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A SAMARSKII-IONKIN PROBLEM FOR TWO-DIMENSIONAL PARABOLIC EQUATION WITH THE CAPUTO FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract: In the work, the authors consider a Samarskii-Ionkin type non-local problem for a fourth-order partial differential equation with the Caputo fractional differential operator in a spatial domain. Applying the method of separation of variables the authors prove the theorem of the existence and uniqueness of the regular solution of these problems.

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1. Problem Definition

At present time, the theory of boundary value problems for fractional differential equations is an emerging part of the general theory of differential equations. The reason of this is application of this theory to viscoelasticity, dynamic processes, control theory, electrochemistry, and diffusion-related processes. More detailed information can be found in [1-3].

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Note, that Samarskii-Ionkin type problems in plane regions were studied in [4-7]. Similar problems (as well as inverse problems) for partial differential equations in three independent variables were considered in [7-9].

Let us consider the equation

$$ {}_C D_{0t}^{\alpha} u + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y,z), \tag{1} $$

where ${}_C D_{0t}^{\alpha} u(x,y,t) = I_{0t}^{1-\alpha}\left(\frac{\partial u(x,y,t)}{\partial t}\right)$, $\alpha \in (0,1]$ is a Caputo fractional differential operator [1, p. 92]. We denote $\Omega = \Omega_{xy} \times (0,p)$,

$$ \Omega_{xy} = \{(x,y):\ 0 < x,y < 1\}, \Omega_{zt} = \{(z,t):\ 0 < z < 1;\ 0 < t < p\}. $$

Problem A. Find the function, $u(x,y,t)$, which:

1) is continuous in domain $\bar{\Omega}$ together with its derivatives mentioned in the boundary conditions;

2) satisfies equation (1) in the domain Ω ;

3) satisfies the following conditions:

$$ u(x,y,0) = \phi(x,y),\ (x,y) \in \bar{\Omega}_{xy}, \tag{2} $$

$$ \frac{\partial^k u}{\partial x^k}\,|_{x=0} = 0,\ \frac{\partial^\ell u}{\partial x^\ell}\,|_{x=0} = \frac{\partial^\ell u}{\partial x^\ell}\,|_{x=1}\ ,\ (y,t) \in \bar{\Omega}_{y,t}, \tag{3} $$

$$ \frac{\partial^k u}{\partial y^k}\,|_{y=0} = \frac{\partial^k u}{\partial y^k}\,|_{y=1} = 0,\ (x,t) \in \bar{\Omega}_{x,t}, \tag{4} $$

where $k = 1,3,\ \ell = 0,2;\ f(x,y,t),\ \phi(x,y)$ are given functions.

2. Investigating Problem A.

As problem is a linear problem we can represent its solutions as the sum of the solutions of the following two problems: the problem for the homogeneous equation

$$ {}_C D_{0t}^{\alpha} u + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0, \tag{5} $$

with conditions (2)-(4) and the problem (1),(3),(4) provided

$$u(x,y,0)=0.$$

Let us consider the problem (5),(2) - (4). We try to find a solution of the form

$$u(x,t)=Z(x,y)\cdot T(t).$$

Substituting this expression into equation (5) and boundary conditions (3) and (4), and separating variables, we obtain the following spectral problem:

$$\frac{\partial^4 Z}{\partial x^4}+\frac{\partial^4 Z}{\partial y^4}-\sigma Z=0,(x,y)\in\Omega_{xy}, \tag{6}$$

$$\frac{\partial^k Z}{\partial x^k}\,|_{x=0}=0,\ \frac{\partial^\ell Z}{\partial x^\ell}\,|_{x=0}=\frac{\partial^\ell Z}{\partial x^\ell}\,|_{x=1},y\in[0,1], \tag{7}$$

$$\frac{\partial^k Z}{\partial y^k}\,|_{y=0}=\frac{\partial^k Z}{\partial y^k}\,|_{y=1}=0,\ x\in[0,1], \tag{8}$$

where $k=1,3,\ \ell=0,2;\ \sigma$ is a spectral parameter.

Let us consider problem (6)-(8). Suppose its solution has the following form

$$Z(x,y)=X(x)\cdot Y(y). \tag{9}$$

Substituting (9) into equation (6) and boundary conditions (7) and (8), we obtain the following problems in unknown functions $X(x),Y(y)$:

$$X^{IV}(x)-\lambda X(x)=0,\ 0<x<1, \tag{10}$$

$$X'(0)=X'''(0)=0,\ X(0)=X(1),\ X''(0)=X''(1), \tag{11}$$

$$Y^{IV}(y)-\mu Y(y)=0,\ 0<y<1, \tag{12}$$

$$Y'(0)=Y'''(0)=Y'(1)=Y'''(1)=0, \tag{13}$$

where $\mu=\sigma-\lambda$.

We know that the problem (12), (13) has eigenvalues and eigenfunctions [10] of the form

$$\mu_k=(k\pi)^4,\ Y_0(y)=1,\ Y_k(y)=\sqrt{2}\cos(k\pi y),\ k\in N. \tag{14}$$

The problem (10), (11) has been investigated in [12], where the authors found eigenvalues and corresponding to them eigenfunctions and associated functions for this problem, which have the following form

$$\lambda_n = (2n\pi)^4, X_0(x) = 1,\ X_{2n-1}(x) = \cos 2\pi nx,$$

$$X_{2n}(x) = x \sin 2\pi nx, n \in N. \tag{15}$$

Therefore, according to (9), the eigenvalues and corresponding to them eigenfunctions of the problem (6)-(8) have the following form

$$\sigma_{nk} = \lambda_n + \mu_k = (2n\pi)^4 + (k\pi)^4,\ n, k \in N_0, \tag{16}$$

$$Z_{0k}(x, y) = Y_k(y),\ Z_{2n-1\,k}(x, y) = X_{2n-1}(x)Y_k(y),$$

$$Z_{2n\,k}(x, y) = X_{2n}(x)Y_k(y). \tag{17}$$

Note that the problem (6)(8) is not self-adjoint in terms of the scalar product, $(\xi, \eta) = \iint_{\Omega_{xy}} \xi(x, y)\eta(x, y)dxdy$. It is easy to verify that its adjoint problem is the following problem:

$$\frac{\partial^4 W}{\partial x^4} + \frac{\partial^4 W}{\partial y^4} - \sigma W = 0, (x, y) \in \Omega_{xy}, \tag{18}$$

$$\frac{\partial^k W}{\partial x^k}\,|_{x=1} = 0,$$

$$\frac{\partial^\ell W}{\partial x^\ell}\,|_{x=0} = \frac{\partial^\ell W}{\partial x^\ell}\,|_{x=1}, y \in [0, 1],\ k = 0, 2,\ \ell = 1, 3, \tag{19}$$

$$\frac{\partial^k W}{\partial y^k}\,|_{y=0} = \frac{\partial^k W}{\partial y^k}\,|_{y=1} = 0,\ x \in [0, 1],\ k = 1, 3. \tag{20}$$

It is not difficult to show that the problem (18)-(20) has eigenvalues of the form (16), the corresponding eigenfunctions and associated functions have the following form:

$$W_{0k}(x, y) = X_0^*(x)Y_k(y),\ W_{2n-1\,k}(x, y) = X_{2n-1}^*(x)Y_k(y),$$

$$W_{2n\,k}(x, y) = X_{2n}^*(x)Y_k(y), \tag{21}$$

where $n, k \in N_0$,

$$X_0^*(x) = 2(1-x),\ X_{2n-1}^*(x) = 4(1-x)\cos 2\pi n x,$$

$$X_{2n}^*(x) = 4\sin 2\pi n x. \tag{22}$$

Note that the completeness and basis properties of the systems (15) and (22) were studied in [12], which implies that these systems are complete and form a Riesz basis in $L_2(0,1)$.

A similar statement holds for the systems (17) and (21).

Lemma 1. *The systems of functions (17) and (21) are biorthogonal systems.*

Lemma 1 can be proved by direct evaluation of the integrals [13].

Lemma 2. *The systems of functions (17) and (21) form a Riesz basis in $L_2(\Omega_{xy})$.*

The proof of the lemma follows from the basis properties of the systems (15) and (22) and from the following lemma [14]:

Lemma 3. *Let for any fixed number $n = 0, 1, 2, ...$, $\{\phi_n^k(x)\}$, $k = 0, 1, 2, ...$ be a complete orthonormal system of functions on $[0, \pi]$, and let the system $\psi_n(y)$, $n = 0, 1, 2, ...$ form a Riesz basis in $L_2[0 \le y \le 2\pi]$. Then the system of functions $u_{n\,k}(x, y) = \phi_n^k(x)\psi_n(y)$, $n = 0, 1, 2, ...$, $k = 0, 1, 2, ...$ forms a Riesz basis in $L_2([0 \le x \le \pi] \times [0 \le y \le 2\pi])$.*

3. Existence and Uniqueness of the Solution of Problem A.

Let us consider the problem (5), (2)-(4). Since the systems (17) and (21) form a Riesz basis in $L_2(\Omega_{xy})$, the function $u(x, y, t)$can be represented in the form of a biorthogonal series

$$u(x,y,t) = \sum_{k=0}^{\infty} T_{0\,k}(t) Z_{0\,k}(x,y) +$$

$$+ \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\Big(T_{2n-1\,k}(t) Z_{2n-1\,k}(x,y) + T_{2n\,k}(t) Z_{2n\,k}(x,y)\Big), \tag{23}$$

where $T_{0\,k}(t)$, $T_{2n-1\,k}(t)$, $T_{2n\,k}(t)$, $n \in N$, $k \in N_0$ are unknown functions.

Using this and taking into account the biorthogonal property of the systems (17) and (21), and the condition (2) from (23) relative to $T_{0\,k}(t)$, $T_{2n-1\,k}(t)$, $T_{2n\,k}(t)$, we obtain the following problems:

$$\begin{cases} {}_C D_{0t}^{\alpha} T_{0\,k}(t) + \mu_k T_{0\,k}(t) = 0 \\ T_{0\,k}(0) = \phi_{0k} \end{cases},$$

$$\begin{cases} {}_C D_{0t}^{\alpha} T_{2n\,k} + \sigma_{nk} T_{2n\,k}(t) = 0 \\ T_{2n\,k}(0) = \phi_{2n\ k} \end{cases},$$

$$\begin{cases} {}_C D_{0t}^{\alpha} T_{2n-1\,k} + \sigma_{nk} T_{2n-1\,k}(t) = 4\sqrt[4]{\lambda_n}^3 T_{2n\ k}(t) \\ T_{2n-1\,k}(0) = \phi_{2n-1\ k} \end{cases},$$

whose solutions are the following functions [1,p.231]:

$$T_{0\,k}(t) = \phi_{0\,k} E_{\alpha}(-\mu_k t^{\alpha}),\ T_{2n\ k}(t) = \phi_{2n\ k} E_{\alpha}(-\sigma_{nk} t^{\alpha}),$$

$$T_{2n\,-1\,k}(t) = \phi_{2n\,-1\,k} E_{\alpha}(-\sigma_{nk} t^{\alpha}) +$$

$$+4\sqrt[4]{\lambda_n}^3 \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^{\alpha}) T_{2n\ k}(\tau)d\tau. \tag{24}$$

Here

$$\begin{array}{l} \phi_{0\,k} = (\phi(x,y), W_{0\,k}(x,y)), \\ \phi_{2n-1\,k} = (\phi(x,y), W_{2n-1\,k}(x,y)) \ , n \in N,\ k \in N_0. \\ \phi_{2n\,k} = (\phi(x,y), W_{2n\,k}(x,y)), \end{array} \tag{25}$$

$E_{\alpha,\beta}(z)$ is a Mittag-Leffler type function [1,p.32], which has the form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\,k+\beta)},\ E_{\alpha,1}(z) = E_{\alpha}(z),\ z,\alpha,\beta \in \mathbb{C},$$

$$Re(\alpha) > 0.$$

Substituting the functions found into (23), we obtain the following result: the solution of the problem (5), (2)-(4) can be represented in the following form

$$u(x,y,t) =$$

$$\sum_{k=0}^{\infty} u_{0\,k}(x,y,t) + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} (u_{2n-1\,k}(x,y,t) + u_{2n\ k}(x,y,t)), \tag{26}$$

where

$$u_{0\,k}(x,y,t) = \phi_{0\,k} E_\alpha(-\mu_k t^\alpha) Z_{0\,k}(x,y), \tag{27}$$

$$u_{2n\,-1\,k}(x,y,t) = \left(\phi_{2n\,-1\,k} E_\alpha(-\sigma_{nk} t^\alpha) + 4\lambda_n^{3/4} \phi_{2n\ k} F_{nk}(t)\right) Z_{2n-1\,k}(t), \tag{28}$$

$$u_{2n\ k}(x,y,t) = \phi_{2n\ k} E_\alpha(-\sigma_{nk} t^\alpha) Z_{2n\ k}(x,y), \tag{29}$$

$$F_{nk}(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^\alpha) E_\alpha(-\sigma_{nk}\tau^\alpha) d\tau,$$

$$n \in N,\ k \in N_0. \tag{30}$$

The following theorem holds.

Theorem 4. *Let a function $\phi(x,y)$ satisfy the following conditions:*

$$\frac{\partial \phi^5(x,y)}{\partial x^4 \partial y} \in C(\bar{\Omega}_{xy}),\ \frac{\partial \phi^5(x,y)}{\partial x \partial y^4} \in C(\bar{\Omega}_{xy}),$$

$$\phi(0,y) = \phi(1,y),\ \phi_{xx}(0,y) = \phi_{xx}(1,y),\ \phi_x(0,y) = \phi_{xxx}(0,y) = 0,$$
$$y \in [0,1],$$
$$\phi_y(x,0) = \phi_y(x,1) = \ \phi_{yyy}(x,0) = \phi_{yyy}(x,1) = 0,\ x \in [0,1].$$

Then the solution of the problem (5), (2)-(4) exists, it is unique, and it can be represented as the sum of the series (23).

Proof. **Uniqueness of the solution.** Let $u_1(x,y,t)$ and $u_2(x,y,t)$ be two solutions of the problem (5), (2)-(4) in the domain Ω. Then the function $u(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$ satisfies equation (2), conditions (3),(4) and $u(x,y,0) = 0$, $(x,y) \in \bar{\Omega}_{xy}$. Taking (24) into account, we have $(\phi(x,y) = 0)$

$$T_{0\,k}(t) = 0;\ T_{2n-1\,k}(t) = 0;\ T_{2n\,k}(t) = 0, n \in N,\ k \in N_0.$$

where

$$T_{0\,k}(t) = (u(x,y,t), W_{0\,k}(x,y)),$$
$$T_{2n-1\,k}(t) = (u(x,y,t), W_{2n-1\,k}(x,y)),$$
$$T_{2n\,k}(t) = (u(x,y,t), W_{2n\,k}(x,y)),$$

it implies that the function $u(x,y,t)$ is orthogonal to the set of functions (21), which is complete and forms a basis in $L_2(\Omega_{xy})$. Hence, $u(x,y,t) = 0$ in Ω . Since $u \in C(\bar{\Omega})$, we have $u(x,y,t) = 0$ in $\bar{\Omega}$.

Existence of the solution. By definition, the function $u(x,y,t)$ satisfies the condition (5), the initial and boundary conditions (2)-(4). First, we are to prove that $u(x,y,t) \in C(\overline{\Omega})$.

(17) and (21) imply that $|Z_{n\,k}(x,y)| \le \sqrt{2}$, $|W_{n\,k}(x,y)| \le 4\sqrt{2}$. Using the properties of the Mittag-Leffler function, we have [10]:

$$E_\alpha(-\sigma_{nk}t^\alpha) \le M_1,\ E_{\alpha,\alpha}(-\sigma_{nk}t^\alpha) \le M_2,$$

where M_1, M_2 are positive numbers.

Then (27) and (29) imply that

$$|u_{0k}(x,y,t)| \le \sqrt{2}M_1|\phi_{0k}|,\ |u_{2n\,k}(x,y,t)| \le \sqrt{2}M_1|\phi_{2n\,k}|\,.$$

Using (31) and taking into account the formula

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}E_{\alpha,\mu}(\lambda t^\alpha)t^{\mu-1}dt = z^{\mu+\alpha-1}E_{\alpha,\mu+\alpha}(\lambda z^\alpha),$$

$$\alpha > 0,\ \mu > 0,$$

from [15,p.120] we can easily show that

$$|F_{nk}(t)| \le \frac{M_3}{\sigma_{nk}},\ M_3 = M_2(1+M_1),\ t \in [0,1].$$

Then using (28) we have

$$|u_{2n\,-1\,k}(x,y,t)| \le M_4\left(|\phi_{2n\,-1\,k}| + \phi_{2n\,k}|\right),$$

$$M_4 = \max\{\sqrt{2}M_1, 4\sqrt{2}M_3\}.$$

Therefore, using (26)-(30) we obtain the following estimate:

$$|u(x,y,t)| \le \sqrt{2}M_1\sum_{k=0}^{\infty}|\phi_{0\,k}| + M_4\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}|\phi_{2n\,-1\,k}| +$$

$$(\sqrt{2}M_1 + M_4)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}|\phi_{2n\,k}|.$$

Using (25), taking into account the conditions imposed on $\phi(x,y)$, and applying Cauchy-Bunyakovskiy and Bessel's inequalities, we obtain the following estimates:

$$\sum_{k=1}^{\infty}|\phi_{0\,k}| \le \frac{\sqrt{6}}{9}\,\|\phi_y\|_{L_2(\Omega_{\mathsf{xy}})},$$

$$\sum_{n=1}^{\infty} |\phi_{2n-1\,0}| \leq \frac{\sqrt{3}}{3} \|\phi\|_{L_2(\Omega_{\mathrm{xy}})} + \frac{\sqrt{3}}{9} \|\phi_x\|_{L_2(\Omega_{\mathrm{xy}})},$$

$$\sum_{n=1}^{\infty} |\phi_{2n\,0}| \leq \frac{\sqrt{3}}{3} \|\phi_x\|_{L_2(\Omega_{\mathrm{xy}})}$$

$$\sum_{n,k=1}^{\infty} |\phi_{2n-1\,k}| \leq \frac{\sqrt{2}}{6} \|\phi_y\|_{L_2(\Omega_{\mathrm{xy}})} + \frac{\sqrt{2}}{18} \|\phi_{xy}\|_{L_2(\Omega_{\mathrm{xy}})},$$

(31)
$$\sum_{n,k=1}^{\infty} |\phi_{2n\,k}| \leq \frac{\sqrt{2}}{6} \|\phi_{xy}\|_{L_2(\Omega_{\mathrm{xy}})}.$$

Hence, the series

$$\sum_{k=0}^{\infty} |\phi_{0\,k}| + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \left(|\phi_{2n-1\,k}| + |\phi_{2n\,k}|\right),$$

majorizing the functional series (26) converges. Then, according to Weyerstrasse's theorem [16,p.20], the series (26) converges absolutely and evenly in the domain $\overline{\Omega}$, and its sum is a continuous function in this domain.

Now we will show that $\frac{\partial^3 u}{\partial x^3} \in C(\bar{\Omega})$. (28) and (29) imply that

$$\frac{\partial^3 u_{2n-1\,0}}{\partial x^3} =$$

$$= \lambda_n^{3/4} \Big(\phi_{2n-1\,0} E_\alpha(-\lambda_n t^\alpha) + 4\lambda_n^{3/4} \phi_{2n\,0} F_{n0}(t) \Big) \sin 2\pi n x,$$

$$\frac{\partial^4 u_{2n-1\,k}}{\partial x^4} =$$

$$= \sqrt{2}\lambda_n^{3/4} \Big(\phi_{2n-1\,k} E_\alpha(-\sigma_{nk} t^\alpha) + 4\lambda_n^{3/4} \phi_{2n\,k} F_{nk}(t) \Big) \sin 2\pi n x \cos \pi n y,$$

$$\frac{\partial^3 u_{2n\,0}}{\partial x^3} = \phi_{2n\,0} E_\alpha(-\lambda_n t^\alpha) \Big(-3\lambda_n^{1/2} \sin 2\pi n x - \lambda_n^{3/4} x \cos 2\pi n x \Big),$$

$$\frac{\partial^3 u_{2n\,k}}{\partial x^3} =$$

$$= \sqrt{2}\phi_{2n\,k} E_\alpha(-\sigma_{nk} t^\alpha) \Big(-3\sqrt{\lambda_n} \sin 2\pi n x - \lambda_n^{3/4} \cos 2\pi n x \Big) \cos k\pi y.$$

Then, taking into account the conditions imposed on the function $\phi(x,y)$, similar to the estimates (31), we can easily obtain the following estimates:

$$\sum_{n=1}^{\infty}\left|\frac{\partial^3 u_{2n\ 0}}{\partial x^3}\right| \le \frac{4\sqrt{3}M_1}{3}\left\|\frac{\partial^4\phi}{\partial x^4}\right\|_{L_2(\Omega_{\mathrm{xy}})},$$

$$\sum_{n=1}^{\infty}\left|\frac{\partial^3 u_{2n\ k}}{\partial x^3}\right| \le \frac{4M_1}{3}\left\|\frac{\partial^5\phi}{\partial x^4\partial y}\right\|_{L_2(\Omega_{\mathrm{xy}})},$$

$$\sum_{n=1}^{\infty}\left|\frac{\partial^3 u_{2n-1\ 0}}{\partial x^3}\right| \le$$

$$\le \frac{5\sqrt{3}M_1+2\sqrt{5}M_3}{15}\left\|\frac{\partial^4\phi}{\partial x^4}\right\|_{L_2(\Omega_{\mathrm{xy}})} + \frac{4\sqrt{3}M_1}{3}\left\|\frac{\partial^3\phi}{\partial x^3}\right\|_{L_2(\Omega_{\mathrm{xy}})},$$

$$\sum_{n=1}^{\infty}\left|\frac{\partial^3 u_{2n-1\ k}}{\partial x^3}\right| \le$$

$$\le \frac{15M_1+4\sqrt{15}M_3}{90}\left\|\frac{\partial^5\phi}{\partial x^4\partial y}\right\|_{L_2(\Omega_{\mathrm{xy}})} + \frac{4M_1}{3}\left\|\frac{\partial^4\phi}{\partial x^3\partial y}\right\|_{L_2(\Omega_{\mathrm{xy}})}.$$

These and (26) imply the result. The proof of ${}_C D_{0t}^{\alpha} u(x,t) \in C(\Omega)$, $\frac{\partial^3 u}{\partial y^3} \in C(\overline{\Omega})$, $\frac{\partial^4 u}{\partial y^4} \in C(\Omega)$ is similar. □

The theorem follows.

Note. The solution of Problem A can be represented as a sum of the series (25), where the functions $T_{0\,k}(t)$, $T_{2n\,-1\,k}(t), T_{2n\ k}(t)$ have the form

$$T_{0\,k}(t) = \phi_{0\,k}E_\alpha(-\mu_k t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{0k}(t-\tau)^\alpha)f_{0\ k}(\tau)d\tau,$$

$$T_{2n\ k}(t) = \phi_{2n\ k}E_\alpha(-\sigma_{nk}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^\alpha)f_{2n\ k}(\tau)d\tau,$$

$$T_{2n\,-1\,k}(t) = \phi_{2n\,-1\,k}E_\alpha(-\sigma_{nk}t^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\sigma_{nk}(t-\tau)^\alpha)$$
$$\left(4\sqrt[4]{\lambda_n}^{\,3}T_{2n\ k}(\tau) + f_{2n\ k}(\tau)\right)d\tau,$$

where $f_{0\,k}(t) = (f(x,y,t), W_{0\,k}(x,y))$; $f_{2n\,k}(t) = (f(x,y,t), W_{2n\,k}(x,y))$, $f_{2n-1\,k}(t) = \Big(f(x,y,t), W_{2n-1\,k}(x,y)\Big)$; and the coefficients $\phi_{0\,k}$, $\phi_{2n-1\,k}$, $\phi_{2n\,k}$ are determined by using the formula (25).

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