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L-CLOSURE OPERATORS AND L-FUZZY PRE-PROXIMITIES

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Abstract: In this paper, we introduce the notions of L-fuzzy pre-proximities and L-interior operators in complete residuated lattices. We obtain the L-fuzzy pre-proximities induced by L-closure operators. Moreover, we investigate the relations between the L-fuzzy pre-proximities and L-closure operators. We give their examples.

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1. Introduction

Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structure. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [3,4,6,15,19]. Höhle [8] introduced L-fuzzy topologies with algebraic structure L(cqm, quantales, MV-algebra).

Katsaras [9-11] introduced the L-fuzzy proximity spaces in complete distributive lattices. Kim [13] extended the the L-fuzzy proximity spaces in strictly two-sided commutative quantales and investigated their topological properties.

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In this paper, we introduce the notions of L-fuzzy pre-proximities and L-closure operators in complete residuated lattices. We obtain the L-closure operators induced by L-fuzzy pre-proximities in Theorem 8 and the L-fuzzy pre-proximities induced by L-closure operators in Theorem 9. Moreover, we study the relations between the L-fuzzy pre-proximities and L-closure operators. We give their examples.

2. Preliminaries

Definition 1. [3,7,8] An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(L2) $(L, \odot, \top)$ is a commutative monoid;

(L3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, \oplus, ^*)$ is complete residuated lattice with an order reversing involution * which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow \perp$$

unless otherwise specified.

For $\alpha \in L, f \in L^X$, we denote $(\alpha \rightarrow f), (\alpha \odot f), \alpha_X \in L^X$ as $(\alpha \rightarrow f)(x) = \alpha \rightarrow f(x)$, $(\alpha \odot f)(x) = \alpha \odot f(x)$, $\alpha_X(x) = \alpha$.

Lemma 2. [3,7,8] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $\top \rightarrow x = x$, $\perp \odot x = \perp$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(3) $x \leq y$ iff $x \rightarrow y = \top$.

(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,

(5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,

(6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,

(7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(8) $(\bigwedge_i x_i) \oplus y = \bigwedge_i (x_i \oplus y)$,

(9) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(10) $x \odot y = (x \rightarrow y^*)^*$ and $x \oplus y = x^* \rightarrow y$,

(11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

(12) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,

(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$.

(14) $x \to y = y^* \to x^*$.

(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.

(16) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$,

(17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,

(18) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

Definition 3. [8] A map $\mathcal{C} : L^X \to L^X$ is called an L-closure operator on X if $\mathcal{C}$ satisfies the following conditions:

(C1) $\mathcal{C}(\perp_X) = \perp_X$,

(C2) $\mathcal{C}(f) \geq f$, for all $f \in L^X$,

(C3) If $f \leq g$, then $\mathcal{C}(f) \leq \mathcal{C}(g)$ for all $f, g \in L^X$,

(C4) $\mathcal{C}(f \oplus g) \leq \mathcal{C}(f) \oplus \mathcal{C}(g)$.

The pair $(X, \mathcal{C})$ is called an L-closure space. An L-closure space is called topological if

(T) $\mathcal{C}(\mathcal{C}(f)) = \mathcal{C}(f), \quad \forall f \in L^X$.

An L-closure space $(X, \mathcal{C})$ is said to be stratified if

(S) $\mathcal{C}(\alpha \odot f) \geq \alpha \odot \mathcal{C}(f)$.

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-closure spaces. $\varphi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is called an L-closure map if, for each $f \in L^X$,

$$\varphi^{\to}(\mathcal{C}_X(f)) \leq \mathcal{C}_Y(\varphi^{\to}(f)).$$

Definition 4. [13,15] A mapping $\delta : L^X \times L^X \to L$ is called an L-fuzzy pre-proximity on X if it satisfies the following axioms.

(P1) $\delta(\perp_X, \top_X) = \delta(\top_X, \perp_X) = \perp$.

(P2) $\delta(f, g) \geq \bigvee_{x\in X}(f(x) \odot g(x))$.

(P3) If $f_1 \leq f_2$ and $g_1 \leq g_2$, then $\delta(f_1, g_1) \leq \delta(f_2, g_2)$.

(P4) $\delta(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta(f_1, g_1) \oplus \delta(f_2, g_2)$.

The pair (X, δ) is called an L-fuzzy pre-proximity space.

An L-fuzzy pre-proximity is called an L-fuzzy quasi-proximity on X if

(PQ) $\delta(f, g) \geq \bigwedge_h \{\delta(f, h) \oplus \delta(h^*, g)\}$.

An L-fuzzy quasi-proximity is called an L-fuzzy proximity on X if (P) $\delta(f, g) = \delta(g, f)$.

Let (X, δ_1) and (Y, δ_2) be L-fuzzy pre-proximity spaces. $\varphi : (X, \delta_1) \to (Y, \delta_2)$ is called L-fuzzy proximity map if

$$\delta_1(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \leq \delta_2(f, g) \;\; \forall\, f, g \in L^Y.$$

Definition 5. [6,14] Let X be a set. A mapping $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions:

(E1) reflexive if $R(x,x) = \top$ for all $x \in X$,

(E2) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$, for all $x,y,z \in X$,

(E3) antisymmetric if $R(x,y) = R(y,x) = \top$, then $x = y$.

Lemma 6. [6,14] For a given set X, define a binary mapping $S : L^X \times L^X \to L$ by

$$S(f,g) = \bigwedge_{x \in X} (f(x) \to g(x)).$$

Then, for each $f,g,h,k \in L^X$, and $\alpha \in L$, the following properties hold.

(1) S is an L-partial order on L^X.

(2) $f \leq g$ iff $S(f,g) = \top$,

(3) If $f \leq g$, then $S(h,f) \leq S(h,g)$ and $S(f,h) \geq S(g,h)$,

(4) $S(f,g) \odot S(k,h) \leq S(f \odot k, g \odot h)$,

(5) $S(g,h) \leq S(f,g) \to S(f,h)$,

(6) $S(f,h) = \bigvee_{g \in L^X} (S(f,g) \odot S(g,h))$.

(7) If $\phi : X \to Y$ is a map, then for $f,g \in L^X$ and $h,k \in L^Y$,

$$S(f,g) \leq S(\phi^{\to}(f), \phi^{\to}(g)),$$

$$S(h,k) \leq S(\phi^{\leftarrow}(h), \phi^{\leftarrow}(k)),$$

and the equalities hold if ϕ is bijective.

3. *L*-Fuzzy Pre-Proximities and *L*-Closure Operators

Lemma 7. *Let $\mathcal{C} : L^X \to L^X$ a map. The following statement are equivalent.*

(1) *For all $f,g \in L^X$, $S(f,g) \leq S(\mathcal{C}(f), \mathcal{C}(g))$.*

(2) *If $f \leq g$, then $\mathcal{C}(f) \leq \mathcal{C}(g)$ and $\mathcal{C}(\alpha \odot h) \geq \alpha \odot \mathcal{C}(h)$ for all $f \in L^X$ and $\alpha \in L$.*

(3) *If $f \leq g$, then $\mathcal{C}(f) \leq \mathcal{C}(g)$ and $\mathcal{C}(\alpha \to h) \leq \alpha \to \mathcal{C}(h)$ for all $f \in L^X$ and $\alpha \in L$.*

Proof. (1) $\Rightarrow$ (2). If $f \leq g$, then $\top = S(f,g) \leq S(\mathcal{C}(f), \mathcal{C}(g))$. Hence $\mathcal{C}(f) \leq \mathcal{C}(g)$. Put $g = \alpha \odot f$. Then $S(f, \alpha \odot f) = \alpha \leq S(\mathcal{C}(f), \mathcal{C}(\alpha \odot f))$. Hence $\alpha \odot \mathcal{C}(f) \leq \mathcal{C}(\alpha \odot f)$.

(2) $\Rightarrow$ (3). Since $\alpha \odot \mathcal{C}(\alpha \to f) \leq \mathcal{C}(\alpha \odot (\alpha \to f)) \leq \mathcal{C}(f)$, $\mathcal{C}(\alpha \to f) \leq \alpha \to \mathcal{C}(f)$.

(3) $\Rightarrow$ (1). Since $S(f,g) \odot f \leq g$ iff $f \leq S(f,g) \to g$, $\mathcal{C}(f) \leq \mathcal{C}(S(f,g) \to g) \leq S(f,g) \to \mathcal{C}(g)$. Hence $S(f,g) \leq S(\mathcal{C}(f), \mathcal{C}(g))$.

From the following theorem, we obtain the L-closure operator induced by an L-fuzzy pre-proximity.

Theorem 8. *Let (X, δ) be an L-fuzzy pre-proximity space. We define a mappings $\mathcal{C}_\delta : L^X \to L^X$ as*

$$\mathcal{C}_\delta(f) = \bigwedge_{g \in L^X} ((S(f, g^*) \odot g) \to \delta(g, g^*))$$

Then $\mathcal{C}_\delta$ is a stratified L-closure operator on X.

Proof. (1) (C1) Since $\delta(\top_X, \bot_X) = \bot$,

$$\begin{array}{l} \mathcal{C}_\delta(\bot_X) = \bigwedge_{g \in L^X} ((S(\bot_X, g^*) \odot g) \to \delta(g, g^*)) \text{ (put } g = \top_X) \\ \leq (S(\bot_X, \bot_X) \odot \top_X) \to \delta(\top_X, \bot_X)) = \bot_X. \end{array}$$

(C2) Since $S(f, g^*) \odot g = S(g, f^*) \odot g \leq f^*$, then

$$\begin{array}{l} \mathcal{C}_\delta(f) = \bigwedge_{g \in L^X} ((S(f, g^*) \odot g) \to \delta(g, g^*)) \geq \bigwedge_{g \in L^X} (f^* \to \delta(g, g^*)) \\ = \bigwedge_{g \in L^X} (\delta^*(g, g^*) \to f) = \bigvee_{g \in L^X} \delta^*(g, g^*) \to f \geq \top \to f = f. \end{array}$$

(C4) Since

$$\begin{array}{l} ((a \to b) \oplus (c \to d))^* = (a \to b)^* \odot (c \to d)^* \\ = (a \odot b^*) \odot (c \odot d^*) = (a \odot c) \odot (b^* \odot d^*), \end{array}$$

we have $(a \to b) \oplus (c \to d) = (a \odot c) \to (b \oplus d)$.

From Lemma 6, we obtain

$$\begin{array}{l} \mathcal{C}_\delta(f) \oplus \mathcal{C}_\delta(h) = (\bigwedge_{g \in L^X} ((S(f, g^*) \odot g) \to \delta(g, g^*))) \\ \oplus (\bigwedge_{k \in L^X} ((S(h, k^*) \odot k) \to \delta(k, k^*))) \\ = \bigwedge_{g,k \in L^X} ((S(f, g^*) \odot g) \odot (S(h, k^*) \odot k) \to (\delta(g, g^*) \oplus \delta(k, k^*))) \\ \geq \bigwedge_{g,k \in L^X} ((S(f \oplus h, g^* \oplus k^*) \odot (g \odot k)) \to (\delta(g \odot k, g^* \oplus k^*))) \\ \geq \mathcal{C}_\delta(f \oplus h). \end{array}$$

(C3) and by Lemma 7, $\mathcal{C}_\delta$ is stratified from:

$S(\mathcal{C}_\delta(f), \mathcal{C}_\delta(h))$
$= \bigwedge_{x\in X} \Big((\bigwedge_{g\in L^X}((S(f,g^*)\odot g) \to \delta(g,g^*)))$
$\to (\bigwedge_{k\in L^X}((S(h,k^*)\odot k) \to \delta(k,k^*))) \Big)$
(by Lemma 2 (16))
$\geq \bigwedge_{x\in X} \bigwedge_{g\in L^X} \Big(((S(f,g^*)\odot g) \to \delta(g,g^*))$
$\to ((S(h,g^*)\odot g) \to \delta(g,g^*)) \Big)$
$\geq \bigwedge_{g\in L^X}(S(h,g^*)\odot g \to S(f,g^*)\odot g)$(by Lemma 2 (18))
$\geq \bigwedge_{g\in L^X}(S(h,g^*) \to S(f,g^*)) = S(f,h).$

Hence $\mathcal{C}_\delta$ is a stratified L-closure operator on X.

From the following theorem, we obtain the L-fuzzy pre-proximity induced by an L-closure operator.

Theorem 9. *Let $(X,\mathcal{C})$ be an L-closure space. Define a mapping $\delta_\mathcal{C} : L^X \times L^X \to L$ by*

$$\delta_\mathcal{C}(f,g) = \bigvee_{x\in X}(f(x)\odot \mathcal{C}(g)(x)).$$

Then we have the following properties.

(1) *$\delta_\mathcal{C}$ is an L-fuzzy pre-proximity.*

(2) *$\delta_\mathcal{C}(f,g) \leq \bigwedge_{h\in L^X}(\delta_\mathcal{C}(f,h)\oplus\delta_\mathcal{C}(g,h^*))$, the equality holds if $\mathcal{C}$ is topological.*

(3) *If $\mathcal{C}$ is topological, then $\delta_\mathcal{C}$ is an L-fuzzy quasi-proximity on X.*

(4) *$\mathcal{C} \leq \mathcal{C}_{\delta_\mathcal{C}}$, the equality holds if $\mathcal{C}$ is topological.*

(5) *If δ is an L-fuzzy pre-proximity on X, then $\delta_{\mathcal{C}_\delta}(f,g) \leq \bigvee_{x\in X}(f(x)\odot (g^*(x)\to\delta(g^*,g)))$, for each $f,g\in L^X$.*

Proof. (1) (P1) Since $\mathcal{C}(\bot_X) = \bot_X$ and $\mathcal{C}(\top_X) = \top_X$, we have

$$\delta_\mathcal{C}(\bot_X,\top_X) = \bigvee_{x\in X}(\bot_X(x)\odot\mathcal{C}(\top_X)(x)) = \bot.$$
$$\delta_\mathcal{C}(\top_X,\bot_X) = \bigvee_{x\in X}(\top_X(x)\odot\mathcal{C}(\bot_X)(x)) = \bot.$$

(P2) Since $\mathcal{C}(g) \geq g$, we have

$$\delta_\mathcal{C}(f,g) = \bigvee_{x\in X}(f(x)\odot\mathcal{C}(g)(x)) \geq \bigvee_{x\in X}(f(x)\odot g(x)).$$

(P3) If $g \leq g_1$, then $\mathcal{C}(g) \leq \mathcal{C}(g_1)$. Thus,

$$\delta_\mathcal{C}(f,g) = \bigvee_{x\in X}(f(x)\odot\mathcal{C}(g)(x))$$
$$\leq \bigvee_{x\in X}(f_1(x)\odot\mathcal{C}(g_1)(x)) = \delta_\mathcal{C}(f_1,g_1).$$

(P4)

$$\begin{aligned}
&\delta_{\mathcal{C}}(f_1, g_1) \oplus \delta_{\mathcal{C}}(f_2, g_2)\\
&= \bigvee_{x\in X}(f_1(x) \odot \mathcal{C}(g_1)(x)) \oplus \bigvee_{x\in X}(f_2(x) \odot \mathcal{C}(g_2)(x))\\
&\geq \bigvee_{x\in X}((f_1(x) \odot \mathcal{C}(g_1)(x)) \oplus (f_2(x) \odot \mathcal{C}(g_2)(x)))\\
&\ \text{(by Lemma 2 (17))}\\
&\geq \bigvee_{x\in X}((f_1(x) \odot f_2(x)) \odot (\mathcal{C}(g_1)(x) \oplus \mathcal{C}(g_2)(x)))\\
&\geq \bigvee_{x\in X}((f_1(x) \odot f_2(x)) \odot \mathcal{C}(g_1 \oplus g_2)(x))\\
&= \delta_{\mathcal{C}}(f_1 \odot f_2, g_1 \oplus g_2)
\end{aligned}$$

Hence $\delta_{\mathcal{C}}$ is an L-fuzzy pre-proximity.

(2)

$$\begin{aligned}
&\delta^*_{\mathcal{C}}(f, h) \odot \delta^*_{\mathcal{C}}(g, h^*)\\
&= \Big(\bigvee_{x\in X}(f(x) \odot \mathcal{C}(h)(x))\Big)^* \odot \Big(\bigvee_{x\in X}(h^*(x) \odot \mathcal{C}(g)(x))\Big)^*\\
&= S(f, \mathcal{C}^*(h)) \odot S(h^*, \mathcal{C}^*(g)) \quad (\text{ Since } \mathcal{C}^*(h) \leq h^*,)\\
&\leq S(f, h^*) \odot S(h^*, \mathcal{C}^*(g)) \leq S(f, \mathcal{C}^*(g)) = \delta^*_{\mathcal{C}}(f, g).
\end{aligned}$$

Hence $\delta_{\mathcal{C}}(f, g) \leq \bigwedge_{h\in L^X}(\delta_{\mathcal{C}}(f, h) \oplus \delta_{\mathcal{C}}(g, h^*))$.

If $\mathcal{C}$ is topological,

$$\begin{aligned}
&\bigvee_{h\in L^X}(\delta^*_{\mathcal{C}}(f, h) \odot \delta^*_{\mathcal{C}}(g, h^*))\\
&= \bigvee_{h\in L^X}(S(f, \mathcal{C}^*(h)) \odot S(h^*, \mathcal{C}^*(g))) \quad (\text{ Put } h = \mathcal{C}(g),)\\
&\geq S(f, \mathcal{C}^*(\mathcal{C}(g))) \odot S(\mathcal{C}^*(g), \mathcal{C}(g^*))\\
&= S(f, \mathcal{C}^*(g)) = \delta^*_{\mathcal{C}}(f, g).
\end{aligned}$$

(3) By (2), it is trivial.

(4)

$$\begin{aligned}
\mathcal{C}^*_{\delta_{\mathcal{C}}}(f) &= \bigvee_{g\in L^X}(\delta^*_{\mathcal{C}}(g, g^*) \odot S(f, g^*) \odot g))\\
&= \bigvee_{g\in L^X}(S(g, \mathcal{C}^*(g^*)) \odot S(f, g^*) \odot g)\\
&\leq \bigvee_{g\in L^X}(S(g, \mathcal{C}^*(g^*)) \odot S(\mathcal{C}(f), \mathcal{C}(g^*)) \odot g)\\
&= \bigvee_{g\in L^X}(S(g, \mathcal{C}^*(g^*)) \odot S(\mathcal{C}^*(g^*), \mathcal{C}^*(f)) \odot g)\\
&\leq \bigvee_{g\in L^X}(S(g, \mathcal{C}^*(f)) \odot g) \leq \mathcal{C}^*(f).
\end{aligned}$$

If $\mathcal{C}$ is topological,

$$\begin{aligned}
&\mathcal{C}^*_{\delta_{\mathcal{C}}}(f) = \bigvee_{g\in L^X}(\delta^*_{\mathcal{C}}(g, g^*) \odot S(f, g^*) \odot g))\\
&\ (\text{ Put } g^* = \mathcal{C}(f),)\\
&\geq S(\mathcal{C}^*(f), \mathcal{C}^*(\mathcal{C}(f))) \odot S(f, \mathcal{C}(f)) \odot \mathcal{C}^*(f)\\
&= S(\mathcal{C}^*(f), \mathcal{C}^*(f)) \odot S(f, \mathcal{C}(f)) \odot \mathcal{C}^*(f) = \mathcal{C}^*(f).
\end{aligned}$$

(5)

$$\delta_{\mathcal{C}_\delta}(f,g) = \bigvee_{x\in X}(f(x)\odot \mathcal{C}_\delta(g)(x))$$
$$= \bigvee_{x\in X}(f(x)\odot \bigwedge_{h\in L^X}(S(h,g^*)\odot h(x)\to \delta(h,h^*)))$$
$$\leq \bigvee_{x\in X}(f(x)\odot (S(g^*,g^*)\odot g^*(x)\to \delta(g^*,g)))$$
$$= \bigvee_{x\in X}(f(x)\odot (g^*(x)\to \delta(g^*,g)))$$

From the following remark, we obtain the L-fuzzy pre-proximity induced by an L-topology.

Remark 10. A subset $\tau\subset L^X$ is called an L-topology (ref[8]) if (O1) $\top_X,\bot_X\in\tau$, (O2) $f\odot g\in\tau$ for $f,g\in\tau$, (O3) $\bigvee_{i\in\Gamma} f_i\in\tau$ for $\{f_i\mid i\in\Gamma\}\subset\tau$. Define $\mathcal{C}_\tau : L^X\to L^X$ as follows:

$$\mathcal{C}_\tau(f) = \bigvee\{g\in L^X \mid g\geq f, g^*\in\tau\}.$$

Then $\mathcal{C}_\tau$ is a topological L-fuzzy closure operator. By the above theorem, we obtain $\delta_{\mathcal{C}_\tau}$ is an L-fuzzy quasi-proximity with $\mathcal{C}_\tau = \mathcal{C}_{\delta_{\mathcal{C}_\tau}}$.

Theorem 11. Let (X,δ_1) and (Y,δ_2) be L-fuzzy pre-proximity spaces, respectively. If $\varphi : (X,\delta_1)\to(Y,\delta_2)$ is an L-fuzzy proximity map, then $\varphi : (X,\mathcal{C}_{\delta_1})\to(Y,\mathcal{C}_{\delta_2})$ is an L-closure map.

Proof. $\varphi : (X,\mathcal{C}_{\delta_1})\to(Y,\mathcal{C}_{\delta_2})$ is an L-closure map from:

$$\varphi^{\leftarrow}(\mathcal{C}_{\delta_2}(g))(x) = \varphi^{\leftarrow}(\bigwedge_{h\in L^Y}(S(g,h^*)\odot h\to \delta_2(g,g^*))(x)$$
$$= \bigwedge_{h\in L^Y}(S(g,h^*)\odot h(\varphi(x))\to \delta_2(g,g^*))$$
(by Lemma 6(7))
$$\geq \bigwedge_{h\in L^Y}(S(\varphi^{\leftarrow}(g),\varphi^{\leftarrow}(h)^*)\odot \varphi^{\leftarrow}(h)(x)\to \delta_1(\varphi^{\leftarrow}(g),\varphi^{\leftarrow}(g)^*))$$
$$\geq \mathcal{C}_{\delta_1}(\varphi^{\leftarrow}(g)).$$

Put $g=\varphi^{\rightarrow}(f)$. Then

$$\mathcal{C}_{\delta_1}(\varphi^{\leftarrow}(\varphi^{\rightarrow}(f)))\leq \varphi^{\leftarrow}(\mathcal{C}_{\delta_2}(\varphi^{\rightarrow}(f)))$$
$$\text{iff } \varphi^{\rightarrow}(\mathcal{C}_{\delta_1}(\varphi^{\leftarrow}(\varphi^{\rightarrow}(f)))\leq \mathcal{C}_{\delta_2}(\varphi^{\rightarrow}(f)).$$

Hence $\varphi^{\rightarrow}(\mathcal{C}_{\delta_1}(f))\leq \mathcal{C}_{\delta_2}(\varphi^{\rightarrow}(f))$.

Theorem 12. Let $(X,\mathcal{C}_X)$ and $(Y,\mathcal{C}_Y)$ be L-closure spaces, respectively. If $\varphi : (X,\mathcal{C}_X)\to(Y,\mathcal{C}_Y)$ is an L-closure map, then $\varphi : (X,\delta_{\mathcal{C}_X})\to(Y,\delta_{\mathcal{C}_Y})$ is an L-fuzzy proximity map.

Proof. Since $\varphi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is an L-closure map,

$$\begin{aligned}
&\delta_{\mathcal{C}_Y}(f,g) = \bigvee_{y\in Y}(f(y)\odot \mathcal{C}_Y(g)(y))\\
&\geq \bigvee_{x\in X}(f(\varphi(x))\odot \mathcal{C}_Y(g)(\varphi(x)))\\
&\geq \bigvee_{x\in X}(\varphi^{\leftarrow}(f)(x)\odot \varphi^{\leftarrow}(\mathcal{C}_Y(g))(x))\\
&\geq \bigvee_{x\in X}(\varphi^{\leftarrow}(f)(x)\odot \mathcal{C}_X(\varphi^{\leftarrow}(g)))(x))\\
&\geq \delta_{\mathcal{C}_X}(\varphi^{\leftarrow}(f),\varphi^{\leftarrow}(g))
\end{aligned}$$

Example 13. Let $([0,1], \odot, \oplus, \to, ^*, 0, 1)$ be a complete residuated lattice (ref.[3,6-8]) as

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}$$

$$x \oplus y = \min\{1, x+y\},\ x^* = 1-x.$$

Let $X = \{x, y, z\}$ and $f \in [0,1]^X$ as follow:

$$f = (0.6, 0.3, 0.6), f \oplus f = (1, 0.6, 1)$$

$$g = (0.5, 0.1, 0.4), f \odot g = (0.1, 0, 0).$$

$$f^* = (0.4, 0.7, 0.4), f^* \oplus f^* = (0.8, 1, 0.8)$$

$$g = (0.5, 0.9, 0.6), f^* \odot g^* = (0.9, 1, 1).$$

(1) Define a $[0,1]$-fuzzy closure operator $\mathcal{C} : [0,1]^X \to [0,1]^X$ as follows:

$$\mathcal{C}(h) = \begin{cases} 0_X, & \text{if } h = 0_X,\\ f, & \text{if } h \leq f,\\ f \odot f & \text{if } h \leq f \oplus f,\ h \not\leq f,\\ 1_X, & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}$ is topological, by Theorem 9, we obtain an $[0.1]$-fuzzy quasi-proximity $\delta_{\mathcal{C}} : [0,1]^X \times [0,1]^X \to [0,1]$ as follows

$$\delta_{\mathcal{C}}(h,g) = \begin{cases} 0, & \text{if } h = 0_X\\ & \text{or } g = 0_X\\ \bigvee_{x\in X}(h(x)\odot f(x)), & \text{if } g \leq f\\ \bigvee_{x\in X}(h(x)\odot (f\oplus f)(x)), & \text{if } g \leq f \oplus f,\\ & g \not\leq f,\\ \bigvee_{x\in X} h(x), & \text{otherwise,} \end{cases}$$

Moreover, since $\mathcal{C}$ is topological, by Theorem 9(4), $\mathcal{C}_{\delta_{\mathcal{C}}} = \mathcal{C}$.

(2) Define $\delta : [0,1]^X \times [0,1]^X \to [0,1]$ as

$$\delta(f,g) = \bigvee_{x\in X} (f(x) \odot g(x).$$

$$\begin{aligned}
&\delta(f_1,g_1) \oplus \delta(f_2,g_2)\\
&= \bigvee_{x\in X}(f_1(x) \odot g_1(x)) \oplus \bigvee_{x\in X}(f_2(x) \odot g_2(x))\\
&\geq \bigvee_{x\in X}\Big((f_1(x) \odot g_1(x)) \oplus (f_2(x) \odot g_2(x))\Big)\\
&\geq \bigvee_{x\in X}\Big((f_1(x) \odot f_2(x)) \odot (g_1(x) \oplus g_2(x))\Big)\\
&= \delta_{\mathcal{C}}(f_1 \odot f_2, g_1 \oplus g_2)
\end{aligned}$$

$$\begin{aligned}
&\bigvee_{h\in L^X}(\delta^*(f,h) \odot \delta^*(g,h^*))\\
&= \bigvee_{h\in L^X}(\bigwedge_{x\in X}(f(x) \to h^*(x)) \odot \bigwedge_{x\in X}(h^*(x) \to g(x)))\\
&= \bigvee_{h\in L^X}(S(f,h^*) \odot S(h^*,g^*)) = S(f,g^*) = \delta^*(f,g).
\end{aligned}$$

$$\begin{aligned}
&\bigwedge_{h\in L^X}(\delta(f,h) \oplus \delta(g,h^*))\\
&= \Big(\bigvee_{h\in L^X}(\delta^*(f,h) \odot \delta^*(g,h^*))\Big)^* = \delta^*(f,g).
\end{aligned}$$

Hence δ is an $[0,1]$-fuzzy quasi-proximity. Since $\bigvee_{g\in L^X} S(g,f^*) \odot g = f^*$, we have

$$\begin{aligned}
&\mathcal{C}_\delta(f) = \bigwedge_{g\in L^X}(S(f,g^*) \odot g \to \delta(g,g^*))\\
&= (\bigvee_{g\in L^X} S(g,f^*) \odot g) \to \bigvee_{x\in X}(g(x) \odot g^*(x)) = f^* \to 0 = f.
\end{aligned}$$

Thus, $\delta_{\mathcal{C}_\delta} = \delta$.

(3) Define a $[0,1]$-topology $\tau = \{0_X, 1_X, f, g, f \odot f, f \odot g\}$.

By Remark 10, we obtain a topological $[0,1]$-closure operator $\mathcal{C}_\tau : [0,1]^X \to [0,1]^X$ as follows:

$$\mathcal{C}_\tau(h) = \begin{cases} 0_X, & \text{if } h = 0_X,\\ f^*, & \text{if } h \leq f^*,\\ g^*, & \text{if } h \leq g^*, h \not\leq f^*\\ f^* \oplus f^* & \text{if } h \leq f^* \oplus f^*,\ h \not\leq f^*,\ h \not\leq g^*\\ f^* \oplus g^* & \text{if } h \leq g^* \oplus f^*,\ h \not\leq f^* \oplus f^*\\ 1_X, & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}$ is topological, by Theorem 9, we obtain an $[0,1]$-fuzzy quasi-proximity

$\delta_{\mathcal{C}_\tau} : [0,1]^X \times [0,1]^X \to [0,1]$ as follows

$$\delta_{\mathcal{C}_\tau}(h,k) = \begin{cases} 0, & \text{if } h = 0_X \\ & \text{or } k = 0_X \\ \bigvee_{x\in X}(h(x) \odot f^*(x)), & \text{if } k \leq f^* \\ \bigvee_{x\in X}(h(x) \odot g^*(x)), & \text{if } k \leq g^*, k \not\leq f^*, \\ \bigvee_{x\in X}(h(x) \odot (f^* \oplus f^*)(x)), & \text{if } k \leq f^* \oplus f^*, \\ & k \not\leq f^*, k \not\leq g^*, \\ \bigvee_{x\in X}(h(x) \odot (f^* \oplus g^*)(x)), & \text{if } k \leq f^* \oplus g^*, \\ & k \not\leq f^* \oplus f^*, \\ \bigvee_{x\in X} h(x), & \text{otherwise}, \end{cases}$$

Moreover, since $\mathcal{C}$ is topological, by Theorem 9(4), $\mathcal{C}_{\delta_{\mathcal{C}_\tau}} = \mathcal{C}$.

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