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ALGORITHMS FOR COMPUTING QUARTIC GALOIS GROUPS OVER FIELDS OF CHARACTERISTIC 0

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Abstract: Let $f(x)$ be an irreducible degree four polynomial defined over a field F and let $K = F(\alpha)$ where α is a root of f in some fixed algebraic closure $\overline{F}$ of F. Several methods appear in the literature for computing the Galois group G of f, most of which rely on forming and factoring resolvent polynomials; i.e., polynomials defining subfields of the splitting field of f. This paper surveys those methods that generalize to arbitrary base fields of characteristic 0. Further, we describe a non-resolvent method that determines if K has a quadratic subfield by counting the number of roots of f that are contained in K, and we also describe how to construct explicitly a polynomial defining a quadratic subfield. We end with a comparison of run times for the various algorithms in the case F is the rational numbers.

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1. Introduction

Galois theory shows it is possible to associate a group structure to a polynomial's roots. This group, called the polynomial's Galois group, is a collection of

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permutations of the roots that encodes arithmetic information concerning the polynomial. For example, the polynomial is solvable by radicals if and only if its Galois group is solvable (see for example [4, p.628]). An important problem in computational algebra therefore is to determine the Galois group of a polynomial $f(x)$ of degree n defined over a field F.

For this paper, we focus on quartic polynomials, surveying methods appearing in the literature that apply to arbitrary base fields of characteristic 0. These methods [1, 3, 4, 7] rely on factoring resolvent polynomials [5, 8, 9], which are polynomials that define subfields of the splitting field of f. We also present a non-resolvent method that relies on factoring the quartic polynomial over the extension field it defines. One advantage of this method is that it allows us to determine when the quartic polynomial's extension field has a quadratic subfield, and it gives us an explicit calculation of a polynomial that defines such a quadratic subfield. As a corollary, we can determine when any irreducible quartic polynomial can be transformed into a biquadratic (i.e., even quartic) polynomial defining the same extension, and hence having the same Galois group.

The remainder of the paper is organized as follows. Section 2 provides a brief overview of the basic notation and terminology related to the Fundamental Theorem of Galois Theory as they relate to our context. In this section, we also show the relationship between the automorphism group of a polynomial's extension field and its Galois group, paving the way for our non-resolvent approach to computing quartic Galois groups. In Section 3 we give an introduction to resolvent polynomials, including a discussion of the most widely-used resolvent; namely, the discriminant. Section 4 details all of the algorithms under consideration that are based on the resolvent method. For each algorithm, we describe its implementation and prove its correctness. Section 5 discusses our non-resolvent algorithm for computing quartic Galois groups. Additionally, we show every biquadratic irreducible quartic polynomial has a Galois group that is a subgroup of D_4 (the dihedral group of order 8). Conversely, if a quartic polynomial's Galois group is a subgroup of D_4, we describe how to construct explicitly a biquadratic polynomial that defines the same extension (and hence has the same Galois group). Our final section includes an analysis of run times for each algorithm considered in this paper in the case the ground field is the rational numbers. Our findings indicate that the most efficient algorithm is the modification of the traditional cubic resolvent method, as outlined in [7].

2. Overview of Galois Theory

In this section, we give a very brief overview of Galois theory in our context. More details can be found in any standard text on abstract algebra (such as [4]).

Let F be a field and let $\overline{F}$ be a fixed algebraic closure. Let $f(x) \in F[x]$ be an irreducible polynomial of degree n with roots $\alpha = \alpha_1, \ldots, \alpha_n \in \overline{F}$. Let K be the **stem field** of f; that is, $K = F(\alpha)$.

Then it is straight forward to verify that K is a vector space over F with the set $\{1, \alpha, \ldots, \alpha^{-1}\}$ serving as a basis. Since the dimension of K as an F-vector space is n, we say K/F is an extension field of **degree** n and we write $[K : F] = n$.

We are interested primarily in automorphisms of K/F, which are field isomorphisms from K to itself that restrict to the identity function on F.

Definition 2.1. An **automorphism** of K/F is a mapping $\sigma : K \to K$ such that,

1. σ is bijective,
2. $\sigma(x + y) = \sigma(x) + \sigma(y)$ for all $x, y \in K$,
3. $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in K$,
4. $\sigma(a) = a$ for all $a \in F$.

The collection of all automorphisms of K/F is denoted $\mathrm{Aut}(K/F)$.

The automorphisms of K/F form a group under function composition. Since automorphisms are field isomorphisms and since a basis for K/F consists of powers of the one root α, it follows that each automorphism is completely determined by where it sends α. Furthermore, since automorphisms act like the identity function on F, we have $0 = \sigma(0) = \sigma(f(\alpha)) = f(\sigma(\alpha))$ for each $\sigma \in \mathrm{Aut}(K/F)$. This shows that automorphisms send α to some other root of f that lies in K.

Therefore, elements in $\mathrm{Aut}(K/F)$ are in a bijective correspondence with the roots of f that are contained in K, and each automorphism permutes these roots. In the case where K contains all n roots of f, $\mathrm{Aut}(K/F)$ is called the **Galois group** of f. Otherwise, the Galois group of f is the automorphism group of the **splitting field** of f; that is, the smallest subfield of $\overline{F}$ that contains F and all n roots of f.

The Fundamental Theorem

The next result is the Fundamental Theorem of Galois Theory, which gives a bijection between the subfields of K/F and the subgroups of $\mathrm{Aut}(K/F)$ when K/F is a Galois extension (see [4, p. 574]).

Theorem 2.2 (Fundamental Theorem of Galois Theory)**.** *Let L be the splitting field of an irreducible polynomial f of degree n defined over F, and let G be the Galois group of f (i.e., the automorphism group of L/F). There is a bijective correspondence between the subfields of L/F and the subgroups of G. Specifically, if K is a subfield of L/F, then K corresponds to the set of all $\sigma \in G$ such that $\sigma(a) = a$ for all $a \in K$. Similarly, if H is a subgroup of G, then H corresponds to the set of all $a \in L$ such that $\sigma(a) = a$ for all $\sigma \in H$.*

The Galois correspondence has the following properties.

1. *If H_1 and H_2 are subgroups of G that correspond to subfields K_1 and K_2 respectively, then $H_1 \leq H_2$ if and only if $K_2 \subseteq K_1$.*

2. *If K defines a subfield of L of degree d over F, then K corresponds to a subgroup $H \leq G$ of index d.*

3. *Let K be a subfield of L/F defined by the polynomial g and let $H \leq G$ be the subgroup corresponding to K. The splitting field of g corresponds to the largest normal subgroup N of G contained inside H; i.e., the kernel of the permutation representation of G acting on G/H. The Galois group of g is isomorphic to G/N.*

4. *If we label the roots of f as $\alpha_1, \dots, \alpha_n$, then G can be identified with a transitive subgroup of S_n, well-defined up to conjugation.*

5. *If we label the roots of f as $\alpha_1, \dots, \alpha_n$, then the stem field of f (generated by α_1) corresponds to G_1, the point stabilizer of 1 inside G.*

Automorphism Groups and Stem Fields

We now prove several facts about subfields of L/F and subgroups of $\mathrm{Aut}(L/F)$. The first considers conjugate fields of a subfield K of L/F. For $\sigma \in \mathrm{Aut}(L/F)$ and a subfield K of L/F, the image $\sigma(K)$ of K under σ is called a **conjugate field** of K. The next result shows that conjugate fields correspond to conjugate subgroups.

Proposition 2.3. *Let L/F be a Galois extension with Galois group G. Let H be a subgroup of G and let K be the fixed field of H. For $\sigma \in G$, the subgroup fixing $\sigma(K)$ is $\sigma H \sigma^{-1}$.*

Proof. First we show $\sigma H \sigma^{-1}$ fixes $\sigma(K)$. Let $\sigma h \sigma^{-1} \in \sigma H \sigma^{-1}$ and let $\sigma(x) \in \sigma(K)$. Then $\sigma h \sigma^{-1}(\sigma(x)) = \sigma h(x) = \sigma(x)$, as desired.

Next we show every element that fixes $\sigma(K)$ belongs to $\sigma H \sigma^{-1}$. Suppose $\tau \in G$ fixes $\sigma(K)$. Then for all $x \in K$, $\tau(\sigma(x)) = \sigma(x)$. Thus $\sigma^{-1}\tau\sigma(x) = x$. Thus $\sigma^{-1}\tau\sigma$ fixes K, which implies $\sigma^{-1}\tau\sigma \in H$. Thus $\tau \in \sigma H \sigma^{-1}$, as desired. □

Proposition 2.4. *Let L/F be a Galois extension with Galois group G. Let H be a subgroup of G and let K be the fixed field of H. Let $\sigma, \tau \in G$. Then $\sigma(x) = \tau(x)$ for all $x \in K$ if and only if $\tau \in \sigma H$.*

Proof. Let $\sigma, \tau \in G$ and suppose $\sigma(x) = \tau(x)$ for all $x \in K$. Then $x = \sigma^{-1}\tau(x)$ for all $x \in K$. Thus $\sigma^{-1}\tau$ fixes K, which is true if and only if $\sigma^{-1}\tau \in H$. This is true if and only if $\tau \in \sigma H$, as desired. □

Next we describe the relationship between the automorphism group of a subfield and the Galois group of the splitting field.

Proposition 2.5. *Let L/F be a Galois extension with Galois group G. Let H be a subgroup of G and let K be the fixed field of H. Then $Aut(K/F) \simeq N/H$, where N is the normalizer of H in G.*

Proof. First we note that every element in $\mathrm{Aut}(K/F)$ corresponds to some restriction to K of an element $\sigma \in G$ [4, p. 575]. Such an element σ will restrict to an automorphism of K/F precisely when $\sigma(K) = K$. By Proposition 2.3, these are restrictions of elements in the normalizer N of H in G. By Proposition 2.4, the elements in N that restrict to distinct functions on K correspond to the cosets of H in N. Therefore $\mathrm{Aut}(K/F)$ is isomorphic to the quotient group N/H, as desired. □

As a corollary, the next result explicitly determines the connection between the automorphism group of a polynomial's stem field and the polynomial's Galois group. This will be the basis for our non-resolvent method for computing quartic Galois groups, as mentioned in Section 5.

Corollary 2.6. *Let L be the splitting field of the irreducible polynomial $f(x)$ of degree n defined over F, and let G be the Galois group of f. Let K/F be the stem field of f. Let G_1 be the point stabilizer of 1 inside G, and let*

Table 1: The 5 conjugacy classes of transitive subgroups of S_4. Generators are for one representative in each conjugacy class.

T	Name	Generators	Size
1	C_4	(1234)	4
2	V_4	(12)(34), (13)(24)	4
3	D_4	(1234), (13)	8
4	A_4	(123), (234)	12
5	S_4	(1234), (12)	24

N be the normalizer of G_1 in G. Then $Aut(K/F) \simeq N/G_1$. Furthermore, the number of roots of f that are contained in K is equal to the index $[N : G_1]$.

Proof. Since K is the stem field, $K \simeq F(\alpha_1)$, where $\alpha_1, \dots, \alpha_n$ are the roots of f in $\overline{F}$. Under the Galois correspondence, the subgroup that corresponds to K consists of those elements that fix K; i.e., all $\sigma \in G$ such that $\sigma(\alpha_1) = \alpha_1$. This is precisely the point stabilizer of 1 inside G. Therefore $\mathrm{Aut}(K/F) \simeq N/G_1$ by Proposition 2.5. In particular, these two groups have the same order. The final statement of the corollary now follows since the number of automorphisms equals the number of roots of f contained in K, and the number of elements in N/G_1 is equal to the index $[N : G_1]$. □

Notice that by item (4) of Theorem 2.2, we must identify the conjugacy classes of transitive subgroups of S_4 in order to determine the group structure of G. This information is well known (see [2]). In Table 1, we give information on the 5 conjugacy classes of transitive subgroups of S_4, including their transitive number (or T-number, as in [6]), generators of one representative, their size, and a more descriptive name based on their structure. The descriptive names are standard: C_4 represents the cyclic group of order n, D_4 the dihedral group of order 8, V_4 the elementary abelian group of order 4, and A_4 and S_4 the alternating and symmetric groups on 4 letters.

3. Resolvent Polynomials and Splitting Fields

The techniques for computing Galois groups this paper is studying are based on the use of absolute resolvent polynomials [8]. In short, this method works as follows. Let $f(x)$ be an irreducible polynomial (over F) of degree n. Let G be the Galois group of f, and let H be a subgroup of S_n. We form a resolvent polynomial $R_H(x)$ whose stem field corresponds to H under the Galois correspondence. By item (3) of Theorem 2.2, the Galois group of $R_H(x)$ is isomorphic to the image of the permutation representation of G acting on the cosets S_n/H. Then as shown in [8], the irreducible factors of $R_H(x)$ therefore correspond to the orbits of this action. In particular, the degrees of the irreducible factors correspond to the orbit lengths.

The most difficult task in the resolvent method is constructing the polynomial $R_H(x)$ that corresponds to a given subgroup H of S_n. The following result gives one method for accomplishing such a task. A proof can be found in [8].

Theorem 3.1. *Let $f(x)$ be an irreducible polynomial of degree n defined over a field F, K the splitting field of f, and $\alpha_1, \ldots, \alpha_n$ the roots of f in $\overline{F}$. Let $T(x_1, \ldots, x_n)$ be a polynomial with coefficients in F, and let H be the stabilizer of T in S_n. That is*

$$H = \{\sigma \in S_n : T(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = T(x_1, \ldots, x_n)\}.$$

Let $S_n//H$ denote a complete set of coset representatives of H in S_n, and define the resolvent polynomial $R_H(x)$ by:

$$R_H(x) = \prod_{\sigma \in S_n//H} (x - T(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})).$$

1. *If $R_H(x)$ is squarefree, its Galois group is isomorphic to the image of the permutation representation of G acting on the cosets S_n/H.*
2. *We can ensure $R_H(x)$ is squarefree by taking a suitable Tschirnhaus transformation of $f(x)$ [3, p.324]. This amounts to randomly picking an element in the stem field of $f(x)$ and computing its characteristic polynomial.*
3. *One choice for T is given by:*

$$T(x_1, \ldots, x_n) = \sum_{\sigma \in H} \left(\prod_{i=1}^{n} x^i_{\sigma(i)} \right).$$

Though this is not the only choice.

We now describe how to construct the resolvents used in the following papers: [1, 3, 7]. To illustrate each of these constructions, we will use the following sample polynomials:

t_1	$x^4 + 5x^2 + 5$
t_2	$x^4 + 1$
t_3	$x^4 + 2$
t_4	$x^4 - 7x^2 - 3x + 1$
t_5	$x^4 - x^3 + 1$

The Discriminant

Perhaps the most well known example of a resolvent polynomial is the discriminant. Recall that the discriminant of a degree n polynomial $f(x)$ is given by

$$\operatorname{disc}(f) = \prod_{1\le i<j\le n} (\alpha_i - \alpha_j)^2,$$

where α_i are the roots of f. In particular, let

$$T = \prod_{1\le i<j\le n} (x_i - x_j).$$

It is well-known that T is stabilized by A_n [4, p.610]. Notice that a complete set of coset representatives of S_n/A_n is $\{\mathrm{id}, (12)\}$. Also notice that applying the permutation (12) to the subscripts of T results in $-T$. In this case, we can form the resolvent polynomial $R(x)$ as follows,

$$R(x) = \prod_{\sigma\in S_{\mathrm{n}}/A_{\mathrm{n}}} (x - \sigma(T)) = x^2 - T^2 = x^2 - \operatorname{disc}(f).$$

In particular, this resolvent factors if and only if the discriminant is a perfect square in the base field. It is well-known that the discriminant can be computed via the resultant of $f(x)$ and its derivative; namely,

$$\operatorname{disc}(f) = (-1)^{n(n-1)}\operatorname{Res}(f, f'),$$

see [3, p. 119].

Furthermore, by the theory of elementary symmetric polynomials, it follows that $\operatorname{disc}(f)$ can be expressed using only the coefficients of f [4, p. 611]. For example, in the case of quartic polynomials,we have:

$$\begin{aligned}\operatorname{disc}(x^4 + ax^3 + bx^2 + cx + d) = {} & -27a^4d^2 + 2a^3c(9bd - 2c^2) \\ & - a^2(4b^3d - b^2c^2 - 144bd^2 + 6c^2d) \\ & - 2ac(40b^2d - 9bc^2 + 96d^2) \\ & + 4b(4b^3 - b^2c^2 - 32bd^2 + 36c^2d) \\ & - 27c^4 + 256d^3.\end{aligned}$$

Here are the discriminants of the five sample polynomials $t_1, \ldots, t_5$; the final column gives the prime factorization of the discriminant over the rational numbers.

Poly	Disc	Factorization
t_1	2000	$2^4.5^3$
t_2	256	2^8
t_3	2048	2^{11}
t_4	33489	$3^2.61^2$
t_5	229	229^1

The Cubic Resolvent

Both the traditional approach to computing quartic Galois groups [4, p. 614] and its refinement [7] rely on factoring a cubic resolvent polynomial. In this case, we let $T = x_1x_3 + x_2x_4$. The stabilizer H of T in S_4 is a dihedral group of order 8. The elements of this particular group are,

$$H = \{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}\ .$$

A complete set of right coset representatives for S_4/H is given by: $\{(1), (12), (14)\}$. We can form the cubic resolvent for $f(x)$ by computing R_H:

$$R_H(x) = (x - (\alpha_1\alpha_3 + \alpha_2\alpha_4))\,(x - (\alpha_2\alpha_3 + \alpha_1\alpha_4))\,(x - (\alpha_4\alpha_3 + \alpha_2\alpha_1))\,.$$

We can also express the cubic resolvent in terms of the coefficients of the original quartic. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$, then the cubic resolvent of f is:

$$R_H(x) = x^3 - bx^2 + (ac - 4d)x + 4bd - a^2d - c^2$$

Here are the cubic resolvents of the five sample polynomials $t_1, \ldots, t_5$; the final column gives the factorization of the cubic resolvent over the rational numbers.

Poly	Cubic Res	Factorization
t_1	$x^3 - 5x^2 - 20x + 100$	$(x^2 - 20)(x - 5)$
t_2	$x^3 - 4x$	$x(x - 2)(x + 2)$
t_3	$x^3 - 8x$	$x(x^2 - 8)$
t_4	$x^3 + 7x^2 - 4x - 37$	$x^3 + 7x^2 - 4x - 37$
t_5	$x^3 - 4x - 1$	$x^3 - 4x - 1$

The Degree 6 Resolvent

The method in [3, §6.3] relies on factoring a degree 6 resolvent polynomial. In this case, we let $T = (x_1 + x_3 - (x_2 + x_4))(x_1 - x_3)(x_2 - x_4)$. The stabilizer H

of T in S_4 is a cyclic group of order 4. The elements of this particular group are,

$$H = \{(1), (1234), (13)(24), (1432)\} .$$

A complete set of right coset representatives for S_4/H is given by:

$$\{(1), (12), (13), (14), (23), (34)\}.$$

We can form the degree 6 resolvent for $f(x)$ by computing R_H:

$$R_H(x) = \prod_{\sigma \in S_4/H} (x - (\alpha_{\sigma(1)} + \alpha_{\sigma(3)} - (\alpha_{\sigma(2)} + \alpha_{\sigma(4)}))(\alpha_{\sigma(1)} - \alpha_{\sigma(3)})(\alpha_{\sigma(2)} - \alpha_{\sigma(4)})).$$

We can also express the degree 6 resolvent in terms of the coefficients of the original quartic. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$, then the degree 6 resolvent of f is:

$$R_H(x) = x^6 + Ax^4 + Bx^2 + C,$$

where:

$$A = -2a^2b^2 + 8b^3 + 6a^3c - 28abc + 36c^2 + 12a^2d - 32bd,$$

$$\begin{aligned} B = {} & a^4b^4 - 8a^2b^5 + 16b^6 - 6a^5b^2c + 52a^3b^3c - 112ab^4c + 9a^6c^2 - 84a^4bc^2 \\ & + 156a^2b^2c^2 + 160b^3c^2 + 124a^3c^3 - 576abc^3 + 432c^4 - 12a^4b^2d \\ & + 96a^2b^3d - 192b^4d + 36a^5cd - 336a^3bcd + 768ab^2cd + 240a^2c^2d \\ & - 1152bc^2d + 144a^4d^2 - 768a^2bd^2 + 768b^2d^2 + 768acd^2 - 1024d^3, \end{aligned} \tag{1}$$

$$\begin{aligned} C = {} & -a^8b^2c^2 + 12a^6b^3c^2 - 48a^4b^4c^2 + 64a^2b^5c^2 + 4a^9c^3 - 50a^7bc^3 + 192a^5b^2c^3 \\ & - 160a^3b^3c^3 - 256ab^4c^3 + 91a^6c^4 - 760a^4bc^4 + 1520a^2b^2c^4 + 256b^3c^4 + 688a^3c^5 \\ & - 2880abc^5 + 1728c^6 + 4a^8b^3d - 48a^6b^4d + 192a^4b^5d - 256a^2b^6d - 18a^9bcd \\ & + 224a^7b^2cd - 864a^5b^3cd + 768a^3b^4cd + 1024ab^5cd + 6a^8c^2d - 480a^6bc^2d \\ & + 3680a^4b^2c^2d - 7168a^2b^3c^2d - 1024b^4c^2d + 96a^5c^3d - 3840a^3bc^3d \\ & + 14336ab^2c^3d + 384a^2c^4d - 9216bc^4d + 27a^10d^2 - 360a^8bd^2 + 1712a^6b^2d^2 \\ & - 3328a^4b^3d^2 + 2048a^2b^4d^2 + 624a^7cd^2 - 5568a^5bcd^2 + 14336a^3b^2cd^2 \\ & - 8192ab^3cd^2 + 4800a^4c^2d^2 - 21504a^2bc^2d^2 + 8192b^2c^2d^2 + 12288ac^3d^2 \\ & - 256a^6d^3 + 2048a^4bd^3 - 4096a^2b^2d^3 - 4096a^3cd^3 + 16384abcd^3 - 16384c^2d^3. \end{aligned} \tag{2}$$

Here are the degree 6 resolvents of the five sample polynomials $t_1, \ldots, t_5$. Note that we need to take a tschirnhaus transformation of t_1, t_2, and t_3, since their degree 6 resolvents are not squarefree. In each case, we give the transformed polynomial, the degree 6 resolvent, and the factorization of the degree 6 polynomial over the rationals. For t_4 and t_5, the resolvent is irreducible, so we only give it.

t_1

tschirnhaus: $x^4 + 8x^3 + 34x^2 + 32x + 181$

degree 6 res: $x^6 - 8388608000x^2 - 214748364800000$

factorization: $(x - 320)(x + 320)(x^4 + 102400x^2 + 2097152000)$

t_2

tschirnhaus: $x^4 - 12x^3 + 62x^2 - 84x + 34$

degree 6 res: $x^6 + 165888x^4 + 6855736320x^2 - 1988235362304$

factorization: $(x^2 - 288)(x^2 + 82944)(x^2 + 83232)$

t_3

tschirnhaus: $x^4 - 12x^3 + 58x^2 - 124x + 99$

degree 6 res: $x^6+648x^5+173360x^4+24478976x^3+1921577216x^2+79384270848x + 1346049994752$

factorization: $(x^2+280x+19856)(x^4+368x^3+50464x^2+3042048x+67790592)$

t_4

degree 6 res: $x^6 + 8x^5 + 8x^4 - 384x^3 - 2176x^2 - 4864x - 6656$

t_5

degree 6 res: $x^6 + x^4 - 30x^3 - 57x^2 - 15x - 4$

The Degree 12 Resolvent

The method in [1] relies on factoring a degree 12 resolvent polynomial. In this case, we let $T = x_1 + x_2 + (x_1 - x_2)(x_3 - x_4)$. The stabilizer H of T in S_4 is a cyclic group of order 2. The elements of this particular group are,

$$H = \{(1), (12)(34)\}\,.$$

A complete set of right coset representatives for S_4/H is given by:

$$\{(1),(12),(13),(14),(23),(24),(123),(124),(132),(142),(13)(24),(1324)\}.$$

We can form the degree 12 resolvent for $f(x)$ by computing R_H:

$$R_H(x) = \prod_{\sigma\in S_4/H} (x - (\alpha_{\sigma(1)} + \alpha_{\sigma(2)} + (\alpha_{\sigma(1)} - \alpha_{\sigma(2)})(\alpha_{\sigma(3)} - \alpha_{\sigma(4)}))).$$

We can also express the degree 12 resolvent in terms of the coefficients of the original quartic. If $f(x) = x^4 + ax^3 + bx^2 + cx + d$, then the degree 12 resolvent of f is:

$$R_H(x) = \sum_{i=0}^{12} A_i x^i,$$

where

$$\begin{aligned}
A_{12} =&1,\\
A_{11} =&-6a,\\
A_{10} =&15a^2 + 4b - 4b^2 + 12ac - 48d,\\
A_9 =&-20a^3 - 20ab + 20ab^2 - 60a^2c + 240ad,\\
A_8 =&15a^4 + 40a^2b + 6b^2 - 42a^2b^2 - 4b^3 + 6b^4 + 2ac + 126a^3c + 6abc\\
&- 36ab^2c + 54c^2 + 54a^2c^2 - 8d - 450a^2d - 240bd + 144b^2d\\
&- 432acd + 864d^2,\\
A_7 =&-6a^5 - 40a^3b - 24ab^2 + 48a^3b^2 + 16ab^3 - 24ab^4 - 8a^2c\\
&- 144a^4c - 24a^2bc + 144a^2b^2c - 216ac^2 - 216a^3c^2 + 32ad\\
&+ 360a^3d + 960abd - 576ab^2d + 1728a^2cd - 3456ad^2,\\
A_6 =&a^6 + 20a^4b + 36a^2b^2 - 32a^4b^2 + 4b^3 - 28a^2b^3 + 4b^4\\
&+ 40a^2b^4 - 4b^5 - 4b^6 + 12a^3c + 96a^5c + 6abc + 46a^3bc - 24ab^2c\\
&- 240a^3b^2c + 38ab^3c + 36ab^4c - 2c^2 + 346a^2c^2 + 360a^4c^2 + 96bc^2\\
&- 78a^2bc^2 - 126b^2c^2 - 110a^2b^2c^2 + 8b^3c^2 + 378ac^3 + 116a^3c^3\\
&- 36abc^3 + 54c^4 - 50a^2d - 42a^4d - 16bd - 1456a^2bd - 288b^2d\\
&+ 834a^2b^2d + 352b^3d + 8a^2b^3d - 176b^4d - 32acd - 2502a^3cd\\
&- 888abcd - 36a^3bcd + 1024ab^2cd - 1512c^2d - 1284a^2c^2d\\
&- 288bc^2d + 64d^2 + 4248a^2d^2 + 54a^4d^2 + 4800bd^2 - 288a^2bd^2\\
&- 1472b^2d^2 + 5568acd^2 - 7424d^3,
\end{aligned}$$

$$
\begin{aligned}
A_5 =& -4a^5b - 24a^3b^2 + 12a^5b^2 - 12ab^3 + 28a^3b^3 - 12ab^4 - 36a^3b^4 + 12ab^5\\
&+ 12ab^6 - 8a^4c - 36a^6c - 18a^2bc - 54a^4bc + 72a^2b^2c + 216a^4b^2c\\
&- 114a^2b^3c - 108a^2b^4c + 6ac^2 - 282a^3c^2 - 324a^5c^2 - 288abc^2\\
&+ 234a^3bc^2 + 378ab^2c^2 + 330a^3b^2c^2 - 24ab^3c^2 - 1134a^2c^3\\
&- 348a^4c^3 + 108a^2bc^3 - 162ac^4 + 38a^3d - 126a^5d + 48abd\\
&+ 1008a^3bd + 864ab^2d - 486a^3b^2d - 1056ab^3d - 24a^3b^3d + 528ab^4d\\
&+ 96a^2cd + 1458a^4cd + 2664a^2bcd + 108a^4bcd - 3072a^2b^2cd\\
&+ 4536ac^2d + 3852a^3c^2d + 864abc^2d - 192ad^2 - 648a^3d^2 - 162a^5d^2\\
&- 14400abd^2 + 864a^3bd^2 + 4416ab^2d^2 - 16704a^2cd^2 + 22272ad^3,\\
A_4 =& 6a^4b^2 - 2a^6b^2 + 12a^2b^3 - 16a^4b^3 + b^4 + 10a^2b^4 + 19a^4b^4 + 4b^5\\
&- 16a^2b^5 + 6b^6 - 14a^2b^6 + 4b^7 + b^8 + 2a^5c + 6a^7c + 18a^3bc + 36a^5bc\\
&+ 6ab^2c - 72a^3b^2c - 114a^5b^2c - 10ab^3c + 148a^3b^3c - 50ab^4c\\
&+ 126a^3b^4c - 46ab^5c - 12ab^6c - 5a^2c^2 + 126a^4c^2 + 171a^6c^2 - 4bc^2\\
&+ 302a^2bc^2 - 300a^4bc^2 + 34b^2c^2 - 345a^2b^2c^2 - 393a^4b^2c^2\\
&+ 64b^3c^2 + 248a^2b^3c^2 + 10b^4c^2 + 58a^2b^4c^2 - 16b^5c^2 + 42ac^3\\
&+ 1342a^3c^3 + 438a^5c^3 - 306abc^3 - 548a^3bc^3 - 180ab^2c^3\\
&- 136a^3b^2c^3 + 120ab^3c^3 + 513c^4 + 1215a^2c^4 + 129a^4c^4 - 540bc^4\\
&- 216a^2bc^4 - 108b^2c^4 + 324ac^5 - 14a^4d + 84a^6d - 52a^2bd - 252a^4bd\\
&- 8b^2d - 974a^2b^2d - 12a^4b^2d - 96b^3d + 1344a^2b^3d + 60a^4b^3d\\
&- 56b^4d - 734a^2b^4d + 144b^5d - 16a^2b^5d + 112b^6d - 8acd - 102a^3cd\\
&+ 36a^5cd + 8abcd - 3522a^3bcd - 270a^5bcd + 240ab^2cd + 4044a^3b^2cd\\
&- 784ab^3cd + 120a^3b^3cd - 944ab^4cd - 168c^2d - 3846a^2c^2d\\
&- 3636a^4c^2d - 2880bc^2d - 3144a^2bc^2d - 216a^4bc^2d + 5040b^2c^2d\\
&+ 2280a^2b^2c^2d + 384b^3c^2d - 6480ac^3d - 1416a^3c^3d - 864abc^3d\\
&- 1296c^4d + 16d^2 + 120a^2d^2 - 2367a^4d^2 + 405a^6d^2 + 320bd^2\\
&+ 14784a^2bd^2 - 2700a^4bd^2 + 5280b^2d^2 + 912a^2b^2d^2 - 108a^4b^2d^2\\
&- 8512b^3d^2 + 384a^2b^3d^2 + 1120b^4d^2 - 2976acd^2 + 14544a^3cd^2\\
&+ 324a^5cd^2 + 12576abcd^2 - 864a^3bcd^2 - 8256ab^2cd^2 + 12960c^2d^2\\
&+ 9792a^2c^2d^2 + 6912bc^2d^2 + 3968d^3 - 15072a^2d^3 - 1296a^4d^3\\
&- 34048bd^3 + 6912a^2bd^3 + 1792b^2d^3 - 33024acd^3 + 33024d^4,
\end{aligned}
$$

$$\begin{aligned}
A_3 =& -4a^3b^3 + 4a^5b^3 - 2ab^4 - 6a^5b^4 - 8ab^5 + 12a^3b^5 - 12ab^6\\
&+ 8a^3b^6 - 8ab^7 - 2ab^8 - 6a^4bc - 10a^6bc - 12a^2b^2c + 24a^4b^2c\\
&+ 36a^6b^2c + 20a^2b^3c - 106a^4b^3c + 100a^2b^4c - 72a^4b^4c\\
&+ 92a^2b^5c + 24a^2b^6c - 34a^5c^2 - 54a^7c^2 + 8abc^2 - 124a^3bc^2\\
&+ 210a^5bc^2 - 68ab^2c^2 + 60a^3b^2c^2 + 236a^5b^2c^2 - 128ab^3c^2\\
&- 456a^3b^3c^2 - 20ab^4c^2 - 116a^3b^4c^2 + 32ab^5c^2 - 84a^2c^3\\
&- 794a^4c^3 - 296a^6c^3 + 612a^2bc^3 + 916a^4bc^3 + 360a^2b^2c^3\\
&+ 272a^4b^2c^3 - 240a^2b^3c^3 - 1026ac^4 - 2160a^3c^4 - 258a^5c^4\\
&+ 1080abc^4 + 432a^3bc^4 + 216ab^2c^4 - 648a^2c^5 + 2a^5d - 18a^7d\\
&+ 24a^3bd - 56a^5bd + 16ab^2d + 508a^3b^2d + 162a^5b^2d + 192ab^3d\\
&- 928a^3b^3d - 80a^5b^3d + 112ab^4d + 588a^3b^4d - 288ab^5d + 32a^3b^5d\\
&- 224ab^6d + 16a^2cd + 44a^4cd - 486a^6cd - 16a^2bcd + 2604a^4bcd\\
&+ 360a^6bcd - 480a^2b^2cd - 2968a^4b^2cd + 1568a^2b^3cd - 240a^4b^3cd\\
&+ 1888a^2b^4cd + 336ac^2d + 132a^3c^2d + 852a^5c^2d + 5760abc^2d\\
&+ 4848a^3bc^2d + 432a^5bc^2d - 10080ab^2c^2d - 4560a^3b^2c^2d\\
&- 768ab^3c^2d + 12960a^2c^3d + 2832a^4c^3d + 1728a^2bc^3d + 2592ac^4d\\
&- 32ad^2 + 80a^3d^2 + 1782a^5d^2 - 540a^7d^2 - 640abd^2 - 5568a^3bd^2\\
&+ 3960a^5bd^2 - 10560ab^2d^2 - 9184a^3b^2d^2 + 216a^5b^2d^2\\
&+ 17024ab^3d^2 - 768a^3b^3d^2 - 2240ab^4d^2 + 5952a^2cd^2\\
&- 1248a^4cd^2 - 648a^6cd^2 - 25152a^2bcd^2 + 1728a^4bcd^2\\
&+ 16512a^2b^2cd^2 - 25920ac^2d^2 - 19584a^3c^2d^2 - 13824abc^2d^2\\
&- 7936ad^3 - 6976a^3d^3 + 2592a^5d^3 + 68096abd^3 - 13824a^3bd^3\\
&- 3584ab^2d^3 + 66048a^2cd^3 - 66048ad^4,\\
A_2 =& a^2b^4 - 2a^4b^4 + a^6b^4 + 4a^2b^5 - 4a^4b^5 + 6a^2b^6 - 2a^4b^6\\
&+ 4a^2b^7 + a^2b^8 + 6a^3b^2c - 6a^7b^2c + 2ab^3c - 14a^3b^3c + 36a^5b^3c\\
&+ 8ab^4c - 58a^3b^4c + 18a^5b^4c + 12ab^5c - 50a^3b^5c + 8ab^6c - 12a^3b^6c\\
&+ 2ab^7c + a^4c^2 + 6a^6c^2 + 9a^8c^2 - 2a^2bc^2 + 18a^4bc^2 - 72a^6bc^2 - 2b^2c^2\\
&+ 28a^2b^2c^2 + 63a^4b^2c^2 - 69a^6b^2c^2 - 12a^2b^3c^2 + 292a^4b^3c^2\\
&+ 20b^4c^2 - 118a^2b^4c^2 + 66a^4b^4c^2 + 40b^5c^2 - 78a^2b^5c^2 + 30b^6c^2\\
&- 2a^2b^6c^2 + 8b^7c^2 - 2ac^3 + 62a^3c^3 + 198a^5c^3 + 114a^7c^3 - 12abc^3
\end{aligned}$$

$$
\begin{aligned}
&- 230a^3bc^3 - 664a^5bc^3 - 24ab^2c^3 + 216a^3b^2c^3 - 192a^5b^2c^3 - 104ab^3c^3\\
&+ 492a^3b^3c^3 - 174ab^4c^3 + 20a^3b^4c^3 - 84ab^5c^3 + 54c^4 + 549a^2c^4\\
&+ 1363a^4c^4 + 225a^6c^4 + 108bc^4 - 1368a^2bc^4 - 822a^4bc^4 + 108b^2c^4\\
&- 216a^2b^2c^4 - 66a^4b^2c^4 + 108b^3c^4 + 288a^2b^3c^4 + 54b^4c^4 + 756ac^5\\
&+ 1242a^3c^5 + 72a^5c^5 + 162abc^5 - 324a^3bc^5 - 324ab^2c^5 - 1458c^6\\
&+ 486a^2c^6 - 4a^4bd + 36a^6bd - 10a^2b^2d - 90a^4b^2d - 84a^6b^2d - 160a^2b^3d\\
&+ 360a^4b^3d + 60a^6b^3d - 32b^4d - 36a^2b^4d - 346a^4b^4d - 128b^5d\\
&+ 376a^2b^5d - 48a^4b^5d - 192b^6d + 270a^2b^6d - 128b^7d + 8a^2b^7d - 32b^8d\\
&- 10a^3cd + 6a^5cd + 252a^7cd - 8abcd - 84a^3bcd - 1098a^5bcd - 270a^7bcd\\
&+ 256ab^2cd + 120a^3b^2cd + 1596a^5b^2cd + 768ab^3cd - 1816a^3b^3cd\\
&+ 360a^5b^3cd + 1088ab^4cd - 2142a^3b^4cd + 936ab^5cd - 84a^3b^5cd\\
&+ 352ab^6cd + 8c^2d - 204a^2c^2d + 1494a^4c^2d + 576a^6c^2d - 384bc^2d\\
&- 5160a^2bc^2d - 4416a^4bc^2d - 648a^6bc^2d - 768b^2c^2d + 7128a^2b^2c^2d\\
&+ 4356a^4b^2c^2d - 448b^3c^2d - 648a^2b^3c^2d + 288a^4b^3c^2d - 168b^4c^2d\\
&- 1284a^2b^4c^2d - 96b^5c^2d + 1440ac^3d - 4744a^3c^3d - 1386a^5c^3d\\
&- 3744abc^3d - 564a^3bc^3d - 324a^5bc^3d - 2160ab^2c^3d + 1704a^3b^2c^3d\\
&+ 288ab^3c^3d - 3024c^4d - 11502a^2c^4d - 468a^4c^4d + 16848bc^4d\\
&+ 1296b^2c^4d - 3888ac^5d + 24a^2d^2 - 66a^4d^2 - 423a^6d^2 + 405a^8d^2\\
&+ 192a^2bd^2 - 180a^4bd^2 - 3240a^6bd^2 + 256b^2d^2 + 7584a^2b^2d^2\\
&+ 9948a^4b^2d^2 - 324a^6b^2d^2 - 14592a^2b^3d^2 + 1260a^4b^3d^2\\
&- 1280b^4d^2 + 1464a^2b^4d^2 + 54a^4b^4d^2 - 1536b^5d^2 - 96a^2b^5d^2\\
&- 512b^6d^2 + 384acd^2 - 4608a^3cd^2 - 2844a^5cd^2 + 972a^7cd^2\\
&+ 1344abcd^2 + 13632a^3bcd^2 - 2430a^5bcd^2 + 14976ab^2cd^2\\
&- 16560a^3b^2cd^2 - 324a^5b^2cd^2 + 12384ab^3cd^2 + 288a^3b^3cd^2\\
&+ 4992ab^4cd^2 - 2880c^2d^2 - 4800a^2c^2d^2 + 6210a^4c^2d^2 + 486a^6c^2d^2\\
&+ 19584bc^2d^2 + 50688a^2bc^2d^2 - 63072b^2c^2d^2 - 11520a^2b^2c^2d^2\\
&- 5760b^3c^2d^2 + 12960ac^3d^2 + 3744a^3c^3d^2 + 15552abc^3d^2 + 7776c^4d^2\\
&- 512d^3 + 5184a^2d^3 + 7824a^4d^3 - 5346a^6d^3 + 2048bd^3 - 27776a^2bd^3\\
&+ 37584a^4bd^3 - 46080b^2d^3 - 71520a^2b^2d^3 + 1296a^4b^2d^3 + 67584b^3d^3\\
&- 5760a^2b^3d^3 + 1024b^4d^3 + 45056acd^3 - 44640a^3cd^3 - 3888a^5cd^3
\end{aligned}
$$

$$
\begin{aligned}
&- 7296abcd^3 + 15552a^3bcd^3 + 16896ab^2cd^3 - 17280c^2d^3 - 30528a^2c^2d^3\\
&- 41472bc^2d^3 - 45056d^4 + 40320a^2d^4 + 7776a^4d^4 + 24576bd^4\\
&- 41472a^2bd^4 + 24576b^2d^4 + 92160acd^4 - 73728d^5,\\
A_1 = &- 2a^2b^3c + 4a^4b^3c - 2a^6b^3c - 8a^2b^4c + 8a^4b^4c - 12a^2b^5c\\
&+ 4a^4b^5c - 8a^2b^6c - 2a^2b^7c - 2a^3bc^2 - 4a^5bc^2 + 6a^7bc^2 + 2ab^2c^2\\
&+ 6a^3b^2c^2 - 30a^5b^2c^2 + 6a^7b^2c^2 + 76a^3b^3c^2 - 68a^5b^3c^2\\
&- 20ab^4c^2 + 128a^3b^4c^2 - 8a^5b^4c^2 - 40ab^5c^2 + 62a^3b^5c^2 - 30ab^6c^2\\
&+ 2a^3b^6c^2 - 8ab^7c^2 + 2a^2c^3 - 20a^4c^3 + 10a^6c^3 - 24a^8c^3 + 12a^2bc^3\\
&- 76a^4bc^3 + 224a^6bc^3 + 24a^2b^2c^3 - 396a^4b^2c^3 + 56a^6b^2c^3\\
&+ 104a^2b^3c^3 - 372a^4b^3c^3 + 174a^2b^4c^3 - 20a^4b^4c^3 + 84a^2b^5c^3\\
&- 54ac^4 - 36a^3c^4 - 310a^5c^4 - 96a^7c^4 - 108abc^4 + 828a^3bc^4 + 606a^5bc^4\\
&- 108ab^2c^4 + 108a^3b^2c^4 + 66a^5b^2c^4 - 108ab^3c^4 - 288a^3b^3c^4\\
&- 54ab^4c^4 - 756a^2c^5 - 918a^4c^5 - 72a^6c^5 - 162a^2bc^5 + 324a^4bc^5\\
&+ 324a^2b^2c^5 + 1458ac^6 - 486a^3c^6 + 2a^3b^2d - 20a^5b^2d + 18a^7b^2d\\
&+ 64a^3b^3d - 72a^5b^3d - 24a^7b^3d + 32ab^4d - 20a^3b^4d + 140a^5b^4d + 128ab^5d\\
&- 232a^3b^5d + 32a^5b^5d + 192ab^6d - 158a^3b^6d + 128ab^7d - 8a^3b^7d + 32ab^8d\\
&+ 2a^4cd - 12a^6cd - 54a^8cd + 8a^2bcd + 92a^4bcd + 240a^6bcd + 108a^8bcd\\
&- 256a^2b^2cd + 120a^4b^2cd - 624a^6b^2cd - 768a^2b^3cd + 1032a^4b^3cd\\
&- 240a^6b^3cd - 1088a^2b^4cd + 1198a^4b^4cd - 936a^2b^5cd + 84a^4b^5cd\\
&- 352a^2b^6cd - 8ac^2d + 36a^3c^2d - 804a^5c^2d - 360a^7c^2d + 384abc^2d\\
&+ 2280a^3bc^2d + 2136a^5bc^2d + 432a^7bc^2d + 768ab^2c^2d - 2088a^3b^2c^2d\\
&- 2076a^5b^2c^2d + 448ab^3c^2d + 1032a^3b^3c^2d - 288a^5b^3c^2d + 168ab^4c^2d\\
&+ 1284a^3b^4c^2d + 96ab^5c^2d - 1440a^2c^3d - 1736a^4c^3d - 30a^6c^3d\\
&+ 3744a^2bc^3d - 300a^4bc^3d + 324a^6bc^3d + 2160a^2b^2c^3d - 1704a^4b^2c^3d\\
&- 288a^2b^3c^3d + 3024ac^4d + 10206a^3c^4d + 468a^5c^4d - 16848abc^4d\\
&- 1296ab^2c^4d + 3888a^2c^5d - 8a^3d^2 - 6a^5d^2 - 162a^9d^2 + 128a^3bd^2\\
&+ 564a^5bd^2 + 1404a^7bd^2 - 256ab^2d^2 - 2304a^3b^2d^2 - 4620a^5b^2d^2\\
&+ 216a^7b^2d^2 + 6080a^3b^3d^2 - 876a^5b^3d^2 + 1280ab^4d^2 - 344a^3b^4d^2\\
&- 54a^5b^4d^2 + 1536ab^5d^2 + 96a^3b^5d^2 + 512ab^6d^2 - 384a^2cd^2\\
&+ 1632a^4cd^2 + 684a^6cd^2 - 648a^8cd^2 - 1344a^2bcd^2 - 1056a^4bcd^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 1566a^6bcd^2 - 14976a^2b^2cd^2 + 8304a^4b^2cd^2 + 324a^6b^2cd^2\\
&- 12384a^2b^3cd^2 - 288a^4b^3cd^2 - 4992a^2b^4cd^2 + 2880ac^2d^2\\
&+ 17760a^3c^2d^2 + 3582a^5c^2d^2 - 486a^7c^2d^2 - 19584abc^2d^2\\
&- 43776a^3bc^2d^2 + 63072ab^2c^2d^2 + 11520a^3b^2c^2d^2 + 5760ab^3c^2d^2\\
&- 12960a^2c^3d^2 - 3744a^4c^3d^2 - 15552a^2bc^3d^2 - 7776ac^4d^2 + 512ad^3\\
&- 1216a^3d^3 - 624a^5d^3 + 4050a^7d^3 - 2048abd^3 - 6272a^3bd^3 - 30672a^5bd^3\\
&+ 46080ab^2d^3 + 73312a^3b^2d^3 - 1296a^5b^2d^3 - 67584ab^3d^3 + 5760a^3b^3d^3\\
&- 1024ab^4d^3 - 45056a^2cd^3 + 11616a^4cd^3 + 3888a^6cd^3 + 7296a^2bcd^3\\
&- 15552a^4bcd^3 - 16896a^2b^2cd^3 + 17280ac^2d^3 + 30528a^3c^2d^3\\
&+ 41472abc^2d^3 + 45056ad^4 - 7296a^3d^4 - 7776a^5d^4 - 24576abd^4\\
&+ 41472a^3bd^4 - 24576ab^2d^4 - 92160a^2cd^4 + 73728ad^5,\\
A_0 =&a^2b^2c^2 - 3a^4b^2c^2 + 3a^6b^2c^2 - a^8b^2c^2 + 8a^2b^3c^2\\
&- 16a^4b^3c^2 + 8a^6b^3c^2 + 22a^2b^4c^2 - 24a^4b^4c^2 + 2a^6b^4c^2\\
&+ 28a^2b^5c^2 - 12a^4b^5c^2 + 17a^2b^6c^2 - a^4b^6c^2 + 4a^2b^7c^2\\
&+ 4a^5c^3 - 8a^7c^3 + 4a^9c^3 - 2abc^3 - 6a^3bc^3 + 42a^5bc^3 - 34a^7bc^3\\
&- 16ab^2c^3 - 46a^3b^2c^3 + 108a^5b^2c^3 - 14a^7b^2c^3 - 44ab^3c^3\\
&- 114a^3b^3c^3 + 86a^5b^3c^3 - 56ab^4c^3 - 114a^3b^4c^3 + 10a^5b^4c^3\\
&- 34ab^5c^3 - 40a^3b^5c^3 - 8ab^6c^3 + c^4 + 17a^2c^4 - 29a^4c^4 + 3a^6c^4\\
&+ 24a^8c^4 + 12bc^4 + 108a^2bc^4 - 92a^4bc^4 - 148a^6bc^4 + 54b^2c^4\\
&+ 250a^2b^2c^4 + 129a^4b^2c^4 - 33a^6b^2c^4 + 116b^3c^4 + 236a^2b^3c^4\\
&+ 130a^4b^3c^4 + 129b^4c^4 + 69a^2b^4c^4 + a^4b^4c^4 + 72b^5c^4 - 8a^2b^5c^4\\
&+ 16b^6c^4 - 54ac^5 + 116a^3c^5 + 122a^5c^5 + 36a^7c^5 - 414abc^5 - 204a^3bc^5\\
&- 138a^5bc^5 - 810ab^2c^5 - 144a^3b^2c^5 - 8a^5b^2c^5 - 594ab^3c^5 + 68a^3b^3c^5\\
&- 144ab^4c^5 + 54c^6 + 459a^2c^6 + 27a^4c^6 + 16a^6c^6 + 324bc^6 + 1134a^2bc^6\\
&- 144a^4bc^6 + 486b^2c^6 + 270a^2b^2c^6 + 216b^3c^6 - 1458ac^7 + 216a^3c^7\\
&- 972abc^7 + 729c^8 + 4a^4b^3d - 8a^6b^3d + 4a^8b^3d - 16a^2b^4d + 48a^4b^4d\\
&- 32a^6b^4d - 64a^2b^5d + 88a^4b^5d - 8a^6b^5d - 96a^2b^6d + 48a^4b^6d\\
&- 64a^2b^7d + 4a^4b^7d - 16a^2b^8d - 2a^3bcd + 2a^5bcd + 18a^7bcd - 18a^9bcd\\
&- 120a^5b^2cd + 152a^7b^2cd + 32ab^3cd + 116a^3b^3cd - 416a^5b^3cd + 60a^7b^3cd\\
&+ 128ab^4cd + 392a^3b^4cd - 368a^5b^4cd + 192ab^5cd + 446a^3b^5cd - 42a^5b^5cd
\end{aligned}
$$

$$
\begin{aligned}
&+ 128ab^6cd + 168a^3b^6cd + 32ab^7cd + 2a^2c^2d - 22a^4c^2d + 78a^6c^2d \\
&+ 6a^8c^2d + 8a^2bc^2d - 180a^4bc^2d - 168a^6bc^2d - 108a^8bc^2d - 32b^2c^2d \\
&+ 36a^2b^2c^2d + 394a^4b^2c^2d + 690a^6b^2c^2d - 256b^3c^2d - 168a^2b^3c^2d \\
&- 640a^4b^3c^2d + 144a^6b^3c^2d - 704b^4c^2d - 358a^2b^4c^2d \\
&- 600a^4b^4c^2d - 896b^5c^2d - 96a^2b^5c^2d - 8a^4b^5c^2d - 544b^6c^2d \\
&+ 64a^2b^6c^2d - 128b^7c^2d - 64ac^3d + 212a^3c^3d + 824a^5c^3d - 132a^7c^3d \\
&- 432abc^3d - 2844a^3bc^3d + 346a^5bc^3d - 162a^7bc^3d + 1312ab^2c^3d \\
&- 2364a^3b^2c^3d + 870a^5b^2c^3d + 4880ab^3c^3d - 892a^3b^3c^3d \\
&+ 68a^5b^3c^3d + 4416ab^4c^3d - 576a^3b^4c^3d + 1216ab^5c^3d + 216c^4d \\
&+ 1362a^2c^4d + 8a^4c^4d - 450a^6c^4d + 1008bc^4d + 2124a^2bc^4d + 2652a^4bc^4d \\
&- 144a^6bc^4d - 648b^2c^4d - 7182a^2b^2c^4d + 1276a^4b^2c^4d - 3456b^3c^4d \\
&- 2328a^2b^3c^4d - 2016b^4c^4d - 7560ac^5d - 2862a^3c^5d + 48a^5c^5d \\
&+ 8424abc^5d - 2340a^3bc^5d + 9504ab^2c^5d + 5832c^6d + 324a^2c^6d - 7776bc^6d \\
&+ a^4d^2 + 9a^6d^2 + 27a^8d^2 + 27a^10d^2 - 68a^4bd^2 - 288a^6bd^2 - 252a^8bd^2 \\
&+ 96a^2b^2d^2 + 918a^4b^2d^2 + 704a^6b^2d^2 - 54a^8b^2d^2 - 768a^2b^3d^2 \\
&- 12a^4b^3d^2 + 228a^6b^3d^2 + 256b^4d^2 - 2368a^2b^4d^2 + 417a^4b^4d^2 \\
&+ 27a^6b^4d^2 + 1024b^5d^2 - 2176a^2b^5d^2 + 24a^4b^5d^2 + 1536b^6d^2 \\
&- 800a^2b^6d^2 + 16a^4b^6d^2 + 1024b^7d^2 - 128a^2b^7d^2 + 256b^8d^2 + 128a^3cd^2 \\
&+ 138a^5cd^2 + 516a^7cd^2 + 162a^9cd^2 - 128abcd^2 - 1584a^3bcd^2 - 3774a^5bcd^2 \\
&- 378a^7bcd^2 + 1536ab^2cd^2 + 8224a^3b^2cd^2 - 4362a^5b^2cd^2 - 162a^7b^2cd^2 \\
&+ 512ab^3cd^2 + 13328a^3b^3cd^2 - 450a^5b^3cd^2 - 6656ab^4cd^2 + 6912a^3b^4cd^2 \\
&- 144a^5b^4cd^2 - 8064ab^5cd^2 + 1216a^3b^5cd^2 - 2560ab^6cd^2 + 128c^2d^2 \\
&+ 816a^2c^2d^2 - 2238a^4c^2d^2 + 1359a^6c^2d^2 + 243a^8c^2d^2 + 3936a^2bc^2d^2 \\
&+ 3900a^4bc^2d^2 + 1134a^6bc^2d^2 - 6656b^2c^2d^2 - 18432a^2b^2c^2d^2 \\
&- 13590a^4b^2c^2d^2 + 270a^6b^2c^2d^2 - 7168b^3c^2d^2 + 7872a^2b^3c^2d^2 \\
&- 2328a^4b^3c^2d^2 + 4992b^4c^2d^2 + 3648a^2b^4c^2d^2 + 5632b^5c^2d^2 \\
&- 9984ac^3d^2 - 13456a^3c^3d^2 + 954a^5c^3d^2 + 216a^7c^3d^2 + 48096abc^3d^2 \\
&+ 14640a^3bc^3d^2 - 2340a^5bc^3d^2 + 3456ab^2c^3d^2 + 11968a^3b^2c^3d^2 \\
&- 26112ab^3c^3d^2 + 15120c^4d^2 + 648a^2c^4d^2 + 3030a^4c^4d^2 - 51840bc^4d^2 \\
&- 16416a^2bc^4d^2 + 27648b^2c^4d^2 + 10368ac^5d^2 - 128a^2d^3 - 424a^4d^3
\end{aligned}
$$

$$
\begin{aligned}
&- 1498a^6d^3 - 1134a^8d^3 + 3072a^2bd^3 + 12400a^4bd^3 + 9396a^6bd^3 \\
&- 2048b^2d^3 - 29696a^2b^2d^3 - 21640a^4b^2d^3 + 1134a^6b^2d^3 + 8192b^3d^3 \\
&+ 6144a^2b^3d^3 - 6336a^4b^3d^3 + 216a^6b^3d^3 + 18432b^4d^3 + 5504a^2b^4d^3 \\
&- 2016a^4b^4d^3 + 4096b^5d^3 + 5632a^2b^5d^3 - 4096b^6d^3 - 4096acd^3 \\
&- 3840a^3cd^3 - 17160a^5cd^3 - 3402a^7cd^3 + 35840abcd^3 + 72160a^3bcd^3 \\
&+ 16200a^5bcd^3 - 972a^7bcd^3 - 67584ab^2cd^3 + 11904a^3b^2cd^3 \\
&+ 9504a^5b^2cd^3 - 51712ab^3cd^3 - 26112a^3b^3cd^3 + 14336ab^4cd^3 \\
&+ 13312c^2d^3 + 14368a^2c^2d^3 - 18504a^4c^2d^3 + 324a^6c^2d^3 - 104448bc^2d^3 \\
&- 56832a^2bc^2d^3 - 16416a^4bc^2d^3 + 133632b^2c^2d^3 + 74240a^2b^2c^2d^3 \\
&- 38912b^3c^2d^3 + 55296ac^3d^3 + 256a^3c^3d^3 - 46080abc^3d^3 - 13824c^4d^3 \\
&+ 4096d^4 + 7168a^2d^4 + 25872a^4d^4 + 9720a^6d^4 + 729a^8d^4 - 49152bd^4 \\
&- 133120a^2bd^4 - 72576a^4bd^4 - 7776a^6bd^4 + 172032b^2d^4 + 145920a^2b^2d^4 \\
&+ 27648a^4b^2d^4 - 81920b^3d^4 - 38912a^2b^3d^4 + 24576b^4d^4 + 40960acd^4 \\
&+ 101376a^3cd^4 + 10368a^5cd^4 - 190464abcd^4 - 46080a^3bcd^4 + 8192ab^2cd^4 \\
&- 55296c^2d^4 + 33792a^2c^2d^4 + 73728bc^2d^4 - 32768d^5 - 92160a^2d^5 \\
&- 13824a^4d^5 + 196608bd^5 + 73728a^2bd^5 - 65536b^2d^5 \\
&- 98304acd^5 + 65536d^6.
\end{aligned}
$$

Here are the degree 12 resolvents of the five sample polynomials $t_1, \dots, t_5$. Note that we need to take a tschirnhaus transformation of t_2 and t_3, since their degree 12 resolvents are not squarefree. In all cases, we give the transformed polynomial (if required), the degree 12 resolvent, and the factorization of the degree 12 polynomial over the rationals. For t_5, the resolvent is irreducible, so we only give it.

t_1

degree 12 res: $x^{12} + 6x^{11} - 21x^{10} - 160x^9 105x^8 + 1446x^7 + 211x^6 - 5166x^5 + 1120x^4 + 12890x^3 - 32546x^2 - 40146x 34441$

factorization: $(x^2 + x - 11)(x^2 + x - 1)(x^4 - 8x^3 + 24x^2 - 37x + 31)(x^4 + 12x^3 + 54x^2 + 113x + 101)$

t_2

tschirnhaus: $x^4 - 12x^3 + 62x^2 - 84x + 34$

degree 12 res: $x^{12} + 72x^{11} - 2504x^{10} - 245280x^9 + 1875080x^8 + 318348672x^7 + 87377344x^6 - 181028775168x^5 - 749536378480x^4 + 34794601092480x^3 + 336921864635776x^2 + 718430161516032x - 194650798165760$

factorization: $(x^2 - 60x + 916)(x^2 - 56x + 802)(x^2 + 8x - 2)(x^2 + 16x + 46)(x^2 + 80x + 1618)(x^2 + 84x + 1780)$

t_3

tschirnhaus: $x^4 - 16x^3 + 160x^2 - 256x + 768$

degree 12 res: $x^{12}+96x^{11}-85632x^{10}-7075840x^9+2777833472x^8+188719038464x^7 - 43620999626752x^6 - 2264865590738944x^5 +36424519215008 9728x^4 + 12636857700441915392x^3 -157395883859115 1931392x^2 - 2682613447679739 2191488x +278509645765100606 9538816$

factorization: $(x^4-480x^3+82304x^2-5920768x+151586816)(x^4 + 32x^3 - 15744x^2 - 256000x + 68194304)(x^4 + 544x^3 + 106880x^2 + 8939520x + 269420544)$

t_4

degree 12 res: $x^{12} + 12x^{11} - 44x^{10} - 880x^9 + 8x^8 + 23296x^7 + 18320x^6-283808x^5-101744x^4+2049472x^3-2031360x^2-9244800x+ 7496000$

factorization: $(x^6+6x^5-40x^4-312x^3+68x^2+4248x+7496)(x^6 + 6x^5 - 40x^4 - 88x^3 + 740x^2 - 1800x + 1000)$

t_5

degree 12 res: $x^{12} + 6x^{11} - 33x^{10} - 220x^9 + 421x^8 + 3070x^7 - 3149x^6 -21182x^5 +18868x^4 +77094x^3 -63598x^2 -106258x -28751$

4. Algorithms

In this section, we describe the algorithms for computing quartic Galois groups as described in [1, 3, 4, 7]. We also describe the simple method in [7] for computing the Galois group of a biquadratic polynomial; i.e., a polynomial of the form $x^4 + ax^2 + b$.

Table 2: The top row contains the transitive subgroups G of S_4, as listed in Table 1. The left column contains the subgroups H referenced in Section 3. Note the group C_2 is generated by (12)(34). For a particular pair (H, G), the entry in the table gives the lengths of the orbits of G acting on the cosets S_4/H. In particular, this list is equivalent to the list of irreducible factors of the resolvent polynomial $R_H(x)$ of a quartic polynomial $f(x)$ where G is the Galois group of f.

H/G	C_4	V_4	D_4	A_4	S_4
C_2	2,2,4,4	2,2,2,2,2,2	4,4,4	6,6	12
C_4	1,1,4	2,2,2	2,4	6	6
D_4	1,2	1,1,1	1,2	3	3
A_4	2	1,1	2	1,1	2

Resolvent Factorizations

As mentioned in the opening paragraph of Section 3, if G is the Galois group of the quartic polynomial f, H is a subgroup of S_4, and $R_H(x)$ is the resolvent polynomial of f corresponding to H, then the Galois group of $R_H(x)$ is isomorphic to the image of the permutation representation of G acting on the cosets S_4/H. The irreducible factors of $R_H(x)$ correspond to the orbits of this action, and therefore the degrees of the irreducible factors correspond to the orbit lengths. Table 2 shows the orbit lengths of the four resolvent polynomials mentioned in Section 3. In particular, the discriminant corresponds to A_4 row, the cubic resolvent corresponds to the D_4 row, the sextic resolvent corresponds to the C_4 row, and the degree 12 resolvent corresponds to the C_2 row.

Cubic Resolvent Algorithm

The following algorithm is found in [4] and is based on the cubic resolvent.

Algorithm 4.1. Given an irreducible quartic polynomial defined over a field F, this algorithm outputs the name of the Galois group of $f(x)$ from among those listed in Table 1.

1. Form the cubic resolvent $g(x)$, as described in Section 3. Factor $g(x)$ over F. Let L be the list of the degrees of the irreducible factors.

2. If $L = \{1, 1, 1\}$, return V_4.

3. Let d be the discriminant of $f(x)$.

4. If $L = \{3\}$, return A_4 if a d is a perfect square in F and S_4 otherwise.

5. Factor $f(x)$ over $K = F(\sqrt{d})$. Return C_4 if $f(x)$ is reducible over K and D_4 otherwise.

Proof. Steps (2) and (4) follow immediately from Table 2. To prove Step (5), we note that $f(x)$ will be reducible over $K = F(\sqrt{d})$ if and only if K defines a quadratic subfield of f's stem field. By the Galois correspondence, quadratic subfields of f's stem field correspond to subgroups H of G of index 2 that contain G_1, the point stabilizer of 1. On the other hand, the subgroup that corresponds to $F(\sqrt{d})$ is $A_4 \cap G$. For the group C_4, these two subgroups are the same; namely, $\{(1), (13)(24)\}$. For D_4, these two groups are different: $A_4 \cap D_4 = V_4$ (the transitive V_4), while the subgroup of D_4 of index 2 containing the point stabilizer of 1 in D_4 is an intransitive $V_4 = \{(1), (13), (24), (13)(24)\}$. This proves that if $G \in \{C_4, D_4\}$, then $f(x)$ is reducible over K if and only if $G = C_4$. □

We note that [7] contains a modification of Algorithm 4.1. The need to factor $f(x)$ over $F(\sqrt{d})$ is replaced by factoring two additional quadratic polynomials over F; these two quadratic polynomials make use of the unique simple root of the cubic resolvent. See their paper for complete details.

Sextic Resolvent Algorithm

The following algorithm is found in [3] and is based on the sextic resolvent.

Algorithm 4.2. Given an irreducible quartic polynomial defined over a field F, this algorithm outputs the name of the Galois group of $f(x)$ from among those listed in Table 1.

1. Form the degree 6 resolvent $g(x)$, as described in Section 3. Factor $g(x)$ over F.

2. If g is squarefree (that is, if $\gcd(g(x), g'(x)) = 1$), let L be the list of the degrees of the irreducible factors. Otherwise, replace f by a Tschirnhaus transformation and repeat step 1.

3. Return C_4, V_4, or D_4 if $L = \{1, 1, 4\}$, $\{2, 2, 2\}$, or $\{2, 4\}$, respectively.

4. Let d be the discriminant of $f(x)$.

5. Return A_4 if d is a perfect square in F and S_4 otherwise.

Proof. Follows immediately from Table 2. □

Degree 12 Resolvent Algorithm

The following algorithm is found in [1] and is based on the degree 12 resolvent.

Algorithm 4.3. Given an irreducible quartic polynomial defined over a field F, this algorithm outputs the name of the Galois group of $f(x)$ from among those listed in Table 1.

1. Form the degree 12 resolvent $g(x)$, as described in Section 3. Factor $g(x)$ over F.
2. If g is squarefree (that is, if $\gcd(g(x), g'(x)) = 1$), let L be the list of the degrees of the irreducible factors. Otherwise, replace f by a Tschirnhaus transformation and repeat step 1.
3. Return C_4, V_4, D_4, A_4, or S_4 if $L = \{2, 2, 4, 4\}$, $\{2, 2, 2, 2, 2, 2\}$, $\{4, 4, 4\}$, $\{6, 6\}$, or $\{12\}$, respectively.

Proof. Follows immediately from Table 2. □

Biquadratic Algorithm

The following algorithm is found in [7].

Algorithm 4.4. Given an irreducible biquadratic polynomial $f(x) = x^4 + ax^2 + b$ defined over a field F, this algorithm outputs the name of the Galois group of $f(x)$ from among those listed in Table 1.

1. Let $d = 16a^4b - 128a^2b^2 + 256b^3$.
2. Return V_4 if d is a perfect square in F.
3. Let $e = a^2 - 4b$
4. Return C_4 if de is a perfect square in F and D_4 otherwise.

Proof. Let G be the Galois group of f and K the stem field of f. Clearly, K has a quadratic subfield, defined by the polynomial $g(x) = x^2 + ax + b$. As noted in the proof of Algorithm 4.1, the presence of a quadratic subfield is equivalent to the existence of a subgroup H of G of index 2 that contains G_1, the point stabilizer of 1. Direct computation shows the groups C_4, V_4, and D_4 have such a subgroup H, while A_4 and S_4 do not. Therefore $G \in \{C_4, V_4, D_4\}$.

Now d equals the discriminant of $f(x)$ and e is the discriminant of $g(x)$. According to Table 2, d is a perfect square if and only if $G = V_4$. This proves Step (2).

Since K is defined by g, $K = F(\sqrt{e})$. Note, $F(\sqrt{d})$ also defines a quadratic extension of F. As shown in the proof of Algorithm 4.1, $F(\sqrt{d}) = K$ if and only if $G = C_4$. But $F(\sqrt{d}) = F(\sqrt{e})$ if and only if de is a perfect square. This proves Step (4). □

5. New Method

In this section, we outline our non-resolvent approach to computing quartic Galois groups. We also show our approach gives rise to an application to biquadratic quartics.

Algorithm 5.1. Given an irreducible quartic polynomial $f(x)$ defined over a field F, this algorithm outputs the name of the Galois group of $f(x)$ from among those listed in Table 1.

1. Let K be the stem field f. Factor f over K and let r denote the number of linear factors.
2. Return D_4 if $r = 2$.
3. Let d be the discriminant of $f(x)$.
4. If $r = 4$, return V_4 if d is a perfect square in F and C_4 otherwise.
5. If $r = 1$, return A_4 if d is a perfect square in F and S_4 otherwise.

Proof. Let G be the Galois group of f, K the stem field of f, G_1 the point stabilizer of 1 in G, and N the normalizer of G_1 in G. By Corollary 2.6, $r = [N : G_1]$. Direct computation on the groups $\{C_4, V_4, D_4, A_4, S_4\}$ shows the corresponding values of $[N : G_1]$ are $\{4, 4, 2, 1, 1\}$, respectively. The algorithm now follows by combining this information with row A_4 in Table 2. □

Biquadratic Polynomials

The proof of Algorithm 4.4 shows if $f(x)$ is biquadratic, then the Galois group of f is either C_4, V_4, or D_4. In this section, we show the converse is true. That is, we show if K is the stem of $f(x)$ where the Galois group of f is C_4, V_4, or D_4, then there exists a biquadratic polynomial $g(x)$ that defines K. Our proof is constructive.

Algorithm 5.2. Given an irreducible quartic polynomial $f(x)$ defined over a field F whose Galois group is either C_4, V_4, or D_4, this algorithm outputs a biquadratic polynomial defining the same extension as f.

1. Let K be the stem field f generated by a root r of f. Factor f over K and let L denote the roots of linear factors; these will necessarily be polynomials in the variable r of degree ≤ 3 (they will be the automorphisms of K/F).

2. Pick $g(r) \in L$ such that $g(r) \neq r$ and $g(g(r)) \equiv r \pmod{f(r)}$. So $g(r)$ is an automorphism of order 2.

3. Let $h(x)$ be the characteristic polynomial of $r - g(r)$. That is, $h(x) = \text{Resultant}_r(x - (r - g(r)), f(r))$.

4. Return $h(x)$ if it is squarefree. Otherwise, replace f by a Tschirnhaus transformation and repeat steps 1–4.

Proof. Let G be the Galois group of $f(x)$, K the stem field of f, and $\text{Aut}(K)$ the automorphism group of K/F. Corollary 2.6 shows $\text{Aut}(K)$ is isomorphic to N/G_1 where G_1 is the point stabilizer of 1 in G and N the normalizer of G_1 in G. The proof of Algorithm 4.4 shows that since $G \in \{C_4, V_4, D_4\}$, K contains a quadratic subfield. Thus $\text{Aut}(K)$ contains a subgroup of order 2, and hence an element of order 2. This proves Step (2) is possible.

Direct computation shows that if $G = V_4$, there are 3 such choices for $g(r)$. As permutations, these automorphisms are given by: $(12)(34), (13)(24), (14)(23)$. Otherwise, there is a unique choice for $g(r)$. If $G = C_4$ or D_4, $g(r)$ is the permutation $(13)(24)$. Note that these permutations depend on the ordering of the roots of f. Different orderings will correspond to conjugate permutations. But the cycle type is invariant under conjugation. So in any case, the permutation $g(r)$ leaves no root fixed.

Let $h(x) = \text{Resultant}_r(x-(r-g(r)), f(r))$. Thus $h(x) \in F[x]$, and therefore the irreducible factors of h define subfields of K. Suppose the roots of f are a, b, c, d. Suppose further that $g(r)$ permutes the roots in the following way: $(ab)(cd)$. Then

$$\begin{aligned} h(x) &= (x - (a - b))(x - (b - a))(x - (c - d))(x - (d - c)) \\ &= (x^2 - (a - b)^2)(x^2 - (c - d)^2) \\ &= x^4 - [(a - b)^2 + (c - d)^2]x^2 + [(a - b)(c - d)]^2. \end{aligned}$$

Thus $h(x)$ is biquadratic. If h is irreducible, then since it defines a subfield of K, it must be the case that the stem field of h is exactly K. □

5.1. Example

For example, let $f(x) = x^4 - x^3 + x^2 - x + 1$. Let $K/\mathbb{Q}$ be the stem field of f. Then factoring f over K shows that K contains all four roots of f. The four automorphisms of K are:

$$r, -r^2, -r^3 + r^2 - r + 1, r^3$$

Direct computation on the 3 nontrivial automorphisms shows that when the 2nd and 4th automorphisms are composed with themselves, they yield the 3rd automorphism. When the 3rd automorphism is composed with itself, it yields r. Note, this proves the Galois group of f is C_4. Let $g(r) = -r^3 + r^2 - r + 1$. Forming the characteristic polynomial $h(x)$ of $r - g(r)$, we obtain $h(x) = x^4 + 5x^2 + 5$, an irreducible biquadratic polynomial.

6. Comparison

In this section, we give an analysis of run times of the various algorithms. In an effort to conduct a timing analysis, we implemented 6 quartic Galois group algorithms into Wolfram Mathematica and used the built-in `Timing` function to calculate run times. The algorithms are the resolvent methods described in Section 3 as well as two non-resolvent methods. We label the resolvent methods as Cubic ([4]), Sextic ([3]), Deg 12 ([1]), and [7]. We label the non-resolvent methods as Algorithm 5.1 and SF. The SF algorithm computes a polynomial defining the splitting field of the original quartic. The splitting field algorithm proceeds by factoring the original quartic over its stem field K. If f does not factor into linear factors, K does not contain all four roots. Thus we adjoin to K a root of a non-linear factor and factor f over this new extension. We continue in this fashion until we obtain an extension which contains all four roots of f.

The sample polynomials for each Galois group are shown below:

- $C4 : x^4 - x^3 + x^2 - x + 1$
- $V4 : x^4 + 1$
- $D4 : x^4 + 2$
- $A4 : x^4 - 2x^3 + 2x^2 - x + 2$
- $S4 : x^4 - x^3 + 1$

Using these polynomials, we applied between one and ten Tschirnhaus transformations in order to test the speed of each algorithm at varying sizes of polynomial coefficients. By doing so, we can test whether certain algorithms are relatively faster or slower for different polynomial sizes. The table below shows the average coefficient size (given as the number of digits) of each polynomial, ranging from one Tschirnhaus transformation to ten Tschirnhaus transformations. Notice that each application of a Tschirnhaus transformation roughly doubles the average number of digits in the polynomial's coefficients.

	V4	**D**4	**C**4	**A**4	**S**4
T1	2	2	2	2	2
T2	4	4	4	4	4
T3	7	8	8	10	9
T4	16	18	17	19	17
T5	34	35	33	37	36
T6	65	67	65	77	72
T7	129	142	137	148	133
T8	269	274	268	308	275
T9	504	575	521	607	551
T10	1021	1047	1016	1173	1145

6.1. Timings

We present the ratios of timings for all methods except [7] (because it is the fastest). For the algorithm in [7], we list its actual run time on our machine (Macbook Pro, 3 GHz, Intel Core i7). For the other methods, their actual run times are divided by the timing of the fastest method. By doing so, we can control for the hardware constraints of the machine on which the algorithms are run. The absolute timings of each algorithm may change from machine to machine, but the ratios should not. Additionally, analyzing ratios allow us to easily interpret the speed of each algorithm by explicitly showing how many times slower each method is than the fastest method. We present the timings of each method below.

Timings when the Galois group is V_4

	[7]	SF	Alg. 5.1	Sextic	Deg 12
T1	0.004	22.13	21.58	361.04	21.79
T2	0.005	14.54	15.44	187.73	13.12
T3	0.0056	9.59	10.51	92.28	7.59
T4	0.0067	9.01	9.93	59.74	5.67
T5	0.0079	10.3	10.78	44.36	4.88
T6	0.0097	13.6	13.09	33.02	4.47
T7	0.0133	19.94	20.72	24.49	4.51
T8	0.0177	39.32	36.77	17.76	5.46
T9	0.0279	84.76	89.18	13.91	8.6
T10	0.0494	239.67	234.11	12.4	16.23

For polynomials with V_4 as Galois group, the method in [7] and the cubic resolvent method are equivalent. They appear to be the fastest, regardless of the size of the polynomial. These methods only take 0.004 seconds for a polynomial with 1 Tschirnhaus transformation and under 0.05 seconds for a polynomial with ten Tschirnhaus transformations. Both the ratios in the timings of the SF and Alg. 5.1 methods become much larger as we increase the number of Tschirnhaus transformations. Interestingly, the sextic resolvent methods get relatively faster compared to [7], but seems to level off compared to [7] for a higher number of Tschirnhaus transformations. Lastly, the degree 12 method appears to be inconsistent, as its timing ratios decrease until six Tschirnhaus transformations and then increase until ten Tschirnhaus transformations. Seeing that the smallest ratio of timings occurs with the degree 12 resolvent method at 6 Tschirnhaus transformations, the method in [7] is at least 4.47 times faster than any other method for polynomials with Galois group V_4.

Timings when the Galois group is D_4

	[7]	SF	Alg. 5.1	Cubic	Sextic	Deg 12
T1	0.0044	15.44	16.57	17.69	308.94	20.4
T2	0.0048	10.56	11.43	10.38	163.39	11.32
T3	0.0059	9.18	9.92	7.51	102.8	8.36
T4	0.0069	9.25	10.26	7	75.05	7.2
T5	0.0083	9.8	11.2	6.16	55.59	6.59
T6	0.0101	11.87	13.33	6.23	42.04	6.36
T7	0.0138	15.31	15.99	6.62	30.83	6.73
T8	0.0204	20.2	21.53	8	23.98	9.29
T9	0.0348	26.04	26.41	9.27	18.9	13.78
T10	0.0694	37.82	37.72	14.55	28.6	33.67

For polynomials with D_4 as Galois group, we can see that the method in [7] is again the quickest regardless of the size of polynomial coefficients. For a polynomial with one Tschirnhaus transformation, this method takes 0.004 seconds to compute the polynomial's Galois group, and for polynomials with ten Tschirnhaus transformations, it takes just under 0.7 seconds. Every other method's timing ratios decrease in size before increasing in size at various Tschirnhaus levels. Since the minimum timing ratio is the cubic method at five Tschirnhaus transformations, we can conclude that the method in [7] is at least 6.16 times faster than any other method for polynomials with Galois group D_4.

Timings when the Galois group is C_4

	[7]	SF	Alg. 5.1	Cubic	Sextic	Deg 12
T1	0.0048	16.19	15.32	19.37	255.32	17.23
T2	0.0055	10.06	9.56	10.72	140.06	10.17
T3	0.0065	8.4	8.26	7.42	84.47	7.42
T4	0.0074	8.3	8	5.83	58.63	5.74
T5	0.0089	8.9	8.67	5.36	45.16	5.26
T6	0.0112	10.67	10.29	5.27	34.43	5.03
T7	0.0146	13.65	12.8	5.55	25.21	5.04
T8	0.0216	17.93	16.82	6.13	18.45	5.72
T9	0.0343	23.21	21.58	7.02	12.72	8.01
T10	0.0703	32.49	32.77	9.37	10.05	15.27

Again, we see that the method in [7] is the fastest method for quartic polynomials with Galois group C_4. This method takes 0.0048 seconds for polynomials with one Tschirnhaus transformation and 0.0703 seconds for polynomials with ten Tschirnhaus transformations. The timing ratios of all methods except the sextic resolvent method become smaller before becoming larger compared to [7] as we increase the size of the polynomial coefficients. The sextic resolvent method becomes relatively faster as we increase coefficient size. We can say that the method in [7] is at least 5 times faster than any other method for polynomials with Galois group C_4, as the smallest ratio is 5.03. This occurs when we use the degree 12 method on polynomials with six Tschirnhaus transformations.

Timings when the Galois group is A_4

	[7]	SF	Alg. 5.1	Sextic	Deg 12
T1	0.004	233.39	17.82	325.28	22.25
T2	0.0045	198.96	13.17	175.58	14.01
T3	0.0055	186.3	10.41	97.92	9.8
T4	0.0065	245.72	10.05	65.04	8.83
T5	0.0077	463.4	11.97	47.33	10.13
T6	0.0094	1026.79	16.11	36.55	14.47
T7	0.0124	3305.62	24.37	28.47	28.25
T8	0.0178	9043.24	44.08	21.43	71.57

For quartic polynomials with Galois group A_4, the method in [7] and the cubic resolvent method are equivalent, and they are the fastest. We only used eight Tschirnhaus transformations for polynomials with this Galois group due to computational constraints of our machine. The ratios of the SF, Alg. 5.1, the degree 12 methods decrease in size initially, and then increase in size as we increase the size of polynomial coefficients. On the other hand, the sextic resolvent method becomes relatively faster as we increase the size of coefficients, when compared to [7]. This method is at least 47% faster than any other method, given a random quartic polynomial with Galois group A_4, as the smallest ratio is 1.47, occurring when we apply the cubic resolvent method to a polynomial with eight Tschirnhaus transformations.

Timings when the Galois group is S_4

	[7]	SF	Alg. 5.1	Sextic	Deg 12
T1	0.0043	229.75	17.27	362.22	22.56
T2	0.0046	181.17	12.36	183.85	12.43
T3	0.0057	179.99	11.01	112.48	8.91
T4	0.0067	227.81	10.8	78.39	7.76
T5	0.0088	376.59	12.37	58.65	7.75
T6	0.0113	742.86	14.24	42.68	9.59
T7	0.0165	1665.3	19.11	31.19	14.28
T8	0.0263	4820.63	25.98	21.33	34.11

Lastly, the method in [7] and the cubic resolvent method are equivalent for polynomials with Galois group S_4. They are the quickest methods in this case. The timing ratios of the SF method appear to increase at an exponential rate. The timing ratios of Alg. 5.1 and the degree 12 methods decrease before increasing five and six Tschirnhaus transformations, respectively. The sextic resolvent method increases in relative speed as we increase the size of the coefficients. The method in [7] is at least 7.75 times faster than any other method, as the smallest timing ratio occurs when we apply the degree 12 resolvent method to a polynomial with 5 Tschirnhaus transformations.

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