

**TRANSFORMATION OPERATORS FOR  
THE FOURTH-ORDER DIFFERENTIAL EQUATIONS  
WITH NON-ANALYTIC COEFFICIENTS**

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**Abstract:** This work is dedicated to constructive method of building cores of transformation operator. In this article a solution of a linear differential equation of the fourth order is investigated. Also, a method of building a solution in the form of integral representation is given.

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## 1. Introduction

When solving inverse problems of spectral analysis for the Sturm-Liouville equation V.A. Marchenko [1, 2] effectively applied special mathematical tool related to the transformation operators. In particular, when reconstructing coefficient of  $q(x)$  and boundary numbers  $h$ ,  $H$  based on spectral data of the problem

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$$-y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < b, \quad (1)$$

$$\begin{cases} y(0) - hy(0) = 0, \\ y(b) - Hy(b) = 0 \end{cases} \quad (2)$$

V.A. Marchenko used the fact that some solution of equation (1) can be represented as following

$$y(x, \lambda) = \omega(x, \lambda) + \int_0^{x_0} K(x, t)\omega(t, \lambda)dt, \quad (3)$$

where  $K(x, t)$  is some function that we will name as the transformation operator's core. Here  $\omega(x, \lambda)$  satisfies equation (1) when  $q(x) \equiv 0$ . His idea was the following: on the first stage the reconstruction of the core  $K(x, t)$  when  $0 \leq t \leq x \leq b$  is performed based on spectral data, then based on  $K(x, t)$  the potential of  $q(x)$  and numbers  $h, H$  is defined.

Presented scheme of V.A. Marchenko [3-5] appeared to be effective also in the inverse scattering problem. Due to the marked successes there is a desire to apply the results of V.A. Marchenko onto differential equations of higher order [6-12]. However, at the beginning of 60s under the efforts of L.A. Sakhnovich, V.I. Matsayev it became clear that solutions of the differential equation

$$y^{(IV)}(x) + (p(x)y)' + qy = \lambda y$$

are not always necessarily connected by the formulas of type (3) while solving the equation  $\omega^{(IV)}(x, \lambda) = \lambda\omega$ . Similar examples are given in the works of L.A. Sakhnovich and V.I. Matsayev. The questions arise: what to do in case when  $q(x)$  is continuous but not differentiable function? What to use in these kinds of cases instead of formula (3)? For the long time there was no any effective replacement of the formula (3). Instead of this efforts the mathematicians were directed towards solving the following question: to find wider class of coefficients  $p(x)$  and  $q(x)$ , so that the representation (3) will stay valid. In this direction the results of I.G. Hachatrian should be acknowledged [8]. He proved that if coefficients  $p(x)$  and  $q(x)$  are holomorphic in some rectangle, then for the solution the formula (3) is true. The question about necessity of the condition that coefficients  $p(x)$  and  $q(x)$  should be holomorphic for the transformation operator of type (3) to exist has been discussed in the works of V. I. Matsayev [9], L.A. Sakhnovich [10, 11] and M.M. Malamud [12]. In [12] it is proved that if part of coefficients of the differential equation

$$y^{(N)}(x) + \sum_{k=0}^{N-2} p_k(x)y^{(k)}(x) = \lambda y(x) \quad (4)$$

is holomorphic and in addition to this has representations (3), then the rest of the coefficients are also holomorphic. Therefore, in order to keep the form (3) of the transformation operator coefficients of differential equation have to be holomorphic. The rejection of this holomorphicity condition results in a problem that has not been solved yet. Other results in this direction could be found in the works of M.K. Fage [13, 14], A.F. Leontiyev [15, 16], and A. P. Khromov [17]. Some further progress is foreshadowed due to the book [18].

In work [18] for the case when  $p_k(x) \in C^k[0; b]$ ,  $b < \infty$  the exponential representation of the particular solutions of differential equation (4) has been proved. In particular, when  $N=4$  the result of the work [18] can be formulated as following.

Main result: *Let differential equation (4) have coefficients  $p_k(x) \in C^k[0; b]$ ,  $k = 0, 1, 2$  and  $N = 4$ . Then, functions  $B(x, \xi)$  and  $A(x, \tau, \xi)$  can be found when  $0 \leq \xi \leq \tau \leq x \leq b$  so that some solution  $y(x, \lambda)$  of differential equation (4) has exponential representation*

$$y(x, \lambda) = \int_0^x B(x, \xi)\psi_1(\xi, \lambda)d\xi + \int_0^x d\xi \int_\xi^x A(x, \tau, \xi)(\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda))d\tau \quad (5)$$

and it is valid under all complex values of the parameter  $\lambda$ .

Here  $\psi_1(\xi, \lambda)$  represents one of the solutions of differential equation (4) with nonzero coefficients  $p_2(x)$ ,  $p_1(x)$  and  $p_0(x)$ . An important piece of the presented statement is the fact that the same cores  $B(x, \xi)$  and  $A(x, \tau, \xi)$  are suitable for all  $\lambda$ . Thus, the right side of the representation (5) is a generalization of the representation (3) in case  $N = 4$ . At the same time it is known from [19] that core  $K(x, t)$  from (3) can be interpreted as a solution to the problem of Gurs for the wave equation

$$\frac{\partial^2 K(x, t)}{\partial x^2} - q(x)K(x, t) = \frac{\partial^2 K(x, t)}{\partial t^2}, \quad 0 \leq t \leq x < b,$$

$$\left. \frac{\partial K(x, t)}{\partial t} \right|_{t=0} = 0, \quad \frac{\partial K(x, x)}{\partial x} = \frac{1}{2}q(x),$$

which is equivalent to integral equation [20]

$$K\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right) = -\frac{1}{4} \int_\eta^\xi d\beta \int_0^\eta q\left(\frac{\beta + \alpha}{2}\right) K\left(\frac{\beta + \alpha}{2}, \frac{\beta - \alpha}{2}\right) d\alpha +$$

$$+ \frac{1}{4} \int_0^\xi q\left(\frac{\alpha}{2}\right) d\alpha + \int_0^\eta \left( q\left(\frac{\beta}{2}\right) - \int_0^\beta q\left(\frac{\alpha+\beta}{2}\right) K\left(\frac{\beta+\alpha}{2}, \frac{\beta-\alpha}{2}\right) d\alpha \right) d\beta$$

that has single solution with a natural priori assessment. Analogous questions, related to cores of the representation (5), have not been investigated yet, and separate work will be dedicated to them. This work is dedicated to constructive method of building cores  $B$  and  $A$  from (5).

## 2. Discussion of Main Results and Elaboration of the Problem

Consider a linear differential equation of the fourth order

$$y^{(IV)}(x) + p_2(x)y'(x) + p_1(x)y(x) + p_0(x)y(x) = \lambda y(x) \quad (6)$$

with initial conditions

$$y_m^{(k-1)}(0) = \delta_{km}, \quad k, m = 1, 2, 3, 4, \quad (7)$$

where we assume that functions  $p_k(x)$ ,  $k = 0, 1, 2$  are  $k$  times continuously differentiable on  $[0, b]$  functions,  $\lambda$  is a parameter and  $\delta_{km}$  is a Kronecker delta. Solutions of the problems (6), (7) are denoted as  $y_1(x, \lambda)$ ,  $y_2(x, \lambda)$ ,  $y_3(x, \lambda)$ ,  $y_4(x, \lambda)$ . When equation (6) has zero coefficients, solutions  $y_j(x, \lambda)$ ,  $j = 1, 2, 3, 4$  are correspondingly functions  $\psi_j(x, \lambda)$ , which can be defined by the following formula

$$\psi_j(x, \lambda) = \frac{1}{4} \left( \frac{\exp(p\omega_0 x)}{(p\omega_0)^{j-1}} + \frac{\exp(p\omega_1 x)}{(p\omega_1)^{j-1}} + \frac{\exp(p\omega_2 x)}{(p\omega_2)^{j-1}} + \frac{\exp(p\omega_3 x)}{(p\omega_3)^{j-1}} \right), \quad j = 1, 2, 3, 4,$$

where different roots of the fourth degree of (6) are denoted through  $\omega_s$ ,  $s = 0, 1, 2, 3$  and  $\theta^4 = \lambda$ .

In the article, only solution of  $y_4(x, \lambda)$  is investigated, although solutions of  $y_1(x, \lambda)$ ,  $y_2(x, \lambda)$ ,  $y_3(x, \lambda)$  can be found in the same way. A constructive method of building  $y_4(x, \lambda)$  using solution of  $\psi_1(x, \lambda)$  is given. Also, the integral operator is found which transfers function  $\psi_1(x, \lambda)$  into solution  $y_4(x, \lambda)$ . It should be noted that the same integral operator is suitable for all complex values of  $\lambda$ . For the first time this integral operator was given in [18]. In this

article the methodical presentation is somewhat different from presentation of work [18]. It is known that Cauchy problem is equivalent to the integral equation relative to  $y_4(x, \lambda)$ .

$$y_4(x, \lambda) = \int_0^x \frac{(x-t)^2}{2} \psi_1(\tau, \lambda) dt + \int_0^x \psi_1(\tau, \lambda) d\tau \int_0^{x-\tau} T(x-\tau, t) y_4(t, \lambda) dt,$$

where

$$T(u, t) = -\frac{(u-t)^2}{2!} (p_2(t) - p_1(t) + p_0(t)) + \frac{(u-t)}{1!} (2p_2(t) - p_1(t) - p_2(t)), \quad u \geq t.$$

Solving the latter integral equation results in the iterations being successively defined by the formulas

$$h_0(x, \lambda) = \int_0^x \frac{(x-t)^2}{2} \psi_1(t, \lambda) dt, \quad B_0(x, t) = \frac{(x-t)^2}{2},$$

$$h_{k+1}(x, \lambda) = \int_0^x \psi_1(\tau, \lambda) d\tau \int_0^{x-\tau} T(x-\tau, t) h_k(t, \lambda) dt, \quad k \geq 0.$$

When coefficients  $p_2(t), p_1(t), p_0(t)$  analytically continue from the interval  $[0, b]$  and extend to some part of complex plane, all iterations  $\{h_k, k \geq 0\}$  have the same structure of the type

$$h_k(x, \lambda) = \int_0^x B_k(x, t) \psi_1(t, \lambda) dt.$$

Consequently, in the case of analyticity of coefficients  $p_2, p_1, p_0$  from the representation of the solution  $y_4(x, \lambda)$  in the form of a series

$$y_4(x, \lambda) = h_0(x, \lambda) + h_1(x, \lambda) + \dots$$

the following exponential representation is obtained

$$y_4(x, \lambda) = \int_0^x B(x, t) \psi_1(t, \lambda) dt.$$

If there is no analyticity we conclude that the structure of the zero approximation

$$h_0(x, \lambda) = \int_0^x \frac{(x-t)^2}{2} \psi_1(t, \lambda) dt$$

is considerably different from the structure of first approximation  $h_1(x, \lambda)$ .

Since equation (6) includes a complex parameter  $\lambda$ , then Cauchy problem (6), (7) is a family of problems. We are trying to find by uniform way the whole family of solutions of mentioned problems, i.e. we will find such an operator  $T$ , which, firstly, does not depend on  $\lambda$ , secondly, it transfers the family of functions  $\psi_j(x, \lambda)$  into a set of solutions  $y_4(x, \lambda)$ .

### 3. Integral Representation of the Cauchy Problem's Solution

It is known from [21] that many problems of differential equations are definitely reduced to integral equations. Let us consider the following Cauchy problem

$$y^{(IV)}(x) + p_2(x)y'(x) + p_1(x)y(x) + p_0(x)y(x) = \lambda y(x),$$

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 0, \quad y''(0, \lambda) = 0, \quad y'''(0, \lambda) = 1$$

and we reduce it to the equivalent integral equation.

Considered problem is reduced to the following integral equation relative to  $y(x, \lambda)$

$$y(x, \lambda) = \int_0^x \frac{(x-t)^2}{2!} \psi_1(t, \lambda) dt + \int_0^x \psi_1(\tau, \lambda) d\tau \int_0^{x-\tau} T(x-\tau, t) y(t, \lambda) dt, \quad (8)$$

where

$$T(u, t) = -\frac{(u-t)^2}{2!} (p_2(t) - p_1(t) + p_0(t)) + \frac{(u-t)}{1!} (2p_2(t) - p_1(t)) - p_2(t), \quad u > t. \quad (9)$$

Equation (8) is a second kind Volterra integral equation [22]. This equation is solved using method of successive approximation [23]. Let the zero solution approximation  $h_0(x, \lambda)$  be defined by formula

$$h_0(x, \lambda) = \int_0^x B_0(x, t) \psi_1(t, \lambda) dt, \quad (10)$$

where  $B_0(x, t) = \frac{(x-t)^2}{2!}$ .

Subsequent approximations are found by formula

$$h_{k+1}(x, \lambda) = \int_0^x \psi_1(\tau, \lambda) d\tau \int_0^{x-\tau} T(x-\tau, t) h_k(t, \lambda) dt, \quad k \geq 0. \quad (11)$$

Then, according to [21] we have uniformly convergent series

$$y(x, \lambda) = h_0(x, \lambda) + h_1(x, \lambda) + \dots + h_n(x, \lambda) + \dots \quad (12)$$

Appropriate assessments of particular approximations are given in [21]. Let us consider the structure of approximations. For example, for the first approximation the following lemma is true.

**Lemma 1.** *Function  $h_1(x, \lambda)$  can be presented as*

$$h_1(x, \lambda) = \int_0^x d\xi \int_{\xi}^x d\tau (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) A_1(x, \tau, \xi) + \int_0^x \psi_1(\xi, \lambda) B_1(x, \xi) d\xi, \quad (13)$$

where

$$A_1(x, \tau, \xi) = \frac{1}{4} \int_{\xi}^{x-\tau+\xi} T(x-\tau+\xi, t) B_0(t, \xi) dt, \quad (14)$$

$$B_1(x, \xi) = \frac{1}{4} \left\{ \int_0^{\xi} d\tau \int_{\xi-\tau}^{x-\tau} T(x-\tau, t) B_0(t, \xi-\tau) dt + \int_0^{\frac{x-\xi}{2}} d\tau \int_{\tau+\xi}^{x-\tau} T(x-\tau, t) B_0(t, \tau+\xi) dt + \int_0^{\frac{x+\xi}{2}} d\tau \int_{\tau-\xi}^{x-\tau} T(x-\tau, t) B_0(t, \tau-\xi) dt \right\}. \quad (15)$$

*Proof.* From formulas (10) and (11) it follows

$$h_1(x, \lambda) = \int_0^x \psi_1(\tau, \lambda) d\tau \int_0^{x-\tau} T(x-\tau, t) \int_0^t \psi_1(\xi, \lambda) \times B_0(t, \xi) d\xi dt. \quad (16)$$

Changing the order of integration in formula (16), we have

$$h_1(x, \lambda) = \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau, \lambda) \psi_1(\xi, \lambda) d\xi \int_{\xi}^{x-\tau} T(x-\tau, t) \times$$

$$\times B_0(t, \xi)dt. \quad (17)$$

Since  $\psi_1$  is a linear combination of exponents, then the multiplication of functions  $\psi_1(\tau, \lambda)$  and  $\psi_1(\xi, \lambda)$  looks like

$$\begin{aligned} \psi_1(\tau, \lambda)\psi_1(\xi, \lambda) &= \frac{1}{4}(\psi_1(\tau + i\xi, \lambda) + \psi_1(\tau - i\xi, \lambda) + \\ &\quad + \psi_1(\tau + \xi, \lambda) + \psi_1(\tau - \xi, \lambda)). \end{aligned}$$

Let us rewrite the formula (17) using this equation

$$\begin{aligned} h_1(x, \lambda) &= \frac{1}{4} \int_0^x d\tau \int_0^{x-\tau} (\psi_1(\tau + i\xi, \lambda) + \\ &\quad + \psi_1(\tau - i\xi, \lambda)) d\xi \int_\xi^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt + \\ &\quad + \frac{1}{4} \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau + \xi, \lambda) d\xi \int_\xi^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt + \\ &\quad + \frac{1}{4} \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau - \xi, \lambda) d\xi \int_\xi^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt. \quad (18) \end{aligned}$$

Let us transform the first integral on the right side of (18). Firstly, we change the order of integration in it, then make a variable substitution  $\tau = \tau_1 - \xi$ :

$$\begin{aligned} &\int_0^x d\tau \int_0^{x-\tau} (\psi_1(\tau + i\xi, \lambda) + \\ &\quad + \psi_1(\tau - i\xi, \lambda)) d\xi \int_\xi^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt = \\ &\quad = \int_0^x d\xi \int_0^{x-\xi} (\psi_1(\tau + \xi, \lambda) + \\ &\quad + \psi_1(\tau - \xi, \lambda)) d\tau \int_\xi^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt = \\ &\quad = \int_0^x d\xi \int_\xi^x (\psi_1(\tau_1 - \xi + i\xi, \lambda) + \\ &\quad + \psi_1(\tau_1 - \xi - i\xi, \lambda)) d\tau_1 \int_\xi^{x-\tau_1+\xi} T(x - \tau_1 + \xi, t) B_0(t, \xi) dt. \quad (19) \end{aligned}$$

After making the following substitutions in the subsequent integrals

$$\xi = \xi_1 - \tau, \quad \xi = \tau - \xi_1$$



and changing the order of integration, we obtain

$$\begin{aligned}
& \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau + \xi, \lambda) d\xi \int_{\xi}^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt = \\
& = \int_0^x d\tau \int_{\tau}^x \psi_1(\xi_1, \lambda) d\xi_1 \int_{\xi_1 - \tau}^{x-\tau} T(x - \tau, t) B_0(t, \xi_1 - \tau) dt = \\
& = \int_0^x \psi_1(\xi_1, \lambda) d\xi_1 \int_0^{\xi_1} d\tau \int_{\xi_1 - \tau}^{x-\tau} T(x - \tau, t) B_0(t, \xi_1 - \tau) dt, \quad (20)
\end{aligned}$$

$$\begin{aligned}
& \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau - \xi, \lambda) d\xi \int_{\xi}^{x-\tau} T(x - \tau, t) B_0(t, \xi) dt = \\
& = \int_0^x d\tau \int_{2\tau-x}^x \psi_1(\xi_1, \lambda) d\xi_1 \int_{\tau - \xi_1}^{x-\tau} T(x - \tau, t) B_0(t, \tau - \xi_1) dt = \\
& = \int_0^x \psi_1(\xi_1, \lambda) d\xi_1 \int_0^{\frac{x-\xi_1}{2}} d\tau \int_{\tau + \xi_1}^{x-\tau} T(x - \tau, t) B_0(t, \tau + \xi_1) dt + \\
& + \int_0^x \psi_1(\xi_1, \lambda) d\xi_1 \int_{\xi_1}^{\frac{x+\xi_1}{2}} d\tau \int_{\tau - \xi_1}^{x-\tau} T(x - \tau, t) B_0(t, \tau - \xi_1) dt. \quad (21)
\end{aligned}$$

Let us substitute expressions (19), (20), (21) into the formula (18), previously removing the indices of  $\xi_1$ ,  $\tau_1$

$$\begin{aligned}
h_1(x, \lambda) = & \\
& = \frac{1}{4} \int_0^x d\xi \int_{\xi}^x (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) d\tau + \\
& + \frac{1}{4} \int_{\xi}^{x-\tau+\xi} T(x - \tau + \xi, t) B_0(t, \xi) dt + \\
& + \int_0^x \psi_1(\xi, \lambda) d\xi \frac{1}{4} \left\{ \int_0^{\xi} d\tau \int_{\xi-\tau}^{x-\tau} T(x - \tau, t) B_0(t, \xi - \tau) dt + \right. \\
& + \int_0^{\frac{x-\xi}{2}} d\tau \int_{\tau+\xi}^{x-\tau} T(x - \tau, t) B_0(t, \tau + \xi) dt + \\
& \left. + \int_0^{\frac{x+\xi}{2}} d\tau \int_{\tau-\xi}^{x-\tau} T(x - \tau, t) B_0(t, \tau - \xi) dt \right\}. \quad (22)
\end{aligned}$$

If into expression (22) we substitute (14), (15), then we obtain a representation (13). Therefore, the formula (13) is proved.  $\square$

It should be noted that when proving lemma 1 the possibility of converting the multiplication of  $\psi_1(\tau, \lambda)\psi_1(\xi, \lambda)$  into the linear combination of the values of function  $\psi_1$  has played very important role. In the subsequent approximations similar multiplications take place again which also can be converted into a linear combinations of function  $\psi_1$ . Therefore, basically each approximation linearly depends on function  $\psi_1$ . Let us prove through mathematical induction that all subsequent approximations  $h_2(x, \lambda)$ ,  $h_3(x, \lambda)$ ,  $\dots$ , have similar representation as it was shown for the first approximation.

**Lemma 2.** *Functions  $h_k(x, \lambda)$  ( $k = 2, 3, \dots$ ) can be written in the form*

$$\begin{aligned} h_k(x, \lambda) &= \\ &= \int_0^x d\xi \int_\xi^x d\tau (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) A_k(x, \tau, \xi) + \\ &\quad + \int_0^x \psi_1(\xi, \lambda) B_k(x, \lambda) d\xi, \quad k = 2, 3, \dots, \end{aligned} \quad (23)$$

where

$$\begin{aligned} B_k(x, \xi) &= \frac{1}{4} \left\{ \int_0^\xi d\tau \int_{\xi-\tau}^{x-\tau} T(x-\tau, t) B_{k-1}(t, \xi-\tau) dt + \right. \\ &\quad + \int_0^{\frac{x-\xi}{2}} d\tau \int_{\tau+\xi}^{x-\tau} T(x-\tau, t) B_{k-1}(t, \tau+\xi) dt + \\ &\quad \left. + \int_\xi^{\frac{x+\xi}{2}} d\tau \int_{\tau-\xi}^{x-\tau} T(x-\tau, t) B_{k-1}(t, \tau-\xi) dt \right\}. \end{aligned} \quad (24)$$

$$\begin{aligned} A_k(x, \tau, \xi) &= \\ &= \frac{1}{4} \left\{ \int_0^{\tau-\xi} d\eta \int_{\tau-\eta}^{x-\eta} T(x-\eta, t) (A_{k-1}(t, \tau-\eta, \xi) + \right. \\ &\quad \left. + A_{k-1}(t, \tau-\eta, \tau-\xi-\eta)) dt + \right. \\ &\quad + \int_0^{\frac{x-\tau}{2}} d\eta \int_{\eta+\tau}^{x-\eta} T(x-\eta, t) (A_{k-1}(t, \eta+\tau, \xi) + \\ &\quad \left. + A_{k-1}(t, \eta+\tau, \eta+\tau-\xi)) dt + \right. \\ &\quad \left. + \int_{\tau-\xi}^{\frac{x+\tau-2\xi}{2}} d\eta \int_{\eta+2\xi-\tau}^{x-\eta} T(x-\eta, t) (A_{k-1}(t, \eta+2\xi-\tau, \xi) + \right. \end{aligned} \quad (25)$$

$$+ A_{k-1}(t, \eta + 2\xi - \tau, \eta - \tau + \xi) dt + \left. + \int_{\xi}^{x-\tau+\xi} T(x - \tau + \xi, t) B_{k-1}(t, \xi) dt \right\}.$$

*Proof.* Since the initial step of induction is already proved in lemma 1, we assume that representation (23) is true for some  $k$ . We will try to obtain formula (23) for  $(k + 1)$ . From formulas (11) and (23) we have

$$\begin{aligned} h_{k+1}(x, \lambda) &= \\ &= \int_0^x d\tau \int_0^{x-\tau} dt \int_0^t d\xi \int_{\xi}^t d\eta (\psi_1(\eta - \xi + i\xi, \lambda) + \\ &+ \psi_1(\eta - \xi - i\xi, \lambda)) A_k(x, \tau, \xi) \psi_1(\tau, \lambda) T(x - \tau, t) A_k(t, \eta, \xi) + \\ &+ \int_0^x d\tau \int_0^{x-\tau} dt \int_0^t d\xi \psi_1(\tau, \lambda) \psi_1(\xi, \lambda) T(x - \tau, t) B_k(t, \xi). \end{aligned} \quad (26)$$

We transform the right side of formula (26).

Let us consider the expression

$$\begin{aligned} I_1 &= \int_0^x d\tau \int_0^{x-\tau} dt \int_0^t d\xi \int_{\xi}^t d\eta (\psi_1(\eta - \xi + i\xi, \lambda) + \\ &+ \psi_1(\eta - \xi - i\xi, \lambda)) \psi_1(\tau, \lambda) T(x - \tau, t) A_k(t, \eta, \xi). \end{aligned}$$

Integrands which depend on parameter  $\lambda$  do not depend on  $t$ , therefore integral over  $t$  is made inner one now

$$\begin{aligned} I_1 &= \int_0^x d\tau \int_0^{x-\tau} d\xi \int_{\xi}^{x-\tau} d\eta (\psi_1(\eta - \xi + i\xi, \lambda) + \\ &+ \psi_1(\eta - \xi - i\xi, \lambda)) \psi_1(\tau, \lambda) \int_{\eta}^{x-\tau} T(x - \tau, t) A_1(t, \eta, \xi) dt. \end{aligned}$$

Using the rule of multiplication of exponents, we have

$$\begin{aligned} (\psi_1(\eta - \xi + i\xi, \lambda) + \psi_1(\eta - \xi - i\xi, \lambda)) \psi_1(\tau, \lambda) &= \\ &= \frac{1}{4} \{ \psi_1(\eta - \xi + \tau + i\xi, \lambda) + \psi_1(\eta - \xi + \tau - i\xi, \lambda) + \\ &+ \psi_1(\eta - \xi - \tau + i\xi, \lambda) + \psi_1(\eta - \xi - \tau - i\xi, \lambda) + \\ &+ \psi_1(\xi + \tau + i(\eta - \xi), \lambda) + \psi_1(\xi + \tau - i(\eta - \xi), \lambda) + \\ &+ \psi_1(\xi - \tau + i(\xi - \eta), \lambda) + \psi_1(\xi - \tau - i(\xi - \eta), \lambda) \}. \end{aligned}$$

Taking into account the latter expression, we divide  $I_1$  into the sum of four integrals

$$\begin{aligned}
I_1 = \frac{1}{4} \left\{ \int_0^x d\tau \int_0^{x-\tau} d\xi \int_\xi^{x-\tau} d\eta (\psi_1(\eta - \xi + \tau + i\xi, \lambda) + \right. \\
+ \psi_1(\eta - \xi + \tau - i\xi, \lambda)) \int_\eta^{x-\tau} T(x - \tau, t) A_k(t, \eta, \xi) dt + \\
+ \int_0^x d\tau \int_0^{x-\tau} d\xi \int_\xi^{x-\tau} d\eta (\psi_1(\eta - \xi - \tau + i\xi, \lambda) + \\
+ \psi_1(\eta - \xi - \tau - i\xi, \lambda)) \int_\eta^{x-\tau} T(x - \tau, t) A_k(t, \eta, \xi) dt + \\
+ \int_0^x d\tau \int_0^{x-\tau} d\xi \int_\xi^{x-\tau} d\eta (\psi_1(\xi + \tau + i(\eta - \xi), \lambda) + \\
+ \psi_1(\xi + \tau - i(\eta - \xi), \lambda)) \int_\eta^{x-\tau} T(x - \tau, t) A_k(t, \eta, \xi) dt + \\
+ \int_0^x d\tau \int_0^{x-\tau} d\xi \int_\xi^{x-\tau} d\eta (\psi_1(\xi - \tau + i(\eta - \xi), \lambda) + \\
\left. + \psi_1(\xi - \tau - i(\eta - \xi), \lambda)) \int_\eta^{x-\tau} T(x - \tau, t) A_k(t, \eta, \xi) dt \right\}. \quad (27)
\end{aligned}$$

Transforming each member of the right side of expression (27), we have

$$\begin{aligned}
I_1 = \int_0^x d\xi \int_\xi^x d\tau (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) \times \\
\times \frac{1}{4} \left\{ \int_0^{\tau-\xi} d\eta \int_{\tau-\eta}^{x-\eta} T(x - \eta, t) (A_k(t, \tau - \eta, \xi) + \right. \\
+ A_k(t, \tau - \eta, \tau - \xi - \eta)) dt + \\
+ \int_0^{\frac{x-\tau}{2}} d\eta \int_{\eta+\tau}^{x-\eta} T(x - \eta, t) A_k(t, \eta + \tau, \xi) dt + \\
+ A_k(t, \eta + \tau, \eta + \tau - \xi)) dt + \\
+ \int_{\tau-\xi}^{\frac{x+\tau-2\xi}{2}} d\eta \int_{\eta+2\xi-\tau}^{x-\eta} T(x - \eta, t) (A_k(t, \eta + 2\xi - \tau, \xi) + \\
\left. + A_k(t, \eta + 2\xi - \tau, \eta - \tau + \xi)) dt \right\}. \quad (28)
\end{aligned}$$

Let us now consider the expression

$$I_2 = \int_0^x d\tau \int_0^{x-\tau} dt \int_0^t d\xi (\psi_1(\tau, \lambda) \psi_1(\xi, \lambda)) T(x - \tau, t) B_k(t, \xi).$$

Rearranging the order of integration, we obtain

$$I_2 = \int_0^x d\tau \int_0^{x-\tau} \psi_1(\tau, \lambda) \psi_1(\xi, \lambda) d\xi \int_\xi^{x-\tau} T(x - \tau, t) B_k(t, \xi) dt.$$

Expression for  $I_2$  and function  $h_1(x, \lambda)$  are similar to each other (look at formula (17)). The difference among them is that instead of function  $B_0(x, \xi)$  in  $h_1(x, \lambda)$  there is a function  $B_k(x, \xi)$  in  $I_2$ . Therefore, expression for  $I_2$  can be written by analogy with  $h_1(x, \lambda)$ .

$$\begin{aligned} I_2 = & \int_0^x d\xi \int_\xi^x (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) d\tau \times \\ & \times \frac{1}{4} \int_\xi^{x-\tau+\xi} T(x - \tau + \xi, t) B_k(t, \xi) dt + \\ & + \int_0^x \psi_1(\xi, \lambda) d\xi \frac{1}{4} \int_0^\xi d\tau \int_{\xi-\tau}^{x-\tau} T(x - \tau, t) B_k(t, \xi - \tau) dt + \\ & + \int_0^{\frac{x-\xi}{2}} d\tau \int_{\eta+\tau}^{x-\eta} T(x - \eta, t) A_k(t, \eta + \tau, \xi) dt + \\ & + A_k(t, \eta + \tau, \eta + \tau - \xi) dt + \\ & + \int_{\tau-\xi}^{\frac{x+\tau-2\xi}{2}} d\eta \int_{\tau+\xi}^{x-\tau} T(x - \eta, t) B_k(t, \tau + \xi) dt + \\ & + \int_\xi^{\frac{x+\xi}{2}} d\tau \int_{\tau-\xi}^{x-\tau} T(x - \tau, t) B_k(t, \tau - \xi) dt. \quad (29) \end{aligned}$$

We substitute transformed expressions (28), (29) of integrals  $I_1$  and  $I_2$  into formula (26) for  $(k+1)$ -st approximation, then we will get the following representation

$$\begin{aligned} h_{k+1}(x, \lambda) = & \\ = & \int_0^x d\xi \int_\xi^x d\tau (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) A_{k+1}(x, \tau, \xi) + \\ & + \int_0^x \psi_1(\xi, \lambda) B_{k+1}(x, \xi) d\xi, \end{aligned}$$

where

$$\begin{aligned}
A_{k+1}(x, \tau, \xi) = & \frac{1}{4} \left\{ \int_0^{\tau-\xi} d\eta \int_{\tau-\eta}^{x-\eta} T(x-\eta, t) (A_k(t, \tau-\eta, \xi) + \right. \\
& + A_k(t, \tau-\eta, \tau-\xi-\eta)) dt + \\
& + \int_0^{\frac{x-\tau}{2}} d\eta \int_{\eta+\tau}^{x-\eta} T(x-\eta, t) (A_k(t, \eta+\tau, \xi) + \\
& + A_k(t, \eta+\tau, \eta+\tau-\xi)) dt + \\
& + \int_{\tau-\xi}^{\frac{x+\tau-2\xi}{2}} d\eta \int_{\eta+2\xi-\tau}^{x-\eta} T(x-\eta, t) (A_k(t, \eta+2\xi-\tau, \xi) + \\
& + A_k(t, \eta+2\xi-\tau, \eta-\tau+\xi)) dt + \\
& \left. + \int_{\xi}^{x-\tau+\xi} T(x-\tau+\xi, t) B_k(t, \xi) dt \right\},
\end{aligned}$$

$$\begin{aligned}
B_{k+1}(x, \xi) = & \frac{1}{4} \left\{ \int_0^{\xi} d\tau \int_{\xi-\tau}^{x-\tau} T(x-\tau, t) B_k(t, \xi-\tau) dt + \right. \\
& + \int_0^{\frac{x-\xi}{2}} d\tau \int_{\tau+\xi}^{x-\tau} T(x-\tau, t) B_k(t, \tau+\xi) dt + \\
& \left. + \int_{\xi}^{\frac{x+\xi}{2}} d\tau \int_{\tau-\xi}^{x-\tau} T(x-\tau, t) B_k(t, \tau-\xi) dt \right\}.
\end{aligned}$$

This way we obtained the necessary formulas for the subsequent approximation  $h_{k+1}$  and also for its coefficients  $A_{k+1}$ ,  $B_{k+1}$ . Consequently, lemma 2 is proved.  $\square$

When  $0 \leq \xi \leq \tau \leq x \leq b$

$$\begin{aligned}
|B_k(x, \xi)| & \leq \left(\frac{3}{8}L\right)^k M \frac{(x+\xi)^k}{k!} \cdot \frac{(x-\xi)^k}{k!}, \\
|A_k(x, \xi)| & \leq \left(\frac{3}{4}L\right)^k K \frac{(x+\tau)^k}{k!} \cdot \frac{(x-\tau)^k}{k!}, \quad k \geq 1,
\end{aligned}$$

where

$$|B_0(x, \xi)| \leq M, \quad |A_1(x, \tau, \xi)| \leq K, \quad |T(u, t)| \leq L.$$

following functions are presented

$$B(x, \xi) = \sum_{k=0} B_k(x, \xi), \quad A(x, \tau, \xi) = \sum_{k=1} A_k(x, \tau, \xi),$$

which do not depend on parameter  $\lambda$ , and assessment of coefficients  $A_{k+1}$ ,  $B_{k+1}$  is proved which guarantee the uniform convergence of series  $\sum_{k=1} A_k$  and  $\sum_{k=1} B_k$ . Thus, by substituting (26) and (10) into the right side of (12) and changing the order of integration and summation, we reach the main theorem.

**Theorem 1.** *Solution  $y(x, \lambda)$  of the problems (6), (7) has integral representation*

$$y(x, \lambda) = \int_0^x d\xi \int_{\xi}^x d\tau (\psi_1(\tau - \xi + i\xi, \lambda) + \psi_1(\tau - \xi - i\xi, \lambda)) A(x, \tau, \xi) + \int_0^x \psi_1(\xi, \lambda) B(x, \xi) d\xi,$$

where

$$A(x, \tau, \xi) = \sum_{k=1} A_k(x, \tau, \xi), \quad B(x, \xi) = \sum_{k=1} B_k(x, \xi).$$

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