International Journal of Pure and Applied Mathematics
Volume 113 No. 4 2017, 5-15
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i4.2
ijpam.eu

LOADED PARABOLIC EQUATIONS AND BOUNDARY VALUE PROBLEMS OF HEAT CONDUCTION IN NON-CYLINDRICAL DEGENERATING DOMAINS

Muvasharkhan T. Jenaliyev
Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN

Abstract: This paper gives an overview of the author's research, his colleagues and pupils for the years 1976–2016. At the beginning the results concerning the loaded differential equations are provided, including the essential-and spectral-loaded equations. Then, the results of research concerning the boundary value problems of heat conduction in non-cylindrical degenerating domains are discussed.

AMS Subject Classification: 35K05, 35K20, 47A75
Key Words: loaded differential equation, boundary value problem, degenerating domain, heat equation, existence and non-unique of solution

1. Introduction

Definition 1. The equation

$$\mathbb{A}u(x) = f(x), \tag{1}$$

given in n–dimensional domain Ω of the Euclidean space of points $x =$

Received: December 15, 2016
Revised: January 30, 2017
Published: April 6, 2017

$(x_1, x_2, \ldots, x_n)$ is called *loaded* [1], if it contains *trace* of some operations from the required solution $u(x)$ on manifolds of dimension $< n$ that belong to the closure $\bar{\Omega}$.

Definition 2. Loaded equation (1) is called *loaded differential equation* [1] in the domain $\Omega \in \mathbb{R}^n$ if it can be represented as

$$\mathbb{A}u \equiv \mathbb{L}u(x) + \mathbb{M}u(x) = f(x), \tag{2}$$

where $\mathbb{L}$ is a differential operator, and $\mathbb{M}$ is a differential or integro-differential operator, including the trace operation from the required solution $u(x)$ on manifolds of nonzero measure strictly less than n that belong to $\overline{\Omega}$.

For example, the problem of the vibrations of a string loaded lumped mass leads to the simplest equation of the form (2). This problem is widely applied in physics and engineering. For the theory of many measuring devices is important to study the torsional vibrations of the thread, by the end of which the mass is hung, for example, mirror.

The problems of such type have acquired the special urgency in connection with the study of stability for vibrations of the wings of the aircraft as to solve this problem it is necessary to compute proper frequency of wing (beam of varying section) loaded with masses (motors). In addition, such problems are encountered in calculating the eigen-oscillations of antennas, loaded with concentrated capacitances and self-inductances.

Also it should be noted that solving many problems for optimal control of the agro-ecosystems, for example, problems of long-term forecasting and control of the groundwater level and soil moisture, are reduced to the study of equations of the form (2).

Main issues arising in the theory of boundary value problems for partial differential equations remain so for the boundary value problems for the loaded equations of the form (2).

2. Overview of the Main Results

Very wide range of issues: the study of the generalized solvability of nonhomogeneous boundary value problems for the loaded differential equations in Sobolev spaces; description of solution spaces and functions that define the right parts and boundary conditions; proof of the a'priori estimates, which provides the correctness of the boundary value problems and the accuracy of the selected spaces was investigated in author's works, his colleagues and pupils [2–26]. We will list the main ones.

1. *Equations with irregular coefficients.* The results on the solvability of boundary value problems for linear loaded equations with irregular coefficients were published in the works [2, 3], in which the theorems on the solvability of boundary value problems for the loaded differential equations with optionally bounded coefficients of loaded terms were proved. Such situations arise, for example, in the study of optimal control problems of equation coefficients.

2. *Problems with time derivatives in the boundary conditions.* Boundary value problems with a time derivative on the boundary for parabolic and hyperbolic equations were studied. Here Jean-L.Lions' problem [27] on the choice of spaces that define the boundary conditions with time derivatives for the parabolic equation was solved; effect of "overdetermination" was detected at setting the initial conditions on the border which reflects the fact that the initial conditions in the domain and on its boundary are set from the class of square integrable functions (which is not necessarily coordinated by the theorem on traces [27]). These results were published in the works [4–7].

3. *Criteria for the unique solvability.* Criteria (necessary and sufficient conditions) of uniquely strong solvability of boundary value problems for loaded by "time" of equations were established. These results were published in the works [8, 9, 15, 16].

4. *On the variational principle.* On the variational principle for the loaded equations. It was built: symmetrization operator for essential unsymmetrical operator of loaded parabolic equation, corresponding Hilbert space of type of K.Fridrihs' space and quadratic functional over it, for which Euler's equation gives a generalized statement of the initial boundary value problem [13, 14].

5. *Spectrally loaded equations.* Boundary and spectral problems for spectrally loaded parabolic operator. In terms of the (complex) spectral parameter λ which is the coefficient of the loaded term, that is not subordinated to the main part of the differential operator, the description of the resolvent is set and a spectrum for the spectrally loaded parabolic operator is obtained, and characteristics of the multiplicity of its eigenfunctions is found in the space of bounded and continuous functions depending on the value of the spectral parameter λ. For spectrally loaded parabolic operator the adjoint operator is found, the solvability of the corresponding boundary value problem in classes of integrable functions is proved. It is shown that the boundary value problem for the spectrally loaded parabolic operator is Noetherian, and has a positive index, depending on the parameter λ. The results have partly been published in the works [17–19].

6. *Boundary value problems in non-cylindrical degenerating domains.* Since 2012 M.T.Jenaliyev together with employees by grant topics carries out research

by boundary value problems of heat conduction in non-cylindrical degenerating domains. It is shown that in these boundary value problems in addition to the trivial solution there exist non-trivial solutions. These results are published in the works [22–26].

By results of research in directions 1–4 monograph [28] was published, and in direction 5 monograph [29] was published.

3. Essentially Loaded Parabolic Equations

The boundary problem (Cauchy problem, Cauchy-Dirichlet problem) were studied for essential-loaded parabolic equation with a load at a fixed time variable. A special feature of these problems is the presence of a loaded term with the derivative of any whole order of the required solution. For each of these tasks two problems are solved: the first problem is to set the dimension of the kernel of the corresponding boundary value problem; the second problem is to find criteria and define classes of strong unique solvability of the corresponding nonhomogeneous problems.

Note that a feature of considered problems lies in the fact that, for example in space L_2 the corresponding differential operators are not closed, because, firstly, the load is not subject to the corresponding differential part of the operator, i.e., is not weak perturbation for its differential part. Secondly, it is known that operators of loads in spaces L_2 are not closable operators.

All this does not allow investigating directly the issues of strong solvability of boundary value problems for the non-closable loaded differential equations [29].

We give some results from [29]. We consider the following Cauchy problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \alpha \dfrac{\partial^k u(x,\overline{t})}{\partial t^k} = f(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{3}$$

where $\alpha \in \mathbb{R}$, $\overline{t} \in \mathbb{R}_+ = (0, +\infty)$ are given values, $k \in \mathbb{N}$.

Theorem 3. *Let be $f(x,t) \equiv 0$, $\varphi(x) \equiv 0$. In order that Cauchy problem (3) has only the trivial solution, it is necessary and sufficient for $\forall s \in \mathbb{R}$ the following conditions hold:*

$$0 \neq \Delta_k(s) = \begin{cases} 1 + \alpha \dfrac{1 - e^{-s^2 \overline{t}}}{s^2}, & k = 0, \\ 1 + (-1)^{k+1} \alpha s^{2(k-1)} e^{-s^2 \overline{t}}, & k \in \mathbb{N}. \end{cases} \tag{4}$$

Theorem 4. *Cauchy problem* (3) *for any*

$$f \in W^k_{2,0}(\mathbb{R}_+; L_2(\mathbb{R})),\ \varphi \in L_2(\mathbb{R}),$$

is uniquely strongly solvable in the space

$$L_{2,\, e^{-\varepsilon t}} \equiv \{v|\, v(x,t) \cdot e^{-\varepsilon t} \in L_2(\mathbb{R} \times \mathbb{R}_+) \quad \forall\, \varepsilon > 0\},$$

if and only if conditions (4) *hold.*

Now let us consider the Cauchy-Dirichlet problem ($t > 0$):

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \alpha \dfrac{\partial^k u(x,\overline{t})}{\partial t^k} = f(x,t), & x \in \mathbb{R}_+, \\ u(0,t) = \psi(t),\ u(x,0) = \varphi(x), & x \in \mathbb{R}_+, \end{cases} \tag{5}$$

where $\alpha \in \mathbb{R}$, $\overline{t} \in \mathbb{R}_+$ are given values, $k \in \mathbb{N}$,

$$f \in W^k_{2,0}(\mathbb{R}_+; L_2(\mathbb{R}_+)),\ \varphi \in L_2(\mathbb{R}_+),\ \psi(t) \in L_2(R_+). \tag{6}$$

Here 0 in the designation of space means

$$\frac{\partial^m f(x,0)}{\partial t^m} = 0,\ m = 0, 1, ..., k-1.$$

Theorem 5. *Let be* $f(x,t) \equiv 0$, $\varphi(x) \equiv 0$, $\psi(t) \equiv 0$. *In order that problem* (5) *has only the trivial solution, it is necessary and sufficient for* $\forall s \in \mathbb{R}_+$ *the following conditions hold*

$$0 \neq \Delta_k(s) = \begin{cases} 1 + \alpha \dfrac{1 - e^{-s^2 \overline{t}}}{s^2}, & k = 0, \\ 1 + (-1)^{k+1} \alpha s^{2(k-1)} e^{-s^2 \overline{t}}, & k \in \mathbb{N}. \end{cases} \tag{7}$$

Theorem 6. *Cauchy-Dirichlet problem* (5) *for all* $\{f, \varphi, \psi\}$, *satisfying the conditions* (6), *is uniquely strongly solvable in the space*

$$L_{2,\, e^{-\varepsilon t}} \equiv \{v|\, v(x,t) \cdot e^{-\varepsilon t} \in L_2(\mathbb{R}_+ \times \mathbb{R}_+) \quad \forall\, \varepsilon > 0\},$$

if and only if conditions (7) *hold.*

4. Spectrally Loaded Parabolic Equations

For spectrally loaded parabolic operator the mutually conjugate boundary value problems are studied in a quarter plane when the load is given by the space variable and, thus, the load point moves with constant and variable speeds. A feature of considered problems is that, firstly, *the spectral parameter is the coefficient of the loaded term*, secondly, *order of the derivative in the loaded term equals to the order of the differential part of the equation* and, thirdly, *the point of load moves* (with constant and variable speeds). New properties of loaded differential operator are identified which are not inherent to operators with a weak perturbation. It is shown that this problems are Noetherian, and for some values of the spectral parameter, strictly described in the complex plane they have a non-zero index, which is determined directly by the modulus and argument of this spectral parameter. Theorems on the solvability of these problems are formulated in naturally introduced functional classes [17–21, 29].

In the domain $Q = \{x \in \mathbb{R}_+,\ t \in \mathbb{R}_+\}$ we consider the following homogeneous boundary value problem ($\lambda \in \mathbb{C}$):

$$u_t - u_{xx} + \lambda u_{xx}(x,t)_{|x=t} = 0, \quad u(x,0) = 0,\ \ u(0,t) = 0. \tag{8}$$

A feature of considered problem is that, firstly, *the spectral parameter is the coefficient of the loaded term*, secondly, *order of the derivative in the loaded term equals to the order of the differential part of the equation* and, thirdly, *the point of load moves* by linear law: $x(t) = t$.

Lines described by the equation $|\lambda| = \exp(|\arg\lambda + 2k\pi|)$, $k \in \mathbb{Z}$, subdivide complex plane of the parameter λ into nonintersecting domains D_m, $m = 0, 1, 2, ...,$ as follows:

$$D_{2n} = \left\{D_n^{(1)} \bigcap D_n^{(2)}\right\} \setminus \bigcup_{k=-1}^{2n-1} D_k, \qquad D_{-1} = \phi, \tag{9}$$

$$D_{2n+1} = \left\{D_n^{(1)} \bigcup D_n^{(2)}\right\} \setminus \bigcup_{k=0}^{2n} D_k,$$

where

$$D_n^{(1)} = \{\lambda : |\lambda| < \exp[(2n+1)\pi - \arg\lambda]\},$$

$$D_n^{(2)} = \{\lambda : |\lambda| < \exp[2n\pi + \arg\lambda]\},\ \ n = 0, 1, 2, ...$$

The external parts of the boundaries ∂D_m, $m = 0, 1, 2, ...,$ of domains D_m are denoted by Γ_m.

Here, it is required the research (for the presence of zeros) of function

$$\mathbf{A}_\lambda(z) \equiv 1 - \lambda \exp\{-\sqrt{i\,z}\},\ z \in \mathbb{C},$$

which in the lower halfplane may have only a finite number of zeros, where

$$-N_1 \leq k \leq N_2,\ N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \tag{10}$$

$$N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

Here the symbol $[a]$ means the integer part of a, and, integer part of a negative number is taken equal zero. Ratios (10) follow from the condition of boundedness solutions to auxiliary integral equations.

We formulate the result on the solvability of boundary value problem (8) as a theorem.

Theorem 7. *If $\lambda \in D_0$ (9), then homogeneous boundary value problem (8) has only the trivial solution. If $\lambda \in \{\mathbb{C} \setminus D_0\} \cap \{D_m \cup \Gamma_{m-1} \setminus \{(-1)^m e^{m\pi}\}\}$ (9), then boundary value problem (8) has the nontrivial solution:*

$$u_{\cdot}(x,t) = \sum_{k=-N_1}^{N_2} c_k u_{\lambda k}(x,t), \tag{11}$$

where c_k are arbitrary constants; $\{u_{\lambda k}(x,t),\ k \in [-N_1, N_2]\}$ is the system of linearly independent nontrivial solutions to (8), and the explicit dependence of their number on value of the spectral parameter is found λ.

Remark 8. The result of this theorem is established for cases when the line of load changes according to a power law [17–21, 29]: $x(t) = t^\omega,\ 0 < \omega < +\infty$.

5. Boundary Value Problems in Non-Cylindrical Degenerating Domains

It is considered the homogeneous boundary problem

$$\frac{\partial u(x,t)}{\partial t} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ \{x,t\} \in G; \tag{12}$$

$$\frac{\partial u(x,t)}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u(x,t)}{\partial x}\Big|_{x=t} + \frac{d\tilde{u}(t)}{dt} = 0; \tag{13}$$

where $\tilde{u}(t) = u(t,t),\ G = \{x,t:\ 0 < x < t,\ t > 0\}$.

Note that problem (12)–(13) is homogeneous case of the problem, studied in [30], which stated that the case of non-homogeneous boundary value problem" ... appears to be useful in the study of some free boundary value problems." In the paper [30] a theorem on the unique solvability of the non-homogeneous boundary value problem in weighted Holder spaces was obtained.

In [30], we set the existence of a nontrivial solution for a constant factor in class of essentially bounded functions with a given weight. We introduce the class as follows:

$$(x + t^{1/2})^{-1}\, u(x,t) \in L_\infty(G),\ \text{i.e. } u(x,t) \in L_\infty(G;\ (x + t^{1/2})^{-1}).$$

Theorem 9. *The boundary value problem (12)–(13) has a nontrivial solution $u(x,t) = C\,\tilde{u}(x,t)$, where $\tilde{u}(x,t) \in L_\infty(G;\ (x + \sqrt{t})^{-1}),\ \ C = const$.*

Theorem 10. *In the class of functions $L_\infty\left(G;\ [x^{1+\alpha} + t^{(1+\alpha)/2}]^{-1}\right)$ the boundary value problem (12)–(13) has only the trivial solution $u(x,t) \equiv 0$.*

Remark 11. In works [22–24, 26] we earlier have studied the boundary value problems for the heat equation in non-cylindrical degenerating domains of the form:

$$\{x,\, t|\, 0 < x < t^\omega,\ t > 0,\ \omega > 1/2\}.$$

We also note publications [31–35] of other authors that are close by category of this item of work.

Brief abstract of this work has been published in the Materials of the workshop "Differential operators and modeling of complex systems" (April 7-8, 2017, Almaty, Kazakhstan) [36].

Acknowledgements

This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0823/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

[1] A.M. Nakhushev, *Loaded equations and their applications* , Nauka, Moscow (2012), (in Russian).

[2] M.T. Jenaliyev (Dzhenaliev), Optimal Control of Linear Loaded Parabolic Equations, *Differential Equations*, **25**, No. 6 (1989), 641–651 (in Russian).

[3] M.T. Jenaliyev (Dzhenaliev), An Solvability of Boundary Value Problems for Linear Loaded Equations with non-regular coefficients, *Differential Equations*, **27**, No. 9 (1991), 1585–1595 (in Russian).

[4] M.T. Jenaliyev (Dzhenaliev), About Boundary Value Problem for Linear Loaded Parabolic Equation with Non-local Boundary Conditions, *Differential Equations*, **27**, No. 10 (1991), 1825–1827 (in Russian).

[5] M.T. Jenaliyev (Dzhenaliev), Boundary Value Problems and Optimal Control Problems for Linear Loaded Hyperbolic Equations, *Differential Equations*, **28**, No. 2 (1992), 232–241 (in Russian).

[6] M.T. Jenaliyev (Dzhenaliev), About one Class of Loaded Elliptic Equations, *Differential Equations*, **28**, No. 3 (1992), 522–524 (in Russian).

[7] M.T. Jenaliyev (Dzhenaliev), Boundary Value Problems for Loaded Parabolic and Hyperbolic Equations with Time-Derivatives in Boundary Conditions, *Differential Equations*, **28**, No. 4 (1992), 661–666 (in Russian).

[8] M.T. Jenaliyev (Dzhenaliev), Nonhomogeneous Boundary Value Problems for Loaded Evolutional Equations of odd Order with Interior Boundary Conditions, *Differential Equations*, **29**, No. 4 (1993), 617–626 (in Russian).

[9] M.T. Jenaliyev (Dzhenaliev), Boundary Value Problems for Differential Operator Equations of odd Order and Loaded Equations, *Differential Equations*, **29**, No. 8 (1993), 1380–1389 (in Russian).

[10] M.T. Jenaliyev (Dzhenaliev), To Generalized Solvability of Loaded Wave Equation, *Differential Equations*, **30**, No. 4 (1994), 723–724 (in Russian).

[11] M.T. Jenaliyev (Dzhenaliev), Boundary Value Problems for Loaded Differential Equations of First and second Order and Their Applications, *Differential Equations*, **31**, No. 1 (1995), 160–162 (in Russian).

[12] M.T. Jenaliyev (Dzhenaliev), About Non-local Boundary Value Problems for Loaded Differential Operator Equations of First Order, *Differential Equations*, **31**, No. 11 (1995), 1902–1906 (in Russian).

[13] M.T. Jenaliyev (Dzhenaliev), On Quadratic Functional in Cauchy Problem for Loaded Differential Operator Equation of First Order. I, *Differential Equations*, **31**, No. 12 (1995), 2029–2037 (in Russian).

[14] M.T. Jenaliyev (Dzhenaliev), On Quadratic Functional in Cauchy Problem for Loaded Differential Operator Equation of First Order. II, *Differential Equations*, **32**, No. 4 (1996), 518–522 (in Russian).

[15] M.T. Jenaliyev, On the boundary value problems for loaded differential equations, *J.Korean Math.Soc. (Seoul)*, **37**, No. 6 (2000), 1031–1042.

[16] M.T. Jenaliyev (Dzhenaliev), On Loaded Equations with Periodic Boundary Conditions, *Differential Equations*, **37**, No. 1 (2001), 48–54 (in Russian).

[17] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On Boundary Value Problem for Spectrally Loaded Heat Operator , *Siberian Mathematical Journal*, **47**, No. 3 (2006), 527–547 (in Russian).

[18] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, About one Boundary Value Problem for Spectrally Loaded Heat Conduction Operator. I, *Differential Equations*, **43**, No. 4 (2007), 498–508 (in Russian).

[19] M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, About one Boundary Value Problem for Spectrally Loaded Heat Conduction Operator. II, *Differential Equations*, **43**, No. 6 (2007), 788–794 (in Russian).

[20] D.M. Akhmanova, M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, On particular integral equation of second kind with spectral parameter, *Siberian Mathematical Journal*, **52**, No. 1 (2011), 3–14 (in Russian).

[21] M.M. Amangaliyeva, D.M. Akhmanova, M.T. Jenaliyev (Dzhenaliev), M.I. Ramazanov, The boundary value problems for spectrally loaded heat conduction operator with proximity loaded curve to times axis at zero or at infinity, *Differential Equations*, **47**, No. 2 (2011), 231–243 (in Russian).

[22] M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov, About Dirichlet boundary value problem for the heat equation in the infinite angular domain, *Boundary Value Problems*, **2014**, 213 (2014), 1–21.

[23] M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov, On a Volterra equation of the second kind with 'incompressible' kernel, *Advances in Difference Equations*, **2015**, 71 (2015), 1–14.

[24] M.M. Amangaliyeva, M.T. Jenaliyev (Dzhenaliev), M.T. Kosmakova, M.I. Ramazanov, On one homogeneous problem for the heat equation in an infinite angular domain, *Siberian Mathematical Journal*, **56**, No. 6 (2015), 982–995 (in Russian).

[25] V.I. Korzyuk, M.T. Jenaliyev (Dzhenaliev), I.S. Kozlovskaya, Boundary Value Problems for a Weakly Loaded Operator of a Second-Order Hyperbolic Equation in a Cylindrical Domain, *Differential Equations*, **51**, No. 12 (2015), 1595–1606 (in Russian).

[26] M.T. Jenaliyev, M.I. Ramazanov, Spectral Volterra Integral Equation of the Second Kind with a Spectral Parameter, In Book *Analytic Methods of Analysis and Differential Equations* (AMADE 2015), Cambridge Scientific Publishers, UK, (2016), 61–76.

[27] J.-L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications, V.2*, Springer-Verlag, Berlin (1972).

[28] M.T. Jenaliyev, *Introduction to the theory of linear loaded differential equations*, Gylym, Almaty (1995) (in Russian).

[29] M.T. Jenaliyev, M.I. Ramazanov, *Loaded equations as perturbation of differential equations*, Gylym, Almaty (2010) (in Russian).

[30] V.A. Solonnikov, A. Fasano, One-dimensional parabolic problem arising in the study of some free boundary problems, *Zapiski nauchnykh seminarov POMI*, **269**, (2000), 322–338 (in Russian).

[31] T. Berroug, H. Ding, R. Labbas, B.-Kh. Sadallah, On a degenerate parabolic problem in Hölder spaces, *Applied Mathematics and Computation*, **162** (2005), 811–833.

[32] R. Labbas, A. Medeghri, B.-Kh. Sadallah, An L_p-approach for the study of degenerate parabolic equations, *Electronic Journal of Differential Equations*, **2005**, 36, (2005), 1–20.

[33] A. Kheloufi, B.-Kh. Sadallah, On the regularity of the heat equation solution in non-cylindrical domains: Two approaches, *Applied Mathematics and Computation*, **218** (2011), 1623–1633.

[34] A. Kheloufi, Existence and uniqueness results for parabolic equations with Robin type boundary conditions in a non-regular domain of $\mathbb{R}^3$, *Applied Mathematics and Computation*, **220** (2013), 756–769.

[35] A. Kheloufi, B.-Kh. Sadallah, Resolution of a high-order parabolic equation in conical time-dependent domains of $\mathbb{R}^3$, *Arab Journal of Mathematical Sciences*, **22** (2016), 165–181.

[36] M.T. Jenaliyev, Loaded parabolic equations and boundary value problems of heat conduction in non-cylindrical degenerating domains, *Kazakh Mathematical Journal*, **17**, No. 1 (2017), 13–23.