International Journal of Pure and Applied Mathematics
Volume 113 No. 4 2017, 65-70
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v113i4.7

ijpam.eu

POLYA TYPE INEQUALITIES FOR THE HEAT OPERATOR IN POLYGONAL CYLINDERS

Tynysbek Sh. Kalmenov[1], Aidyn Kassymov[2], Durvudkhan Suragan[3]
[1,2,3]Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN
[2]Al-Farabi Kazakh National University
Almaty, KAZAKHSTAN

Abstract: In this note we prove Polya type inequalities for the Cauchy-Dirichlet heat operator in polygonal cylindric domains of a given volume. That is, in particular, we prove that the s_1-number of the Cauchy-Dirichlet heat operator is minimized in the equilateral triangular cylinder among all triangular cylinders of given volume, which means that the operator norm of the inverse operator is maximized in the equilateral triangular cylinder among all triangular cylinders of a given volume.

AMS Subject Classification: 35P05, 58J50
Key Words: heat operator, s-numbers, Polya inequality

1. Introduction

Isoperimetric inequalities for the eigenvalues of the Laplacian and related self-adjoint operators have been extensively investigated during the last decades. We refer to the review paper [1] and recent papers [4], [5], [6] and [7] for general discussions in this subject. Our aim (see, e.g. [3]) is to extend those known results to non-self adjoint operators, namely, the Cauchy-Dirichlet heat operator. Thus, the main motivation of the present note is Polya's result (inequality) which asserts that the equilateral triangle is a minimizer of the first eigenvalue of the Dirichlet Laplacian among all triangles of a given area.

Received: December 15, 2016
Revised: January 30, 2017
Published: April 6, 2017

Let $D_1 = \Omega_1 \times (0,T)$ and $D_2 = \Omega_2 \times (0,T)$ be cylindrical domains, where $\Omega_1 \subset \mathbb{R}^2$ is a triangle and $\Omega_2 \subset \mathbb{R}^2$ is a quadrilateral. We consider the Cauchy-Dirichlet heat operator by the formula

$$\diamondsuit u(x,t) := \begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta_x u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,T). \end{cases} \tag{1}$$

Here and after Ω is Ω_1 or Ω_2 as well as $\partial\Omega$ is the boundary of Ω. The operator $\diamondsuit$ is compact, but it is a non-selfadjoint operator in $L^2(D)$. An adjoint operator $\diamondsuit^*$ to the operator $\diamondsuit$ can be presented as

$$\diamondsuit^* v(x,t) = \begin{cases} -\frac{\partial v(x,t)}{\partial t} - \Delta_x v(x,t), \\ v(x,T) = 0, \ \ x \in \Omega, \\ v(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,T). \end{cases} \tag{2}$$

In general, if A is a compact operator, then the eigenvalues of the operator $(A^*A)^{1/2}$, where A^* is the adjoint operator of A, are called s-numbers of the operator A. A direct calculation gives that

$$\diamondsuit^*\diamondsuit u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t), \\ u(x,0) = 0, \ \ x \in \Omega, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ \ x \in \Omega, \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ \ x \in \partial\Omega, \ \forall t \in (0,T). \end{cases} \tag{3}$$

In the following section we present main results of this note and their proofs.

2. Main Results and Their Proofs

We denote an equilateral triangular cylinder by $C_\triangle = \triangle \times (0,T)$, where $\triangle \subset \mathbb{R}^2$ is an equilateral triangle with $|\triangle| = |\Omega_1|$, and a quadratic cylinder $C_\square = \square \times (0,T)$, where $\square \subset \mathbb{R}^2$ is a square with $|\square| = |\Omega_2|$. Here and after $|\Omega|$ is the area of the domain Ω.

Let us introduce operators T, $L : L^2(\Omega) \to L^2(\Omega)$ by the formulae

$$Tz(x) = \begin{cases} -\Delta z(x) = \mu z(x), \\ z(x) = 0, \ \ x \in \partial\Omega. \end{cases} \tag{4}$$

and

$$Lz(x) = \begin{cases} \Delta^2 z(x) = \lambda z(x), \\ z(x) = 0, \ \ x \in \partial\Omega, \\ \Delta z(x) = 0, \ \ x \in \partial\Omega. \end{cases} \tag{5}$$

Lemma 1. *The first eigenvalue of the operator L is minimized in the equilateral triangle (square) among all triangles (quadrilaterals) with $|\triangle| = |\Omega|$ ($|\square| = |\Omega|$).*

Proof. Pólya theorem ([8]) for the operator T says that the equilateral triangle (square) is minimizer of the first Dirichlet Laplacian eigenvalue among all triangles (quadrilaterals) Ω_1 (Ω_2) of the same area with $|\triangle| = |\Omega_1|(|\square| = |\Omega_2|)$. Let us calculate T^2 from (4),

$$T^2 z(x) = \begin{cases} \Delta^2 z(x) = \mu^2 z(x), \\ z(x) = 0, \ \ x \in \partial\Omega, \\ \Delta z(x) = 0, \ \ x \in \partial\Omega. \end{cases} \tag{6}$$

That is, $T^2 = L$ and $\mu^2 = \lambda$. Thus, we establish $\lambda_1(\triangle) = \mu_1^2(\triangle) \leq \mu_1^2(\Omega_1) = \lambda_1(\Omega_1)$ and $\lambda_1(\square) = \mu_1^2(\square) \leq \mu_1^2(\Omega_2) = \lambda_1(\Omega_2)$. Then, $\lambda_1(\triangle) \leq \lambda_1(\Omega_1)$ and $\lambda_1(\square) \leq \lambda_1(\Omega_2)$. $\square$

Theorem 1. *The s_1-number of the operator $\diamondsuit$ is minimized in the equilateral triangular cylinder among all triangular cylinders of given volume, that is,*

$$s_1(C_\triangle) \leq s_1(D_1),$$

with $|D_1| = |C_\triangle|$.

Proof. Let u be a nonnegative, measurable function on $\mathbb{R}^2$, and let V be a line through the origin of $\mathbb{R}^2$. Choose an orthogonal coordinate system in $\mathbb{R}^2$ such that the x^1-axis is perpendicular to $V = x^2$.

Definition 1 ([2])**.** A nonnegative, measurable function $u^\star(x|V)$, $x = (x^1, x^2)$, on $\mathbb{R}^2$ is called a Steiner symmetrization with respect to V of the function $u(x)$, if $u^\star(x^1, x^2)$ is a symmetric decreasing rearrangement with respect to x^1 of $u(x^1, x^2)$ for each fixed x^2.

The Steiner symmetrization (with respect to the x^1-axis) $\Omega^\star$ of a measurable set Ω is defined in the following way: if we write $x = (x^1, y)$ with $y \in \mathbb{R}^2$, and let $\Omega_z = \{x^1 : (x^1, y) \in \Omega\}$, then

$$\Omega^\star := \{(x^1, y) \in \mathbb{R} \times \mathbb{R}^2 : x^1 \in \Omega_y^*\},$$

where Ω_y^* is a symmetric rearrangement of Ω_y

Consider the following spectral problem

$$\diamondsuit^*\diamondsuit u = su,$$

$$\diamondsuit^*\diamondsuit u(x,t) = \begin{cases} -\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_x^2 u(x,t) = su(x,t), \\ u(x,0) = 0, \ \ x \in \Omega_1, \\ \frac{\partial u(x,t)}{\partial t}|_{t=T} - \Delta_x u(x,t)|_{t=T} = 0, \ \ x \in \Omega_1, \\ u(x,t) = 0, \ \ x \in \partial\Omega_1, \ \forall t \in (0,T), \\ \Delta_x u(x,t) = 0, \ \ x \in \partial\Omega_1, \ \forall t \in (0,T). \end{cases} \tag{7}$$

The domain $D_1 = \{(x,t)|x \in \Omega_1 \subset \mathbb{R}^2, \ t \in (0,T)\}$ is a cylindrical domain and we can have $u(x,t) = X(x)\varphi(t)$, so that $u_1(x,t) = X_1(x)\varphi_1(t)$ is the first eigenfunction of the operator $\diamondsuit^*\diamondsuit$. We can also restate above fact (7) as

$$-\varphi_1''(t)X_1(x) + \varphi_1(t)\Delta^2 X_1(x) = s_1\varphi_1(t)X_1(x). \tag{8}$$

By the variational principle for the operator $\diamondsuit^*\diamondsuit$ and after a straightforward calculation, we obtain

$$s_1(D_1) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(\Delta X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(-\mu_1(\Omega_1)X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \mu_1^2(\Omega_1)\int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx},$$

where $\mu_1(\Omega_1)$ is the first eigenvalue of the operator Dirichlet Laplacian.

For each non-negative function $X \in L^2(\Omega_1)$, we obtain

$$\int_{\Omega_1} |X_1(x)|^2 dx = \int_{\triangle} |X_1^*(x)|^2 dx. \tag{9}$$

Applying Lemma (1) and (9), we get

$$s_1(D_1) = \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_{\Omega_1} X_1^2(x)dx + \mu_1^2(\Omega_1)\int_0^T \varphi_1^2(t)dt \int_{\Omega_1}(X_1(x))^2 dx}{\int_0^T \varphi_1^2(t)dt \int_{\Omega_1} X_1^2(x)dx}$$

$$\geq \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\triangle (X_1^*(x))^2dx + \mu_1^2(\triangle)\int_0^T \varphi_1^2(t)dt \int_\triangle (X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_\triangle (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\triangle (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_\triangle (-\mu_1(\triangle)X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_\triangle (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \varphi_1''(t)\varphi_1(t)dt \int_\triangle (X_1^*(x))^2dx + \int_0^T \varphi_1^2(t)dt \int_\triangle (\Delta X_1^*(x))^2dx}{\int_0^T \varphi_1^2(t)dt \int_\triangle (X_1^*(x))^2dx}$$

$$= \frac{-\int_0^T \int_\triangle \frac{\partial^2 u_1^*(x,t)}{\partial t^2}dxdt + \int_0^T \int_\triangle (\Delta_x^2 u_1^*(x,t))^2dxdt}{\int_0^T \int_\triangle (u_1^*(x,t))^2dxdt}$$

$$\geq \inf_{z(x,t)\neq 0} \frac{-\int_0^T \int_\triangle z_t(x,t)z(x,t)dxdt + \int_0^T \int_\triangle (\Delta_x z(x,t))^2dxdt}{\int_0^T \int_\triangle z^2(x,t)dxdt} = s_1(C_\triangle).$$

The proof is complete. □

Corollary 1. *The norm of the operator* $\diamondsuit^{-1}$ *is maximized in the equilateral triangular cylinder* $C_\triangle$ *among all triangular cylinders of a given volume, i.e.* $\|\diamondsuit^{-1}\|_{D_1} \leq \|\diamondsuit^{-1}\|_{C_\triangle}$.

Theorem 2. *The* s_1-*number of the operator* $\diamondsuit$ *is minimized in the quadratic cylinder among all quadrangular cylinders of a given volume, i.e.*

$$s_1(C_\square) \leq s_1(D_2),$$

with $|D_2| = |C_\square|$.

Proof. The proof is similar to the proof of Theorem (1). □

Brief abstract of this work has been published in the Materials of the workshop "Differential operators and modeling of complex systems" (April 7-8, 2017, Almaty, Kazakhstan) [9].

Acknowledgements

This research is financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 4075/GF4). This publication is supported by the target program 0085/PTSF-14 from the Ministry of Science and Education of the Republic of Kazakhstan.

References

[1] R.D. Benguria, H. Linde and B. Loewe, Isoperimetric inequalities for eigenvalues of the Laplacian and the Schrödinger operator, *Bull. Math. Sci.*, **2**, No. 1 (2012), 1–56.

[2] H.J. Brascamp, E.H. Lieb and J.M. Luttinger, A general rearrangement inequality for multiple integrals, *J. Funct. Anal.*, **17** (1974), 227–237.

[3] A. Kassymov and D. Suragan, Some Spectral Geometry Inequalities for Generalized Heat Potential Operators, *Complex Anal. Oper. Theory*, to appear (2016), doi:10.1007/s11785-016-0605-9

[4] G. Rozenblum, M. Ruzhansky and D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials, *J. Funct. Anal.*, **271** (2016), 224–239.

[5] M. Ruzhansky and D. Suragan, Isoperimetric inequalities for the logarithmic potential operator, *J. Math. Anal. Appl.*, **434** (2016), 1676–1689.

[6] M. Ruzhansky and D. Suragan, Schatten's norm for convolution type integral operator, *Russ. Math. Surv.*, **71** (2016), 157–158.

[7] M. Ruzhansky and D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries, *Bull. Math. Sci.*, **6** (2016), 325–334.

[8] G. Pólya, On the characteristic frequencies of a symmetric membrane, *Math. Z.*, **63** (1955), 331–337.

[9] T.Sh. Kal'menov, A. Kassymov, D. Suragan, Polya type inequalities for the heat operator in polygonal cylinders, *Kazakh Mathematical Journal*, **17**, No. 1 (2017), 33-34.