoted with $\lambda_{B_{m,1}}$, or with $\lambda_{B_{m,2}}$, and so on up to $\lambda_{B_{m,r_m}}$. Similarly, $\lambda_{ki_1 \dots i_m}$ is denoted with $\lambda_{B_{m+1,1}}$, or with $\lambda_{B_{m+1,2}}$, and so on up to $\lambda_{B_{m+1,r_{m+1}}}$. Consequently, the left hand side of $(3)_3$ with $m \geq 1$ causes, with respect to the expression (5), a rise of one unity of the index r_m ; similarly, the right hand side of $(3)_3$ with $m \geq 1$ causes, with respect to the expression (4), a rise of one unity of the index r_{m+1} . More precisely, we obtain for both sides the following expression

$$\sum_{\substack{0 \dots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ m + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ odd}}} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot$$

$$\frac{\left[m + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{\left[m + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot$$

$$\cdot \left\{ \sum_{r_0=1}^{\infty} C_{\left(\sum_{i=0}^{N+2} p_i - r_0, m + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right)} \cdot \right.$$

$$\cdot \frac{\partial^{1 + \sum_{i=1}^N r_i}}{\partial \lambda^{1 + \sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{1 + \sum_{i=0}^N r_i} \left(1 + \sum_{i=0}^N r_i \right)!$$

$$\cdot C_{\left(1 + \sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, m + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right)} (\lambda)^{r_0} \left. \right\} \cdot$$

$$\cdot \delta^{(ki_1 \dots i_m A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} (\mu)^{p_0} \cdot$$

$$\cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot$$

- Let us conclude verifying $(3)_3$ in the case $m = 0$.

We have the same situation of the above case for its right hand side, while for its left hand side we have to take simply the derivative of (5) with respect to λ . More precisely, we obtain for both sides the following expression

$$\sum_{\substack{0 \dots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ odd}}} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \frac{\left[\sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{\left[\sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot \left\{ \sum_{r_0=1}^{\infty} C_{\left(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right)} \cdot \frac{\partial^{1+\sum_{i=1}^N r_i}}{\partial \lambda^{1+\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{1+\sum_{i=0}^N r_i} \left(1 + \sum_{i=0}^N r_i \right)! \cdot C_{\left(1+\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right)} (\lambda)^{r_0} \right\} \cdot \delta^{(kA_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} (\mu)^{p_0} \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot$$

So we have finished to verify that (4), (5) is a solution of (3)₁₋₄ and there remains to prove that it satisfies also (3)₅.

4. Proof of the Solution (4) and (5) - Part II

We prove now that (4), (5) is a solution of (3)₅. To this regard we note firstly that in this condition we can put under an unique summation the coefficients of $\mu \dots$, from the first coefficient of a $\lambda \dots$ we can isolate the term with $n = 1$ and in the other ones we can change the index of the summation according to the

law $n = 2 + s$. In this way $(3)_5$ can be rewritten as

$$\sum_{n=0}^{N+1} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} (n+1) \mu_{ij_1 \dots j_n} + \frac{\partial h'}{\partial \mu_{ki}} 2\lambda + \sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_{s+1}i}} + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{kh_1 \dots h_s}} (s+1) \lambda_{ih_1 \dots h_s} + h' \delta^{ki} = 0. \tag{6}$$

- Now, for the first term of this relation we can use (4) and the derivation causes a presence of the factor p_{n+1} , the substitution of $\mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}}}$ with $\mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}}$ and the new free indexes $kj_1 \dots j_n$ in the expression of δ^{\dots} ; more precisely, that term becomes

$$\sum_{n=0}^{N+1} (n+1) \mu_{ij_1 \dots j_n} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} = \sum_{n=0}^{N+1} (n+1) p_{n+1} \cdot \sum_{0 \dots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot p_0, \dots, p_{N+2}, r_1, \dots, r_N \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even} \cdot \frac{[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i} \frac{1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i}{[1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i]!!} \cdot \left\{ \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} (\lambda)^{r_0} \right\} \cdot \delta^{jn+1i} \delta^{(kj_1 \dots j_n A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}-1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}})}$$

$$\begin{aligned}
 & (B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N}) \cdot \\
 & \cdot (\mu)^{p_0} \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{n+1,1}} \cdots \mu_{A_{n+1,p_{n+1}-1}} \cdot \\
 & \cdot \mu_{j_1 \cdots j_{n+1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} ,
 \end{aligned}$$

where we have also substituted $\mu_{i j_1 \cdots j_n}$ with $\mu_{j_1 \cdots j_{n+1}} \delta^{j_{n+1} i}$. Now in this expression we can insert a symmetrization over all the indexes of the set

$$j_1 \cdots j_{n+1} A_{n+1,1} \cdots A_{n+1,p_{n+1}-1}$$

because that expression remains the same if we exchange two of these indexes. This fact is evident if the two indexes are taken between $j_1 \cdots j_{n+1}$; for the proof in the other cases, let us consider the shortened expression

$$\delta^{j_{n+1} i} \delta^{(k j_1 \cdots j_n k_1 \cdots k_n k_{n+1} \cdots)} \mu_{j_1 \cdots j_{n+1}} \mu_{k_1 \cdots k_{n+1}} .$$

Here we can exchange the names of the indexes j with those of the k ., so that the above shortened expression becomes

$$\delta^{k_{n+1} i} \delta^{(k k_1 \cdots k_n j_1 \cdots j_n j_{n+1} \cdots)} \mu_{k_1 \cdots k_{n+1}} \mu_{j_1 \cdots j_{n+1}} .$$

Now we can exchange the indexes $k_1 \cdots k_n$ with the indexes $j_1 \cdots j_n$ in the expression of $\delta^{(k k_1 \cdots k_n j_1 \cdots j_n j_{n+1} \cdots)}$ because this is a symmetric tensor.

We obtain $\delta^{k_{n+1} i} \delta^{(k j_1 \cdots j_n k_1 \cdots k_n j_{n+1} \cdots)} \mu_{k_1 \cdots k_{n+1}} \mu_{j_1 \cdots j_{n+1}}$. By comparing this result with the expression which we started from, it is the same if he had exchanged the indexes j_{n+1} and k_{n+1} . This completes the proof of the fact that the expression (7) remains the same if we exchange two indexes of the set $j_1 \cdots j_{n+1} A_{n+1,1} \cdots A_{n+1,p_{n+1}-1}$; so we can insert there a symmetrization over those indexes and (7) becomes

$$\begin{aligned}
 & \sum_{n=0}^{N+1} (n+1) \mu_{i j_1 \cdots j_n} \frac{\partial h'}{\partial \mu_{k j_1 \cdots j_n}} = \sum_{n=0}^{N+1} (n+1) p_{n+1} . \tag{8} \\
 & \cdot \sum_{0 \cdots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot \\
 & \frac{p_0, \cdots, p_{N+2}, r_1, \cdots, r_N}{\sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i \text{ even}} \\
 & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i \right] !!}{1 + \sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i} \frac{1 + \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i}{\left[1 + \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i \right] !!} .
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \right. \\
 & \cdot \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot \\
 & \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} (\lambda)^{r_0} \left. \right\} \cdot \\
 & \cdot \underline{\delta^{j_{n+1}i}} \delta^{(kj_1 \dots j_n A_{1,1} \dots A_{1,p_1} \dots \underline{A_{n+1,1} \dots A_{n+1,p_{n+1}-1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}}} \\
 & \hspace{15em} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} \cdot \\
 & \cdot (\mu)^{p_0} \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{n+1,1}} \dots \mu_{A_{n+1,p_{n+1}-1}} \mu_{j_1 \dots j_{n+1}} \dots \\
 & \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}},
 \end{aligned}$$

where underlined indexes denote symmetrization over these indexes. Now we observe that $(n + 1)p_{n+1}$ is exactly the number of the indexes of the set $j_1 \dots j_{n+1} A_{n+1,1} \dots A_{n+1,p_{n+1}-1}$ and that, thanks to the summation $\sum_{n=0}^{N+1}$ the index near i in $\delta^{j_{n+1}i}$ can be every index of the set $A_{1,1} \dots A_{1,p_1} \dots A_{n+1,1} \dots A_{n+1,p_{n+1}} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}}$. These facts allow to rewrite (8) as

$$\begin{aligned}
 & \sum_{n=0}^{N+1} (n + 1) \mu_{ij_1 \dots j_n} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} = \left[\sum_{n=0}^{N+1} (n + 1) p_{n+1} \right] \cdot \tag{9} \\
 & \cdot \sum_{\substack{0 \dots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even}}} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \cdot \frac{[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i} \frac{1 + \sum_{i=0}^{N+2} (i - 2)p_i + \sum_{i=1}^N ir_i}{[1 + \sum_{i=0}^{N+2} (i - 2)p_i + \sum_{i=1}^N ir_i]!!} \cdot \\
 & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \right.
 \end{aligned}$$

$$\begin{aligned} & \cdot \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot \\ & \cdot C_{\left(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i \quad , \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1} \right)} (\lambda)^{r_0} \} \cdot \\ & \cdot \underline{\delta^{j_{n+1}i} \delta^{(kj_1 \cdots j_n A_{1,1} \cdots A_{1,p_1} \cdots A_{n+1,1} \cdots A_{n+1,p_{n+1}-1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}})} } \\ & \hspace{15em} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N}) . \end{aligned}$$

$$\begin{aligned} & \cdot (\mu)^{p_0} \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{n+1,1}} \cdots \mu_{A_{n+1,p_{n+1}-1}} \cdot \\ & \cdot \mu_{j_1 \cdots j_{n+1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} . \end{aligned}$$

- For the fourth term of (6) we can do similar passages (the difference is that we have the λ ... instead of the μ ..., $N - 2$ instead of N and s instead of n); in this way that term becomes

$$\begin{aligned} & \sum_{s=0}^{N-1} (s+1) \lambda_{ih_1 \cdots h_s} \frac{\partial h'}{\partial \lambda_{kh_1 \cdots h_s}} = \left[\sum_{s=0}^{N-1} (s+1) r_{s+1} \right] \cdot \tag{10} \\ & \cdot \sum_{\substack{0 \cdots \infty \\ p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even}}} \frac{1}{p_0!} \frac{1}{p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot \\ & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i} \frac{1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i}{\left[1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot \\ & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{\left(\sum_{i=0}^{N+2} p_i - r_0 \quad , \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1} \right)} \cdot \right. \\ & \cdot \frac{\partial^{\sum_{i=1}^N r_i}}{\partial \lambda^{\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot \\ & \left. \cdot C_{\left(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i \quad , \quad \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1} \right)} (\lambda)^{r_0} \right\} \cdot \end{aligned}$$

$$\cdot \delta^{\underline{h_{s+1}^i}} \delta^{(kh_1 \dots h_s A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}})}$$

$$\underline{B_{1,1} \dots B_{1,r_1} \dots B_{s+1,1} \dots B_{s+1,r_{s+1}-1} \dots B_{N,1} \dots B_{N,r_N}} \cdot$$

$$\cdot (\mu)^{p_0} \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{s+1,1}} \dots \lambda_{B_{s+1,r_{s+1}-1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot$$

- If we look at the last term of (6), we see that it can be written together with (9) and (10) and they become

$$\sum_{n=0}^{N+1} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_n}} (n+1) \mu_{ij_1 \dots j_n} + \sum_{s=0}^{N-1} \frac{\partial h'}{\partial \lambda_{kh_1 \dots h_s}} (s+1) \lambda_{ih_1 \dots h_s} + \tag{11}$$

$$+ h' \delta^{ki} = \left[1 + \sum_{n=0}^{N+1} (n+1) p_{n+1} + \sum_{s=0}^{N-1} (s+1) r_{s+1} \right] \cdot$$

$$\sum_{p_0, \dots, p_{N+2}, r_1, \dots, r_N}^{0 \dots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot$$

$$\sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i \text{ even}$$

$$\cdot \frac{\left[1 + \sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i \right]!!}{1 + \sum_{i=1}^{N+2} i p_i + \sum_{i=1}^N i r_i} \frac{1 + \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i}{\left[1 + \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i \right]!!} \cdot$$

$$\cdot \left\{ \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i - 1)} \cdot \right.$$

$$\cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot$$

$$\cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2) p_i + \sum_{i=1}^N i r_i - 1)} (\lambda)^{r_0} \left. \right\} \cdot$$

$$\cdot \delta^{ki} \delta^{\underline{(A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})}} \cdot$$

$$\begin{aligned} & \cdot (\mu)^{p_0} \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot \\ & \cdot \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} , \end{aligned}$$

where also the index k has been put under the symmetrization and we recall that a property of symmetrization is $\delta^{ik} \delta^{\overline{(\dots)}} = \delta^{ik} \delta^{\overline{(\dots)}} = \delta^{(ik \dots)}$. It is interesting to see that the coefficient in square bracket at the beginning of the right hand side of eq. (11) has become equal to the denominator of the half factorial of that right hand side!

- For the second term of (6), we can use (4) and the derivation causes somehow the rising of one unity of the index p_2 ; more precisely we have

$$\begin{aligned} & \frac{\partial h'}{\partial \mu_{ki}} 2 \lambda = \tag{12} \\ & = 2 \lambda \sum_{\substack{p_0, \dots, p_{N+2}, r_1, \dots, r_N \\ \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even}}}^{0 \cdots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \cdots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \cdots \frac{1}{r_N!} \cdot \\ & \cdot \left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right] !! \frac{1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i}{\left[1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right] !!} \cdot \\ & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{(1+\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \cdot \right. \\ & \cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot \\ & \cdot C_{(1+\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} (\lambda)^{r_0} \left. \right\} \cdot \\ & \cdot \delta^{(ki A_{1,1} \cdots A_{1,p_1} \cdots A_{N+2,1} \cdots A_{N+2,p_{N+2}} B_{1,1} \cdots B_{1,r_1} \cdots B_{N,1} \cdots B_{N,r_N})} . \\ & \cdot (\mu)^{p_0} \mu_{A_{1,1}} \cdots \mu_{A_{1,p_1}} \cdots \mu_{A_{N+2,1}} \cdots \mu_{A_{N+2,p_{N+2}}} \cdot \\ & \cdot \lambda_{B_{1,1}} \cdots \lambda_{B_{1,r_1}} \cdots \lambda_{B_{N,1}} \cdots \lambda_{B_{N,r_N}} , \end{aligned}$$

- There remains to consider the third term of (6); for it we can use (4) and the derivation causes somehow the rising of one unity of the index p_{s+3} ; more precisely we have

$$\sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_{s+1} i}} = \sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \tag{13}$$

$$\cdot \sum_{s+3 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even}}^{0 \dots \infty} \frac{1}{p_0! p_1!} \dots \frac{1}{p_{N+2}! r_1! r_2!} \dots \frac{1}{r_N!} \cdot$$

$$\frac{[s + 4 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i]!!}{s + 4 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i} \cdot$$

$$\frac{1}{[s + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i]!!} \cdot$$

$$\cdot \left\{ \sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i + s)} + \right.$$

$$\cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \cdot$$

$$\cdot C_{(1 + \sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i + s)} (\lambda)^{r_0} \left. \right\} \cdot$$

$$\cdot \delta^{(kj_1 \dots j_{s+1} i A_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} (\mu)^{p_0} \cdot$$

$$\cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot$$

$$\cdot \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}},$$

In this expression, we can substitute $\lambda_{j_1 \dots j_{s+1}}$ with $\lambda_{B_{s+1}, r_{s+1}+1}$ and $j_1 \dots j_{s+1}$ with $B_{s+1}, r_{s+1}+1$; moreover, we substitute $\frac{1}{r_{s+1}!}$ with $\frac{r_{s+1}+1}{(r_{s+1}+1)!}$ and, after that, decrease r_{s+1} of one unity (which is equivalent to a

change of index). So this expression becomes

$$\begin{aligned}
 \sum_{s=0}^{N-1} 2\lambda_{j_1 \dots j_{s+1}} \frac{\partial h'}{\partial \mu_{kj_1 \dots j_{s+1}i}} &= \sum_{s=0}^{N-1} 2r_{s+1}. \tag{14} \\
 &\cdot \sum_{0 \dots \infty} \frac{1}{p_0!} \frac{1}{p_1!} \dots \frac{1}{p_{N+2}!} \frac{1}{r_1!} \frac{1}{r_2!} \dots \frac{1}{r_N!} \cdot \\
 & \quad p_0, \dots, p_{N+2}, r_1, \dots, r_N \\
 & 2 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \text{ even} \\
 & \cdot \frac{\left[1 + \sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i \right]!!}{\left[-1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i \right]!!} \cdot \\
 & \cdot \left\{ \sum_{r_0=1}^{\infty} C_{(1+\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \cdot \right. \\
 & \cdot \frac{\partial^{-1+\sum_{i=1}^N r_i}}{\partial \lambda^{-1+\sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{-1+\sum_{i=0}^N r_i} \left(-1 + \sum_{i=0}^N r_i \right)! \cdot \\
 & \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} (\lambda)^{r_0} \left. \right\} \cdot \\
 & \cdot \delta^{(kiA_{1,1} \dots A_{1,p_1} \dots A_{N+2,1} \dots A_{N+2,p_{N+2}} B_{1,1} \dots B_{1,r_1} \dots B_{N,1} \dots B_{N,r_N})} (\mu)^{p_0} \cdot \\
 & \cdot \mu_{A_{1,1}} \dots \mu_{A_{1,p_1}} \dots \mu_{A_{N+2,1}} \dots \mu_{A_{N+2,p_{N+2}}} \cdot \\
 & \lambda_{B_{1,1}} \dots \lambda_{B_{1,r_1}} \dots \lambda_{B_{N,1}} \dots \lambda_{B_{N,r_N}} \cdot
 \end{aligned}$$

It is true that with change of index we have in the summation the extra term with $r_{s+1} = 0$; but it doesn't effect the result for the presence of the coefficient r_{s+1} .

If we look at the expressions (11), (12) and (14) we conclude that to prove (6) it is sufficient that the following relation holds

$$0 = \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} \cdot \tag{15}$$

$$\begin{aligned}
 & \cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \\
 & \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1})} (\lambda)^{r_0} + \\
 & + 2\lambda \sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1})} \cdot \\
 & \cdot \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + 2\lambda \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left(\sum_{i=0}^N r_i \right)! \\
 & \cdot C_{(1 + \sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1})} (\lambda)^{r_0} + \\
 & + \sum_{s=0}^{N-1} 2r_{s+1} \left\{ \sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, -1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \cdot \right. \\
 & \left. \frac{\partial^{-1 + \sum_{i=1}^N r_i}}{\partial \lambda^{-1 + \sum_{i=1}^N r_i}} \left[\left(\frac{-1}{2\lambda} \right)^{r_0} \right] + \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{-1 + \sum_{i=0}^N r_i} \left(-1 + \sum_{i=0}^N r_i \right)! \cdot \right. \\
 & \left. \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_{i-1})} (\lambda)^{r_0} \right\} .
 \end{aligned}$$

Now the terms in lines 3, 5 of this relation can be written together as

$$\begin{aligned}
 & \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[2\lambda \sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, -1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \cdot \right. \\
 & \left. \cdot \left(\frac{-1}{2\lambda} \right)^{r_0} \right] = \\
 & = - \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, -1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \cdot \right. \\
 & \left. \cdot \left(\frac{-1}{2\lambda} \right)^{r_0-1} \right] = \\
 & = - \frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\sum_{\psi=0}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - \psi, -1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \cdot \right.
 \end{aligned}$$

$$= -\frac{\partial \sum_{i=1}^N r_i}{\partial \lambda \sum_{i=1}^N r_i} \left[\sum_{\psi=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - \psi, -1 + \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i)} \left(\frac{-1}{2\lambda}\right)^\psi \right] =$$

where in the second passage we have changed index with the law $r_0 = \psi + 1$ and in the last passage we have excluded the value with $\psi = 0$ because the derivative of a constant is zero; obviously this excludes the case with $r_1 + r_2 + \dots + r_N = 0$. The result is exactly the opposite of the first term in eq. (15) in the case $r_1 + r_2 + \dots + r_N \neq 0$. Instead of this, if $r_1 + r_2 + \dots + r_N = 0$ (that is $r_1 = 0, r_2 = 0, \dots, r_N = 0$) the term in line 5 of eq. (15) is not present and, while those in lines 1, 3 reduce to

$$\begin{aligned} 0 &= \sum_{r_0=1}^{\infty} C_{(\sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i - 1)} \left(\frac{-1}{2\lambda}\right)^{r_0} + \\ &- \sum_{r_0=1}^{\infty} C_{(1 + \sum_{i=0}^{N+2} p_i - r_0, \sum_{i=0}^{N+2} (i-2)p_i - 1)} \left(\frac{-1}{2\lambda}\right)^{r_0-1} = \\ &= -C_{(\sum_{i=0}^{N+2} p_i, \sum_{i=0}^{N+2} (i-2)p_i - 1)} \end{aligned}$$

where in the second passage we have changed the index of the second summation with the law $r_0 = \psi + 1$. So, up to now, we have found that the terms in lines 1, 3, 5 of eq. (15) are present only if $r_1 = 0, r_2 = 0, \dots, r_N = 0$ and, in this case, their sum is

$$-C_{(\sum_{i=0}^{N+2} p_i, \sum_{i=0}^{N+2} (i-2)p_i - 1)}.$$

Now the term in line 4 of eq. (15) can be transformed by substituting $\frac{1}{r_0!}$ with $\frac{r_0+1}{(r_0+1)!}$ and then decreasing r_0 of an unity; in this way it becomes

$$\begin{aligned} &2 \sum_{r_0=0}^{\infty} \frac{r_0}{r_0!} (-2)^{-1 + \sum_{i=0}^N r_i} \left(-1 + \sum_{i=0}^N r_i\right)! \cdot \\ &\cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N ir_i - 1)} (\lambda)^{r_0}. \end{aligned}$$

It is true that in this way we have added the term with $r_0 = 0$, but this term gives no contribute, thanks to the factor r_0 . The result, jointly with the term

in line 6 of eq. (15) becomes

$$2 \sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{-1+\sum_{i=0}^N r_i} \left(\sum_{s=0}^N r_s \right) \cdot \left(-1 + \sum_{i=0}^N r_i \right)! \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N i r_{i-1})} (\lambda)^{r_0} .$$

Consequently, the sum of the elements in lines 2, 4, 6 of eq. (15) is

$$\sum_{r_0=0}^{\infty} \frac{1}{r_0!} (-2)^{\sum_{i=0}^N r_i} \left[\left(\sum_{i=0}^N r_i \right)! - \left(\sum_{s=0}^N r_s \right) \cdot \left(-1 + \sum_{i=0}^N r_i \right)! \right] \cdot C_{(\sum_{i=0}^{N+2} p_i + \sum_{i=0}^N r_i, \sum_{i=0}^{N+2} (i-2)p_i + \sum_{i=1}^N i r_{i-1})} (\lambda)^{r_0}$$

which is zero if $r_1 + r_2 + \dots + r_N \neq 0$, while it is equal to

$$C_{(\sum_{i=0}^{N+2} p_i, \sum_{i=0}^{N+2} (i-2)p_{i-1})} ,$$

if $r_1 = 0, r_2 = 0, \dots, r_N = 0$, as a consequence of the identity

$$r! - r(r-1)! = \begin{cases} 0 & r > 0 \\ 1 & r = 0 \end{cases} .$$

So we have found that the sum of the lines 2, 4, 6 of eq. (15) is the opposite of the sum of its lines 1, 3, 5. This concludes the proof of eq. (15) and of all the arguments which we had to prove.

5. Comparison with Previously Known Results

- Let us begin with $N = 0$, that is the eleven moments model.

We see that in the expression (4) of h' , the second index of C_{\cdot} is negative so that we have $h' = 0$.

In the expression (5) of h'^k , the second index of C_{\cdot} is not negative iff $p_0 = 0, p_1 = 0$ so that

$\sum_{i=1}^2 i p_i$ cannot be odd; it follows $h'^k = 0$.

These results confirm the closure found in [17] where the contribute (4) and (5) for h' and h'^k is not present.

- Let us consider the case $N = 1$, that is the 24 moments model.

We firstly note that in (4) and (5) we have to impose that the second index of C_{\cdot} must be non negative. For this reason $\sum_{p_0=0}^{\infty}$ transforms in $\sum_{p_0=0}^{\frac{r+s-p-2}{2}}$ in the expression for h' which is present in eq. (27) of [17] and in $\sum_{p_0=0}^{\frac{r+s-p-1}{2}}$ in the expression for h'^k which is present in eq. (26) of [17].

After that, we note that in the line 4 of the present eq. (4) is present the term

$C_{p_0+p_1+p_2+p_3+r_0+r_1, -2p_0-p_1+p_3+r_1-1}$ whose first index is greater than the second one. Then this term is zero. There remains only the term in line 3. In order not to have the first index negative we have to impose $r_0 \leq p_0+p_1+p_2+p_3$; in order not to have the first index greater than the second one, we have to impose $r_0 \geq 3p_0 + 2p_1 + p_2 - r_1 + 1$. For this reason $\sum_{r_0=1}^{\infty}$ transforms in $\sum_{\vartheta=\max\{1, 3p_0+2p_1+p_2-r_1+1\}}^{p_0+p_1+p_2+p_3}$ in the expression for h' which is present in eq. (27) of [17].

For what concerns h'^k , we note that in the line 4 of the present eq. (5) is present a term C_{\cdot} whose first index is not greater than the second one only if $p_0 = 0, p_1 = 0, p_2 = 0, r_0 = 0$ so that $\sum_{i=1}^{N+2} ip_i + \sum_{i=1}^N ir_i = 3p_3 + r_1$. It is easy to identify this term with that in the last line, eq. (26) of [17] with

$$F_{r,s} = \frac{(r + 3s)!!}{(r + s)!!} (-2)^r r! C_{r+s, r+s}.$$

For what concerns the term in line 3, in order not to have the first index negative we have to impose $r_0 \leq p_0 + p_1 + p_2 + p_3$; in order not to have the first index greater than the second one, we have to impose $r_0 \geq 3p_0 + 2p_1 + p_2 - r_1$. For this reason $\sum_{r_0=1}^{\infty}$ transforms in $\sum_{\vartheta=\max\{1, 3p_0+2p_1+p_2-r_1\}}^{p_0+p_1+p_2+p_3}$ in the expression for h'^k which is present in eq. (26) of [17].

Under every other aspect, the present (4) and (5) are the same thing than (27) and (26) of [17].

So in the present article we have seen how these terms originated from the 1-Model are transmitted also to the models with subsequent values of N .

Conclusions: In this paper we have found the general exact solution for the macroscopic approach to extended thermodynamics of polyatomic gases with many moments. In fact, we can take for h' the sum of the present expression (4), of the expression (4) of [2] and of (5) in [4]. Similarly, we can take for h'^k the sum of the present expression (5), of the expression (5) of [2] and of (6) in [4]. The result is also a solution of the conditions (3) and, in the particular case $N = 1$, it is equal to (4) and (5) of [2]. There is no possible further additional term for h' and h'^k , as outlined in the conclusions of [1]. This confirms the fact

that the present results completely determine the requested solution for h' and h'^k .

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