International Journal of Pure and Applied Mathematics
Volume 113 No. 4 2017, 47-52
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i4.5

ijpam.eu

THE CAUCHY PROBLEM FOR THE MC KENDRICK-VON FOERSTER LOADED EQUATION

Anatoly Kh. Attaev
Institute of Applied Mathematics and Automation
RUSSIA

Abstract: We study the Cauchy problem for equations of type $u_x + u_t + c\,u = \lambda u(x_0, t)$ with initial value given in a characteristic and a non-characteristic manifolds. Necessary conditions for the existence and uniqueness of the solution are found, and their explicit analytical representations are given.

AMS Subject Classification: 35F10
Key Words: Cauchy problem, loaded equation, infuence domain, definition domain, dependence domain

1. Introduction

It is known [1, p. 244] the model equation describing closed population dynamics when an individual is withdrawn at each time t at age x_0 has the form of

$$u_x + u_t + c\,u = \lambda u(x_0, t), \tag{1}$$

where c, λ, x_0 are given real constants, and $\lambda \neq 0$.

Equation (1) is of the class of loaded differential equations [2, p. 17]. The loaded part of a differential equation has a significant effect on the correct setting of certain initial value problems. In this regard, we note only some works in which this thesis is confirmed [4]-[9]. Especially this effect is evident for the loaded hyperbolic equations, as well as for the first order partial differential equations. So there is an interest to issues related to the dependence domain

Received: December 15, 2016
Revised: January 30, 2017
Published: April 6, 2017

of an arbitrary point (x,t), to the influence domain of the initial values and the domain of solution $u(x,t)$ defined by the initial values, as well as to the description of the manifolds that can be support for initial values. In this paper, we discuss these issues using equation (1) as a model. We refer to equation (1) as Mc Kendrick-Von Foerster equation [1] when $\lambda = 0$ and $c = c(x,t)$.

2. Solution representation to the equation (1)

Introducing the notation $u(x_0,t) = f(t)$, equation (1) can be written as follows

$$u_x + u_t + c\,u = \lambda f(t).$$

Obviously, any regular solution of the last equation has the form of

$$u(x,t) = g(x-t)e^{-\frac{c}{2}(x+t)} + \lambda \int\limits_0^t e^{c(\xi-y)} f(\xi)d\xi, \tag{2}$$

where $g(x)$ is an arbitrary, continuously differentiable function.

Substituting (2) into (1), after elementary transformations, we obtain a representation for the function $g(y)$:

$$g(t) = e^{\frac{c}{2}(2x_0-t)} f(x_0-t) - \lambda \int\limits_0^{x_0-t} e^{c(\xi-x_0+y)} f(\xi)d\xi.$$

Now we have to substitute the obtained representation for $g(t)$ in (2). The result is

$$u(x,t) = e^{c(x_0-x)} f(x_0-x+t) + \lambda \int\limits_{x_0-x+t}^{t} e^{c(\xi-t)} f(\xi)d\xi. \tag{3}$$

We have shown that any regular solution of equation (1) has the form (3) for any arbitrary continuously differentiable function $f(x)$. Moreover, it should be noted, that it is consistent with our notation introduced above.

Indeed, if we substitute $x = x_0$ in (3), then we obtain $u(x_0,t) = f(t)$.

3. The Cauchy problem

We seek a regular solution to equation (1) in case when the initial values support is the line $y = 0$, i.e. non-characteristic manifold, and in case when the initial values support is the line $y = x$, i.e. characteristic manifold. We call these problems problem 1 and problem 2, respectively.

Problem 1. Find a regular solution of equation (1) which satisfies

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty. \tag{4}$$

Theorem 1. *Let $\lambda \neq 0$, $\varphi(x) \in C^1(]-\infty, \infty[)$, then the unique solution to equation* (1) *can be represented as follows*

$$u(x,t) = e^{-ct}[\varphi(x-t) + \lambda \int\limits_{x_0-t}^{x_0} e^{\lambda(t+\xi-x_0)}\varphi(\xi)d\xi. \tag{5}$$

Proof. Let equation (3) satisfy condition (4), then we get

$$e^{c(x_0-x)} f(x_0 - x) + \lambda \int\limits_{x_0-x}^{0} e^{c\xi} f(\xi)d\xi = \varphi(x),$$

hence

$$e^{cx} f(x) - \lambda \int\limits_{0}^{x} e^{c\xi} f(\xi)d\xi = \varphi(x_0 - x).$$

In order to find $f(x)$ we differentiate the last equation with respect to x, and obtain the following first order ordinary differential equation

$$\left[e^{(c-x)} f(x)\right]' = -e^{-\lambda x}\varphi'(x_0 - x).$$

In view of $f(0) = \varphi(x_0)$ we have

$$f(x) = e^{-cx}\left[\varphi(x_0 - x) + \lambda \int\limits_{0}^{x} e^{\lambda(x-t)}\varphi(x_0 - t)dt\right].$$

Substituting the last expression for $f(x)$ in (4), we get

$$u(x,t) = e^{-ct}\left[\varphi(x-t) + \lambda \int\limits_{0}^{x_0-x+t} e^{\lambda(x_0+x+t-\xi)}\varphi(x_0 - \xi)d\xi\right] +$$

$$+\lambda\int\limits_{x_0-x+t}^{t} e^{-ct}\varphi(x_0-\xi)d\xi+\lambda^2e^{-ct}\int\limits_{x_0-x+t}^{t}\int\limits_{0}^{\xi} e^{\lambda(\xi-\eta)}\varphi(x_0-\eta)d\eta\, d\xi.$$

Changing the integration order in the double integral and performing obvious derivations, we obtain (5). □

Problem 2. Find a regular solution to equation (1) satisfying the condition

$$u(x,x)=\psi(x),\quad -\infty<x<\infty. \tag{6}$$

Theorem 2. *Let $\psi(x)\in C^2(]-\infty,\infty[)$, $\lambda\neq 0$ and*

$$(c-\lambda)\psi(x_0)+\psi'(x_0)=0. \tag{7}$$

Then there exists a unique solution of Problem 2.

Proof. When (3) meets (6), we can obtain

$$e^{c(x_0-x)}f(x_0)+\lambda\int\limits_{x_0}^{x} e^{c(\xi-x)}f(\xi)d\xi=\psi(x). \tag{8}$$

Differentiating (8) with respect to x, and, summing term-wise the obtained result and equation (8) multiplied by c, we get

$$\lambda f(x)=c\psi(x)+\psi'(x).$$

From the last relation and formula (3), it follows that

$$u(x,t)=\psi(t)-e^{c(x_0-x)}\psi(x_0-x+t)+$$
$$+\frac{1}{\lambda}e^{c(x_0-x)}[c\,\psi(x_0-x+t)+\psi'(x_0-x+t)]. \tag{9}$$

Thus it is proved that if there is a solution of problem 2, then it can be represented by (9).

It remains to establish that the function given by (9) is the solution of Problem 2. It follows from (6) that

$$u(x,x)=\psi(x)-e^{c(x_0-x)}\psi(x_0)+\frac{1}{\lambda}e^{c(x_0-x)}[c\,\psi(x_0)+\psi'(x_0)]=$$
$$=\psi(x)-e^{c(x_0-x)}[-\psi(x_0)+\frac{c}{\lambda}\psi(x_0)+\frac{1}{\lambda}\psi'(x_0)]=$$
$$=\psi(x)-e^{c(x_0-x)}\frac{1}{\lambda}[(c-\lambda)\psi(x_0)+\psi'(x_0)].$$

Hence function (9) satisfies condition (6) if and only if condition (7) is held. One can verify that function (9) is a regular solution of equation (1). □

4. Domain of the dependence, domain of the influence and domain of the definition

In accordance with [3, p. 158], we introduce the notions for the domains of dependence, influence and determination. The set of points of the line on which the initial values are set, and which possesses the property that the value of the solution of equation (1) at the point (x, t) of the space $\mathbb{R}^2$ is completely determined by these initial values, is called the domain of dependence for the point (x, t). Now let the initial value support be bounded by the segment $[0, l]$. The set of points of $\mathbb{R}^2$ in which the value of $u(x, t)$ are effected by the initial values given in $[0, l]$ is called the domain of influence. Finally, the set of points of $\mathbb{R}^2$ in which the values of $u(x, t)$ are determined by the given initial values bounded in $[0, l]$ is called the domain of definition. Denote these sets for Problem 1 by Ω_0^0, Ω_1^0, Ω_2^0 when $\lambda = 0$, and by Ω_0^1, Ω_1^1, Ω_2^1 when $\lambda \neq 0$, and, for the problem 2, by Ω_0^2, Ω_1^2, Ω_2^2 in case when $\lambda \neq 0$.

As we know the solution of the Cauchy problem for equation (1) when $\lambda = 0$ and (4) has the form of

$$u(x, t) = e^{-ct}\varphi(x - t). \tag{10}$$

It follows from formula (5) when $\lambda = 0$.

The domains of dependence

$$\Omega_0^0 = \{x - t\},$$

$$\Omega_1^0 = \{x - t\} \cup [x_0 - t, x_0],$$

$$\Omega_2^0 = \{t\} \cup \{x_0 - x + t\}.$$

The domain of influence as we can see from formulas (10), (5) and (9) respectively has the form of:

$$\Omega_0^1 = \{(x, t) : 0 < x - t < l\},$$

$$\Omega_1^1 = \{(x, t) : 0 < x - t < l\} \cup \{(x, t) : x_0 - l < t < x_0\},$$

$$\Omega_2^1 = \{(x, t) : x_0 - l < x - t < x_0\} \cup \{(x, t) : 0 < t < l\}.$$

The domain of definition as it is seen again from formulas (10), (5) and (9) respectively has the form of:

$$\Omega_0^2 = \{(x, t) : 0 < x - t < l\},$$

$$\Omega_1^2 = \{(x, t) : 0 < x - t < l\} \cap \{(x, t) : x_0 - l < t < x_0\},$$

$$\Omega_2^2 = \{(x,t) : x_0 - l < x - t < x_0\} \cap \{(x,t) : 0 < t < l\}.$$

Brief abstract of this work has been published in the Materials of the workshop "Differential operators and modeling of complex systems" (April 7-8, 2017, Almaty, Kazakhstan) [10].

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