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SOLVABILITY AND CONSTRUCTION OF A SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR LINEAR INTEGRAL AND DIFFERENTIAL EQUATIONS WITH RESTRICTIONS

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Abstract: The necessary and sufficient conditions for the solvability of boundary value problems for linear integral - differential equations with phase and integral constraints are obtained. The method of constructing the solution of the boundary value problem with constraints is developed by constructing minimizing sequences. The base of the proposed method for solving the boundary value problem is the principle of immersion. The principle of immersion has been created by building the general solution of a class of Fredholm integral equations of the first kind.

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1. Problem Statement

We consider the following boundary value problem for linear integral-differential equations

$$\dot{x} = A_0(t)x + B_0(t)\int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau + \mu(t), \quad t \in I = [t_0, t_1], \tag{1}$$

with boundary conditions

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$$(x(t_0) = x_0, \ \ x(t_1) = x_1) \in S \subset R^{2n}, \tag{2}$$

at phase constraints

$$\begin{gathered} x(t) \in G(t) = \{x \in R^n / \alpha(t) \leq L(t)x(t) \leq \beta(t), \ \ t \in I; \\ \alpha_1(t) \leq \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau \leq \beta_1(t), \ \ t \in I\}, \end{gathered} \tag{3}$$

as well as the integral constraints

$$g_j(x) \leq c_j, \ \ j = \overline{1, m_1}, \ \ g_j(x) = c_j, \ \ j = \overline{m_1 + 1, m_2} \tag{4}$$

$$g_j(x) = \int_{t_0}^{t_1} \left[\langle a_j(t), x(t) \rangle + \left\langle b_j(t), \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau \right\rangle \right] dt, \tag{5}$$

$$j = \overline{1, m_2}.$$

Here $A_0(t)$, $B_0(t)$, $K(t,\tau)$, $L(t)$, $t \in I$, $\tau \in I$ are prescribed matrixes with piecewise continuous elements of the dimension $n \times n$, $n \times m$, $m \times n$, $s \times n$, respectively, $\mu(t)$, $t \in I$ is given n – dimensional vector function with piecewise continuous components, S is given convex closed set, $a_j(t) = (a_{1j}(t), \ldots, a_{nj}(t))$, $b_j(t) = (b_{1j}(t), \ldots, b_{nj}(t))$, $t \in I$, $j = \overline{1, m_2}$ are known vector functions with piecewise continuous elements, $\alpha(t) = (\alpha_1(t), \ldots, \alpha_s(t))$, $\beta(t) = (\beta_1(t), \ldots, \beta_s(t))$, $\alpha_1(t) = (\alpha_{11}(t), \ldots, \alpha_{m1}(t))$, $\beta_1(t) = (\beta_{11}(t), \ldots, \beta_{m1}(t))$, $t \in I$ are given continuous function. The values c_j, $j = \overline{1, m_2}$ are prescribed constants.

The following problems are set:

Problem 1. Find necessary and sufficient conditions for the existence of solutions of the boundary problem (1) – (5).

Problem 2. Construct a solution of the boundary value problem (1) – (5).

Particular cases of the boundary value problem (1) – (5) in the absence of phase and integral restrictions with affine set S are studied in the works [1, 2, 3]. This work is a continuation of research of [4, 5].

Constructive theory of boundary value problems with phase and integral constraints for ordinary differential equations, as well as for the parabolic equations are presented in [6, 7, 8, 9, 10].

2. Transformation

Introducing the additional variables $d = (d_1, \ldots, d_{m_1}) \in R^{m_1}$, $d \geq 0$, the relations (4), (5) can be presented as

$$g_j(x) = \int_{t_0}^{t_1} [\langle a_j(t), x(t)\rangle +$$

$$+ \left\langle b_j(t), \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau \right\rangle \Bigg] dt = c_j - d_j, \;\; j = \overline{1, m_1},$$

where $d \in D = \{d \in R^{m_1}/d \geq 0\}$.

Let the vector $\overline{c} = (\overline{c}_1, \ldots, \overline{c}_{m_2})$ has the components $\overline{c}_j = c_j - d_j$, $j = \overline{1, m_1}$, $\overline{c}_j = c_j$, $j = \overline{m_1 + 1, m_2}$. We introduce the vector functions $\eta(t) = (\eta_1(t), \ldots, \eta_{m_2})$, $t \in I$ by equality

$$\eta_j(t) = \int_{t_0}^{t} \left[\langle a_j(\tau), x(\tau)\rangle + \left\langle b_j(\tau), \int_{t_0}^{t_1} K(\tau,\rho)x(\rho)d\rho \right\rangle \right] d\tau, t{\in}I,$$

then

$$\dot{\eta}_j(t) {=} < a_j(t), x(t) > + < b_j(t), \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau >, \; j{=}\overline{1, m_2},$$

$$\eta_j(t_0) = 0, \;\; \eta_j(t_1) = \overline{c}_j, \; j{=}\overline{1, m_2}, \; d{\in}D.$$

It follows that

$$\dot{\eta}(t) = A_1(t)x(t) + B_1(t) \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau, \;\; t \in I.$$

Now the original boundary value problem (1) – (5) is written as

$$\dot{\xi} = A(t)\xi + B(t) \int_{t_0}^{t_1} K(t,\tau)x(\tau)d\tau + \mu_1(t), \;\; t \in I, \tag{6}$$

$$\xi(t_0) = \xi_0 = (x_0, O_{m_2,1}), \;\; \xi(t_1) = \xi_1 = (x_1, \overline{c}), \tag{7}$$

$$(x_0, x_1) \in S, \;\; d \in D, \;\; P\xi(t) \in G(t), \;\; t \in I, \tag{8}$$

where

$$\xi(t) = \begin{pmatrix} x(t) \\ \eta(t) \end{pmatrix}, \;\; A(t) = \begin{pmatrix} A_0(t) & O_{nm_2} \\ A_1(t) & O_{m_2m_2} \end{pmatrix}, \;\; B(t) = \begin{pmatrix} B_0(t) \\ B_1(t) \end{pmatrix},$$

$$\mu_1(t) = \begin{pmatrix} \mu(t) \\ O_{m_21} \end{pmatrix}, \;\; P = (I_n, \; O_{nm_2}), \;\; P\xi = x,$$

O_{jk} is $j \times k$ matrix with zero elements, $\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+m_2})$, I_n is an identity matrix of order $n \times n$.

3. Linear Control System

Along with the differential equation (6) with boundary conditions (7) we consider the linear controlled system

$$\dot{y} = A(t)y + B(t)w(t) + \mu_1(t), \quad t \in I, \tag{9}$$

$$y(t_0) = \xi_0 = (x_0, O_{m_2,1}), \quad y(t_1) = \xi_1 = (x_1, \bar{c}), \tag{10}$$

$$(x_0, x_1) \in S, \quad d \in D, \quad w(\cdot) \in L_2(I, R^m), \tag{11}$$

where $A(t)$, $B(t)$ are matrixes with piecewise continuous elements of the order $(n+m_2)\times(n+m_2)$, $(n+m_2)\times m$, respectively. It is easy to make sure that the control $w(\cdot) \in L_2(I, R^m)$ that transfers the trajectory of the system (9) from any initial state ξ_0 to any desired final state ξ_1, is a solution of the integral equation

$$\int_{t_0}^{t_1} \Phi(t_0, t)B(t)w(t)dt = a, \tag{12}$$

where $\Phi(t, \tau) = \theta(t)\theta^{-1}(\tau)$, $\theta(t)$ is a fundamental matrix of solutions of the linear homogeneous system $\dot{\omega} = A(t)\omega$, vector

$$a = a(\xi_0, \xi_1) = \Phi(t_0, t_1)\xi_1 - \xi_0 - \int_{t_0}^{t_1} \Phi(t_0, t)\mu_1(t)dt.$$

Theorem 1. *The integral equation* (12) *at any fixed* $a \in R^{n+m_2}$ *has a solution if and only if* $(n+m_2)\times(n+m_2)$ *matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^*(t)\Phi^*(t_0, t)dt$$

is positive definite, where $(*)$ *denotes transposition.*

The proof of the theorem is given in [11].

Theorem 2. *Let the matrix* $W(t_0, t_1)>0$. *Control* $w(\cdot)\in L_2(I, R^m)$ *transforms the trajectory of system* (9) *from any starting point* $\xi_0 \in R^{n+m_2}$ *to any final state* $\xi_1 \in R^{n+m_2}$ *if and only if*

$$w(t) \in W = \{w(\cdot)\in L_2(I, R^m)/w(t) = v(t) + \lambda_1(t, \xi_0, \xi_1) + \\ +N_1(t)z(t_1, v), t \in I, \quad v(\cdot) \in L_2(I, R^m)\}, \tag{13}$$

where the function $z(t) = z(t, v)$, $t \in I$ *is the solution of the differential equation*

$$\dot{z} = A(t)z + B(t)v(t), \quad z(t_0) = 0, \quad v(\cdot) \in L_2(I, R^m). \tag{14}$$

Moreover the solution of the differential equation (9)*, corresponding to the control* $w(t) \in W$ *is defined by equality*

$$y(t) = z(t) + \lambda_2(t, \xi_0, \xi_1) + N_2(t)z(t_1, v), \quad t \in I. \tag{15}$$

Here $v(\cdot) \in L_2(I, R^m)$ *is any function,* $\lambda_1(t, \xi_0, \xi_1)$, $N_1(t)$, $\lambda_2(t, \xi_0, \xi_1)$, $N_2(t)$ *are defined by formulas*

$$\lambda_1(t, \xi_0, \xi_1) = B^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_1)a, \; N_1(t) = -B^*(t)\Phi^*(t_0, t)\times$$

$$\times W^{-1}(t_0, t_1)\Phi(t_0, t_1), \; \lambda_2(t) = \Phi(t, t_0)W(t, t_1)W^{-1}(t_0, t_1)\xi_0 +$$

$$+\Phi(t, t_0)W(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1)\xi_1 + \int_{t_0}^{t} \Phi(t, \tau)\mu_1(\tau)d\tau -$$

$$-\Phi(t, t_0)W(t_0, t)W^{-1}(t_0, t_1)\int_{t_0}^{t_1} \Phi(t_0, \tau)\mu_1(\tau)d\tau, N_2(t) = \Phi(t_1, t_0)\times$$

$$\times W(t_0, t)W^{-1}(t_0, t_1)\Phi(t_0, t_1), \quad W(t, t_1) = \int_{t}^{t_1} \Phi(t_0, \tau)B(\tau)\times$$

$$\times B^*(\tau)\Phi^*(t_0, \tau)d\tau, \quad W(t_0, t) = \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau)d\tau.$$

The proof of the analogous theorem can be found in [6].

4. Optimization Problem

We consider the following optimization problem: minimize the functional

$$J(v, u, p, x_0, x_1, d) = \int_{t_0}^{t_1} \Big[|w(t) - u(t)|^2 + |p(t) - L(t)Py(t)|^2 + \\ + \left| w(t) - \int_{t_0}^{t_1} K(t, \tau)Py(\tau)d\tau \right|^2 \Big] dt \to \inf \tag{16}$$

under conditions

$$\dot{z} = A(t)z + B(t)v(t), \quad z(t_0) = 0, \quad v(\cdot) \in L_2(I, R^m), \tag{17}$$

$$u(t) \in U(t) = \{u(\cdot) \in L_2(I, R^m)/\alpha_1(t) \le u(t) \le \beta_1(t), \quad t \in I\}, \tag{18}$$

$$p(t) \in V(t) = \{p(\cdot) \in L_2(I, R^s)/\alpha(t) \le p(t) \le \beta(t), \quad t \in I\}, \tag{19}$$

$$(x_0, x_1) \in S, \quad d \in D, \tag{20}$$

where the functions $w(t)$, $y(t)$, $t \in I$ are defined by the formulas (13) – (15), respectively.

We denote

$$X{=}L_2(I,R^m){\times}U(t){\times}V(t){\times}S{\times}D{\subset}H{=}L_2(I,R^m){\times}L_2(I,R^m){\times}$$

$$\times L_2(I,R^s) \times R^{2n} \times R^{m_1},\ J_* = \inf_{\theta\in X} J(\theta),\ \theta = (v(t),u(t),p(t),x_0,$$

$$x_1,d) \in X,\ X_* = \{\theta_* \in X/J(\theta_*) = \inf_{\theta\in X} J(\theta) = \min_{\theta\in X} J(\theta)\}.$$

We introduce the following notations

$$F_0(q(t),t) = |w(t) - u(t)|^2 + |p(t) - L(t)Py(t)|^2 =$$

$$= q^*(t)Q(t)q(t) + 2q^*(t)\overline{a}(t) + \overline{b}(t) \geq 0,$$

where

$$Q(t) = Q^*(t) \geq 0,\ \ t \in I,\ \ q(t) = (\theta(t), z(t,v), z(t_1,v));$$

$$F_1(q(t),t) = \left| w(t) - \int_{t_0}^{t_1} K(t,\tau)Py(\tau)d\tau \right|^2,$$

where $w(t) = v(t) + \overline{T}_1(t)x_0 + \overline{T}_2(t)x_1 + \overline{T}_3(t)d + \overline{\mu}_2(t) + \overline{N}_1(t)z(t_1,v)$, $q(t) = (v(t), x_0, x_1, d, z(t,v),\ z(t_1,v))$, $t \in I$, $\tau \in I$.

Now the problem (16) – (20) can be written as

$$J(v,u,p,x_0,x_1,d){=}\int_{t_0}^{t_1} F_0(q(t),t)dt{+}\int_{t_0}^{t_1} F_1(q(t),t)dt{\to}\inf \qquad (21)$$

under conditions (17) – (20).

Lemma 1. *Let $S \subset R^{2n}$ be a convex set. Then:*

1) functional (21) under conditions (17) – (20) is convex;

2) the partial derivatives of the function $F_0(q,t)$, $F_1(q,t)$ in the variable $q = (v,u,p,x_0,x_1,d,\ z, z(t_1)) \in R^N$, $N = 2m + S + 2n + m_1 + 2(n + m_2)$ satisfy Lipschitz conditions.

Proof. Since $F_0(q,t) = q^*Q(t)q + 2q^*\overline{a}(t) + \overline{b}(t)$, $t \in I$, $q \in R^N$, then $\partial^2 F_0(t,q)/\partial^2 q = 2Q(t) \geq 0$, the function $F_0(t,q)$ with respect to variable $q \in R^N$ is convex. The function $F_1(q_1,t)$ is convex in the variables $(v,x_0,x_1,d,z(t_1)) = q_1$, due to the fact that $\overline{w}^*\overline{w} = \overline{q}_1^* Q_1 \overline{q}_1$, where $Q_1 = Q_1^* \geq 0$.

From the convexity of the function $F_0(q,t)$, $F_1(q,t)$, with taking into account, that

$$z(t,\alpha v_1{+}(1-\alpha)v_2){=}\alpha z(t,v_1){+}(1-\alpha)z(t,v_2), t \in I, \forall v_1,v_2{\in}L_2(I,R^m),$$

we obtain

$$J(\alpha\theta_1 + (1-\alpha)\theta_2) = \int_{t_0}^{t_1} F_0(\alpha\overline{q} + (1-\alpha)\overline{\overline{q}}, t)dt + \int_{t_0}^{t_1} F_1(\alpha\overline{q}_1 +$$

$$+(1-\alpha)\overline{\overline{q}}_1, t)dt \leq \alpha J(\theta_1) + (1-\alpha)J(\theta_2), \ \ \forall\theta_1, \theta_2 \in X.$$

Consequently, the function (21) under conditions (17) – (20) is convex.

The partial derivatives of the function $F(v, u, p, x_0, x_1, d, z, z(t_1)) = F_0(q, t) + F(q_1, t)$ equal to:

$$F_{0q}(q, t) = 2Q(t)q + 2\overline{a}(t), \ F_{1q_1}(q_1, t) = 2Q_1q_1 + 2\overline{a}_1(t). \tag{22}$$

As it follows from (22), the partial derivatives $F(q, t) = F_0(q, t) + F_1(q, t)$ satisfy Lipschitz conditions. Lemma is proved. □

Theorem 3. *Let the matrix* $W(t_0, t_1) > 0$. *Then the functional* (21) *under conditions* (17) – (20) *is continuously Frechet differentiable, the gradient of the functional*

$$J'(\theta) = (J'_v(\theta), J'_u(\theta), J'_p(\theta), J'_{x_0}(\theta), J'_{x_1}(\theta), J'_d(\theta)) \in H$$

at any point $\theta \in X$ *is calculated by the formula*

$$J'_v(\theta) = \frac{\partial F(q,t)}{\partial v} - B^*(t)\psi(t), \ J'_u(\theta) = \frac{\partial F(q,t)}{\partial u}, \ J'_p(\theta) = \frac{\partial F(q,t)}{\partial p},$$

$$J'_{x_0}(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial x_0}dt, \ J'_{x_1}(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial x_1}dt, \tag{23}$$

$$J'_d(\theta) = \int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial d}dt,$$

where $z(t) = z(t, v)$, $t \in I$ *is a solution of differential equation* (17), *and function* $\psi(t)$, $t \in I$ *is a solution of the adjoint system*

$$\dot{\psi} = \frac{\partial F(q,t)}{\partial z} - A^*(t)\psi(t), \ \ \psi(t_1) = -\int_{t_0}^{t_1} \frac{\partial F(q,t)}{\partial z(t_1)}dt. \tag{24}$$

In addition, the gradient $J'(\theta) \in H$ *satisfies Lipschitz condition*

$$\|J'(\theta_1) - J'(\theta_2)\|_H \leq K\|\theta_1 - \theta_2\|_H, \ \ \forall\theta_1, \ \theta_2 \in X. \tag{25}$$

Proof. Let $\theta(t) \in X$, $\theta(t)+\Delta\theta(t) \in X$. Then the increment of the functional

$$\begin{gathered}\Delta J{=}J(\theta{+}\Delta\theta){-}J(\theta){=}\int_{t_0}^{t_1}[F(q(t)+\Delta q(t),t)-\\ -F(q(t),t)]dt=\int_{t_0}^{t_1}[h^*(t)F_v(q(t),t)+\Delta u^*(t)F_u(q(t),t)+\\ +\Delta p^*(t)F_p(q,t)+\Delta x_0^*F_{x0}(q,t)+\Delta x_1^*F_{x_1}(q,t)+\\ +\Delta d^*F_d(q,t)+\Delta z^*(t)F_z(q,t)+\Delta z^*(t_1)F_{z(t_1)}(q,t)]dt+\sum_{i=1}^{8}R_i,\end{gathered} \tag{26}$$

where $\Delta q(t) = (h(t), \Delta u(t), \Delta p(t), \Delta x_0, \Delta x_1, \Delta d, \Delta z, \Delta z(t_1))$.

Hence, by the fact that

$$|\Delta z(t)| \leq \int_{t_0}^{t_1} \|\Phi(t,\tau)B(\tau)\||h(\tau)|d\tau \leq c_1\|h\|_{L_2}.$$

The increment of the functional (26) can be represented as

$$\begin{gathered}\Delta J=\int_{t_0}^{t_1}\{h^*(t)[F_v(q(t),t)-B^*(t)\psi(t)]+\Delta u^*(t)F_u(q(t),t)+\\ +\Delta p^*(t)F_p(q,t)+\Delta x_0^*F_{x_0}(q(t),t)+\Delta x_1^*F_{x_1}(q(t),t)+\\ +\Delta d^*F_d(q(t),t)\}dt+\sum_{i=1}^{8}R_i.\end{gathered} \tag{27}$$

Further, taking into consideration that the partial derivatives $F(q,t)$ satisfy Lipschitz condition, we obtain

$$\sum_{i=1}^{8}|R_i| \leq c_2\|\Delta\theta\|^2, \quad \Delta\theta = (h, \Delta u, \Delta p, \Delta x_0, \Delta x_1, \Delta d).$$

Then from (27) it follows that the gradient $J'(\theta)$ is defined by formula (23), where $\psi(t)$, $t \in I$ is solution (24). Let $\theta_2 = \theta$, $\theta_1 = \theta + \Delta\theta$. Then from (23) it follows

$$\begin{gathered}|J'(\theta_1)-J'(\theta_2)| \leq L_1|\Delta q(t)|+L_2|\Delta\psi(t)|+L_3\|\Delta q\|,\\ \|J'(\theta_1)-J'(\theta_2)\|^2 \leq L_4\|\Delta q\|^2+L_5\int_{t_0}^{t_1}|\Delta\psi(t)|^2dt.\end{gathered} \tag{28}$$

Since

$\Delta\dot{\psi}(t) = [F_z(q(t)+\Delta q(t),t)-F_z(q(t),t)]-A^*(t)\Delta\psi(t),\ \ t \in I,$

$\Delta\psi(t_1) = -\int_{t_0}^{t_1}[F_{z(t_1)}(q(t)+\Delta q(t),t)-F_{z(t_1)}(q(t),t)]dt,$

then by using Granuolla's lemma, we obtain

$$|\Delta\psi(t)| \leq L_6\|\Delta q\|,\ \ t \in I. \tag{29}$$

Equation (25) follows from estimations (28), (29). Theorem is proved. □

For solving the applied problems one can assume that

$$v(t) \in V_1 = \{v(\cdot) \in L_2(I, R^m)/|v(t)| \le \gamma_0,\ \gamma_0 < \infty,\ \text{a.e. } t \in I\},$$

$$d \in D_1 = \{d \in R^m/|d_1| \le \gamma_1 < \infty\},$$

where $\gamma_0 > 0$, $\gamma_1 > 0$ are sufficiency large numbers.

Lemma 2. *Let $v(t) \in V_1$, $d \in D_1$, S be bounded convex closed set. Then the functional* (21) *under conditions* (17) – (20) *gets a lower bound on the set*

$$X_1 = V_1 \times U(t) \times V(t) \times S \times D_1 \subset H,$$

$$J_* = \inf_{\theta \in X} J(\theta) = \inf_{\theta \in X_1} J(\theta) = \min_{\theta \in X_1} J(\theta) = J(\theta_*),\ \ \theta_* \in X_1.$$

Proof. Since the set X_1 is bounded convex closed set in reflexible Banach space H, then X_1 is weakly bicompactly [12]. Continuous and convex functional (21) on the convex set X_1 is weakly semicontinuous below. Then according to Weierstrass' theorem the weakly semicontinuous functional gets a lower bound on the weakly bicompact set. Lemma is proved.

We construct the sequences $\{\theta_n\} \subset X_1$ by the rules:

$$\begin{gathered} v_{n+1} = P_{V_1}[v_n - \alpha_n J'_v(\theta_n)],\ \ u_{n+1} = P_{U_1}[u_n - \alpha_n J'_u(\theta_n)], \\ p_{n+1} = P_V[p_n - \alpha_n J'_p(\theta_n)],\ \ x_{0n+1} = P_{S_1}[x_{0n} - \alpha_n J'_{x_0}(\theta_n)], \\ x_{1n+1} = P_{S_1}[x_{1n} - \alpha_n J'_{x_1}(\theta_n)],\ \ d_{n+1} = P_{D_1}[d_n - \alpha_n J'_d(\theta_n)], \\ 0 < \varepsilon_0 \le \alpha_n \le \frac{2}{K+2\varepsilon_1},\ \ \varepsilon_1 > 0,\ \ n = 0, 1, 2, \dots, \end{gathered} \tag{30}$$

where $P_{W_1}[\cdot]$ is a projection of the point on the set W_1, $K = const$ is a Lipschitz constant (25). In particular, for $\varepsilon_0 = \frac{1}{K}$, $\varepsilon_1 = \frac{K}{2}$, the value $\alpha_n - \frac{1}{K} = const >$ 0. □

Theorem 4. *Let $W(t_0, t_1) > 0$ be the matrix, let the sequence $\{\theta_n\} \subset X_1$ be defined by formula* (30). *Then*

1) the sequence $\{\theta_n\} = \{v_n, u_n, p_n, x_{0n}, x_{1n}, d_n\} \subset X_1$ is minimizing;

2) the sequence $\{\theta_n\} \subset X_1$ weakly converges to the set $X_* \subset X_1 \subset X$, where

$$X_* = \{\theta_* = (v_*, u_*, p_*, x_{0*}, x_{1*}, d_*) \in X_1/J(\theta_*) =$$

$$= J_* = \inf_{\theta \in X_1} J(\theta) = \min_{\theta \in X_1} J(\theta)\};$$

3) $J(\theta_n) - J_* \le \frac{c}{n}$, $\ c = const > 0$, $\ n = 1, 2, \dots$.

Proof. From (30), with taking into account the property of the point projection on the set, we obtain

$$\langle J'_v(\theta_n), v_n - v_{n+1}\rangle_{L_2} \geq \frac{1}{\alpha_n}\|v_n - v_{n+1}\|^2, \tag{31}$$

$$\left\langle J'_u(\theta_n), u_n - u_{n+1}\right\rangle_{L_2} \geq \frac{1}{\alpha_n}\|u_n - u_{n+1}\|^2,$$

$$\left\langle J'_p(\theta_n), p_n - p_{n+1}\right\rangle_{L_2} \geq \frac{1}{\alpha_n}\|p_n - p_{n+1}\|^2,$$

$$\left\langle J'_{x_0}(\theta_n), x_{0n} - x_{0n+1}\right\rangle_{R^n} \geq \frac{1}{\alpha_n}\|x_{0n} - x_{0n+1}\|^2,$$

$$\left\langle J'_{x_1}(\theta_n), x_{1n} - x_{1n+1}\right\rangle_{R^n} \geq \frac{1}{\alpha_n}\|x_{1n} - x_{1n+1}\|^2,$$

$$\left\langle J'_d(\theta_n), d_n - d_{n+1}\right\rangle_{L_2} \geq \frac{1}{\alpha_n}\|d_n - d_{n+1}\|^2.$$

Since the functional $J(\theta) \in C^{1,1}(X_1)$, then the inequality is valid

$$J(\theta_n) - J(\theta_{n+1}) \geq \left\langle J'(\theta_n), \theta_n - \theta_{n+1}\right\rangle - \frac{K}{2}\|\theta_n - \theta_{n+1}\|^2. \tag{32}$$

Then from (31), (32) it follows that

$$J(\theta_n) - J(\theta_{n+1}) \geq \left(\frac{1}{\alpha_n} - \frac{K}{2}\right)\|\theta_n - \theta_{n+1}\|^2 \geq \varepsilon_1\|\theta_n - \theta_{n+1}\|^2, \tag{33}$$

where $\frac{1}{\alpha_n} \geq \frac{K+2\varepsilon_1}{2}$, $\frac{1}{\alpha_n} - \frac{K}{2} \geq \varepsilon_1$. From (33) follows, that the numeric sequence $\{J(\theta_n)\}$ decreases strictly. Since the value of the functional $J(\theta_n)$ is bounded from below, i.e. $J(\theta_n) \geq 0$, $\forall\theta$, $\theta \in X_1$, then the numeric sequence $\{J(\theta_n)\}$ is converged. Consequently, $\lim\limits_{n\to\infty}[J(\theta_n) - J(\theta_{n+1})] = 0$. Then by transferring to the limit from (33) we get $\|\theta_n - \theta_{n+1}\| \to 0$, at $n \to \infty$.

We show, that the sequence $\{\theta_n\} \subset X_1$ is minimizing. As it follows from lemma 1, the functional $J(\theta) \in C^{1,1}(X_1)$ is convex. Then necessarily and sufficiently the inequality is satisfied

$$J(\theta_2) - J(\theta_1) \leq \left\langle J'(\theta_2), \theta_2 - \theta_1\right\rangle_{L_2}, \ \ \forall\theta_1,\ \theta_2 \in X_1.$$

From the inequality at $\theta_1 = \theta_* \in X_* \subset X_1$, $\theta_2 = \theta_n \in X$, we get

$$\begin{gathered} J(\theta_n) - J(\theta_*) \leq \langle J'(\theta_n), \theta_n - \theta_*\rangle_{L_2} = \\ \langle J'(\theta_n), \theta_n - \theta_{n+1}\rangle - \langle J'(\theta_n), \theta_* - \theta_{n+1}\rangle . \end{gathered} \tag{34}$$

From (31) at $\theta = \theta_n$, we obtain

$$\left\langle J'(\theta_n), \theta_* - \theta_{n+1} \right\rangle \geq \frac{1}{\alpha_n} \left\langle \theta_n - \theta_{n+1}, \theta_* - \theta_{n+1} \right\rangle . \tag{35}$$

From (34), (35), we get

$$J(\theta_n) - J(\theta_*) \leq l\|\theta_n - \theta_{n+1}\|, \quad l = const > 0, \tag{36}$$

where r is a diameter of the set X_1, $\|\theta_* - \theta_{n+1}\| \leq r$, $\frac{1}{\alpha_n} \leq \frac{1}{\varepsilon_0}$, $0 \leq \varepsilon_0 \leq \alpha_n$.

Since $\|\theta_n - \theta_{n+1}\| \to 0$ at $n \to \infty$, then from (36) $\lim\limits_{n\to\infty} J(\theta_n) = J(\theta_*) = J_* = \inf\limits_{\theta \in X_1} J(\theta)$ follows. This means, that the sequence $\{\theta_n\} \subset X_1$ is minimizing.

We show, that the sequence $\{\theta_n\} \subset X_1$ weakly converges to the point $\theta_* \in X_*$. In fact, since the set X_1 is weak bicompactly, then the sequence $\{\theta_n\} \subset X_1$. Consequently, the sequence $\{\theta_n\} \subset X_1$ has at least one subsequence $\{\theta_{k_m}\} \subset X_1$ such that $\theta_{k_m} \to \theta_*$ at $m \to \infty$, moreover $\theta_* \in X_1$. Since the sequence $\{J(\theta_n)\}$ converges to $J(\theta_*)$, then the numeric sequence $J(\theta_{k_m})$ converges to the number $J(\theta_*)$ i.e. $\lim\limits_{m\to\infty} J(\theta_{k_m}) = J(\theta_*)$.

From the inequality (33), (36) it follows, that

$$a_n \leq l\|\theta_n - \theta_{n+1}\|, \quad a_n - a_{n+1} \geq \varepsilon_1\|\theta_n - \theta_{n+1}\|, \quad a_n = J(\theta_n) - J(\theta_*).$$

Then $a_n \leq \frac{1}{A_n}$, $n = 1, 2, \ldots$, $A = \frac{\varepsilon_1}{l^2} > 0$ (see. [5]). Hence it follows the third statement of the lemma. Theorem is proved. □

5. Solution Existence

Let $\theta_* = (v_*(t), u_*(t), p_*(t), x_{0*}, x_{1*}, d_*) \in X_1$ be a solution of the optimization problem (16) – (20). Then

$$w_*(t) = v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t)z(t_1, v_*), \; t \in I, \; v_*(t) \in V_1, t \in I,$$

$$y_*(t) = z(t, v_*) + \lambda_2(t, \xi_{0*}, \xi_{1*}) + N_2(t)z(t_1, v_*), \;\; p_*(t) \in V(t), \;\; t \in I,$$

where

$$\xi_{0*} = (x_{0*}, O_{m_2,1}), \quad \xi_{1*} = (x_1^*, \overline{c}_*),$$

$$\overline{c}_* = (c_1 - d_{1*}, \ldots, c_{m_1} - d_{m_1*}, c_{m_1+1}, \ldots, c_{m_2}),$$

$$d_* = (d_{1*}, \ldots, d_{m_1*}) \in D_1, \quad (x_{0*}, x_{1*}) \in S.$$

We notice, that the value $J(\theta) \geq 0$, $\forall \theta$, $\theta \in X_1$. In particular, the value $J(\theta_*) = 0$.

Theorem 5. *Let the matrix be* $W(t_0, t_1) > 0$. *For existence of a solution of the boundary value problem* (1) – (5) *it is necessary and sufficient that the value* $J(\theta_*) = 0$, *where* $\theta_* \in X_1$ *is the solution of the optimization problem* (16) – (20).

Proof. Let the value be $J(\theta_*) = 0$. We show, that the boundary value problem (1) – (5) has a solution $x_*(t) = Py_*(t)$, $t \in I$. As it follows from optimization problem (16) – (20) the value $J(\theta_*) = 0$ if and only if

$$w_*(t) = v_*(t) + \lambda_1(t, x_{0*}, x_{1*}) + N_1(t)z(t_1, v_*) = u_*(t), \quad t \in I,$$

$$p_*(t) = L(t)Py_*(t), \quad t \in I, \quad w_*(t) = \int_{t_0}^{t_1} K(t,\tau)Py_*(\tau)d\tau,$$

where $u_*(t) \in U(t)$, $p_*(t) \in V(t)$, $(x_{0*}, x_{1*}) \in S$, $d_* \in D_1$. The function $y_*(t)$, $t \in I$ is a solution of the linear controllable system (9) – (11). Consequently, the equality is valid

$$\dot{y}_*(t) = A(t)y_*(t) + B(t)u_*(t) + \mu_1(t), \quad t \in I, \tag{37}$$

$$y_*(t_0) = \xi_{0*} = (x_{0*}, O_{m_2,1}), \quad y_*(t_1) = \xi_{1*} = (x_{1*}, \overline{c}_*), \tag{38}$$

$$(x_{0*}, x_{1*}) \in S, \quad d_* \in D_1, \quad u_*(t) \in U(t), \quad t \in I. \tag{39}$$

From the equality

$$w_*(t) = u_*(t) = \int_{t_0}^{t_1} K(t,\tau)Py_*(\tau)d\tau \in U(t), \quad t \in I$$

and relations (37) – (39) it follows that $\xi_*(t) = y_*(t)$, $Py_*(t) = x_*(t)$, $t \in I$, where $\xi_*(t) = (x_*(t), \eta_*(t))$, $x_*(t_0) = x_{0*}$, $x_*(t_1) = x_{1*}$, $\eta_*(t_0) = 0$, $\eta_*(t_1) = \overline{c}_*$. Since $P\xi_*(t) = Py_*(t) = x_*(t)$, $t \in I$, then the function

$$x_*(t) = Py_*(t) = P[z(t, v_*) + \lambda_2(t, \xi_{0*}, \xi_{1*}) + N_2(t)z(t_1, v_*)], \quad t \in I$$

is the solution of the boundary value problem (1) – (5). We notice, that from inclusion $p_*(t) \in V(t)$, $t \in I$, $u_*(t) \in U(t)$ follows the function $x_*(t) \in G(t)$, $t \in I$. From the conditions that the function $\eta_*(t)$, $t \in I$ satisfies to the conditions $\eta_*(t_0) = 0$, $\eta_*(t_1) = \overline{c}_*$ follows satisfication of the integral constraints (4), (5). Theorem is proved. □

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