

STABILITY OF CIRCULAR ORBITS IN SCHWARZSCHILD SPACETIME

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Abstract: The Jacobi stability of circular orbits in the Schwarzschild spacetime is investigated and it is shown that condition for Jacobi (un)stable circular orbits is the same known Liapunov (in)stability criteria.

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1. Introduction

The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold equipped with a Riemannian or Finslerian metrics to the non-metric manifold. The Jacobi (also called KCC) stability studies the robustness of a second-order differential equation where robustness is a measure of insensitivity and adaptation to change of the system internal parameters and the environment. Jacobi stability analysis of dynamical systems has been recently studied by several authors [5, 8, 10, 15, 16, 17] using the Kosambi-Cartan-Chern (KCC) theory.

The KCC theory studies the deviation of close by trajectories, which permits one to estimate the perturbation permitted around the steady states solutions of the second-order differential equation. This is of interest in physical applications when one needs to identify the "robust arrest" regions, i.e., the regions where one has both Liapunov and Jacobi stability.

Initiated from the works of D. D. Kosambi [13], E. Cartan [11] and S. S. Chern [12], and hence the abbreviation KCC (Kosambi-Cartan-Chern), the KCC theory has applications in engineering, physics, chemistry and biology. Recent developments have been made in KCC applications in gravitation and cosmology. Boehmer *et al.* [17] have analyzed Jacobi stability and its relations with the linear Liapunov stability analysis of dynamical systems, and presented a comparative study of these methods in the fields of gravitation and astrophysics. Nevertheless, there are apparent discrepancies observed in their studies in the sense that in some cases investigated, Liapunov stability is not in agreement with Jacobi stability. Most recently [1, 2] demonstrated that these two types of stability concepts are in agreement in two cases of torque-free rigid body motion around a stationary point and circular orbits in a central force field.

The purpose of the present paper is to present a specific example from classical general relativity, i.e., the stability of circular orbits in the Schwarzschild spacetime where Liapunov stability condition is entirely in agreement with Jacobi stability. We review the KCC theory to analyze its relations with the linear Liapunov stability analysis of dynamical systems, and to present a comparative study of these methods for circular orbits stability. We also give a *full dynamical systems treatment* of the condition for the stability of circular orbits in the Schwarzschild spacetime which is rarely presented in the literature.

The present paper is organized as follows. In Section 2 we review briefly the Liapunov stability analysis. Section 3 is devoted to the KCC theory and the Jacobi stability analysis of the dynamical systems. The equation of motion for a particle in the Schwarzschild spacetime following a geodesics is briefly reviewed in Section 4. A full dynamical systems approach of the motion of a particle in the Schwarzschild spacetime leading to stability analysis of (Liapunov and Jacobi) circular orbits are given in Section 5. A discussion and final remarks appears in Section 6.

2. Liapunov Stability

2.1. Brief Survey of Techniques in Dynamical Systems

In this section we give precise mathematical expression to the concepts of stability of dynamical systems used in this research [14, 19].

Definition. A *fixed point* of a system of autonomous ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (1)$$

is a point $\bar{\mathbf{x}} \in \mathbf{R}^n$ such that $\mathbf{f}(\bar{\mathbf{x}}) = 0$ where $\mathbf{f}$ is a C^1 vector field in $\mathbf{R}^n$. In the present paper, the fixed points are also called *equilibrium point* interchangeably.

Definition. Let $\bar{\mathbf{x}}$ be a fixed point of the DE (1). The point $\bar{\mathbf{x}}$ is called a *hyperbolic* fixed point if $\operatorname{Re}(\lambda_i) \neq 0$ for all eigenvalues, λ_i , of the Jacobian of the vector field $\mathbf{f}(\mathbf{x})$ evaluated at $\bar{\mathbf{x}}$. Otherwise the point is called *non-hyperbolic*.

Definition. (Liapunov Stability) $\bar{\mathbf{x}}(t)$ is said to be stable (or Liapunov stable) if, given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any other solution, $\mathbf{y}(t)$, of (1) satisfying $|\bar{\mathbf{x}}(t_0) - \mathbf{y}(t_0)| < \delta$, then $|\bar{\mathbf{x}}(t) - \mathbf{y}(t)| < \epsilon$ for $t > t_0$, $t_0 \in \mathbf{R}$. A solution which is not stable is said to be *unstable*.

Definition. (Asymptotic Stability) $\bar{\mathbf{x}}(t)$ is said to be *asymptotically stable* if it is Liapunov stable and for any other solution, $\mathbf{y}(t)$, of (1), there exists a constant $b > 0$ such that, if $|\bar{\mathbf{x}}(t_0) - \mathbf{y}(t_0)| < b$, then $\lim_{t \rightarrow \infty} |\bar{\mathbf{x}}(t) - \mathbf{y}(t)| = 0$ as $t \rightarrow \infty$.

Theorem. (Liapunov Stability Theorem) Consider the vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$. Let $\bar{\mathbf{x}}$ be a equilibrium point of the above vector field and let $V : U \rightarrow \mathbf{R}$ be a C^1 function, called the *Liapunov function*, defined on some neighborhood U of $\bar{\mathbf{x}}$ such that

i) $V(\bar{\mathbf{x}}) = 0$ and $V(\mathbf{x}) > 0$ if $\mathbf{x} \neq \bar{\mathbf{x}}$.

ii) $\dot{V}(\mathbf{x}) \leq 0$ in $U - \bar{\mathbf{x}}$.

Then $\bar{\mathbf{x}}$ is stable. Moreover, if

iii) $\dot{V}(\mathbf{x}) < 0$ in $U - \bar{\mathbf{x}}$

Then $\bar{\mathbf{x}}$ is asymptotically stable.

It can be shown that if V is a Liapunov function for a fixed point c of the vector field $\mathbf{f}(\mathbf{x})$, then near c the integral curves of $\mathbf{f}(\mathbf{x})$ are tangent to the level hypersurface $\{\mathbf{x} \in \mathbf{R}^n | V(\mathbf{x}) = k = \text{const.}\}$, namely $\nabla U \cdot \mathbf{f}(\mathbf{x}) = 0$, or cross these hypersurfaces toward their interiors (direction of decreasing values of V).

One must begin a qualitative analysis of a dynamical system by locating its fixed points. Once the fixed points are obtained, one should consider the dynamics in a local neighbourhood of each of the points. Assuming that the vector field $\mathbf{f}(\mathbf{x})$ is of class C^1 , then the local behaviour of the orbits can be determined by linear approximation of the vector field in the local neighbourhood of the fixed point $\bar{\mathbf{x}}$. In this neighborhood,

$$\mathbf{f}(\mathbf{x}) \approx D\mathbf{f}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) \quad (2)$$

where $D\mathbf{f}(\bar{\mathbf{x}})$ is the Jacobian of the vector field at the fixed point $\bar{\mathbf{x}}$. The system (2) is referred to as the *linearization of the DE at the fixed point*. The classification of the fixed points can then be done by studying the eigenvalues of the Jacobian of the linearized vector field at the point. The classification then follows from the fact that if the fixed point is hyperbolic in nature the orbits of the non-linear system and its linear approximation are *topologically equivalent* in a neighborhood of the fixed point. The following important theorem supports this fact:

Theorem. (Hartman-Grobman Theorem) *Consider a DE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ where the vector field $\mathbf{f}$ is of class C^1 . If $\bar{\mathbf{x}}$ is a hyperbolic fixed point of the DE then there exists a neighbourhood of $\bar{\mathbf{x}}$ on which the flow is topologically equivalent to the flow of the linearization of the DE at $\bar{\mathbf{x}}$.*

Given a linear system of ODEs:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad (3)$$

where A is a $n \times n$ square matrix with constant coefficients, it is a straightforward to show when eigenvalues of A have negative real parts, then the fixed point is a hyperbolic sink (stable), since all solutions converge to the fixed point $\bar{\mathbf{x}} = 0$; a linear system whose eigenvalues have positive real parts is called to have a hyperbolic source (unstable), because the solutions in the neighborhood of $\bar{\mathbf{x}} = 0$ all diverge from that point; and when one eigenvalue is negative and the other is positive the fixed point is said to be a hyperbolic saddle (unstable), where some orbits are attracted to the fixed point, and some are repelled away. If the eigenvalues are complex conjugates with non-zero real parts then the fixed point is of type spiral (stable or unstable depends on the sign of $Re(\lambda_i)$) and if $Re(\lambda_i) = 0$, then the fixed point is of type center.

In the non-linear case, the topological equivalence of flows allows for a similar classification of the fixed points. The equivalence only applies in directions where the eigenvalue has non-zero real parts. If the linearization of the dynamical systems at the fixed point has purely imaginary eigenvalues, then the

stability type of equilibrium point cannot be deduced from the linear approximation. In such case one must use other methods such as Liapunov stability analysis to analyze the case. This brief introduction concludes our treatment of stability of dynamical systems which is used in this paper.

3. Kosambi-Cartan-Chern (KCC) Theory and Jacobi Stability

3.1. General Theory

We now introduce the main ideas of KCC-theory [4, 5, 8, 10]. Let $\mathcal{M}$ be a real, smooth n -dimensional manifold and let $T\mathcal{M}$ be its tangent bundle. Let $\mathbf{u} = (\mathbf{x}, \mathbf{y})$ be a point in $T\mathcal{M}$, where $\mathbf{x} = (x^1, x^2, \dots, x^n)$, and $\mathbf{y} = (y^1, y^2, \dots, y^n)$ which means $y^i = \frac{dx^i}{dt}$, $i = 1, 2, \dots, n$. Consider the second order differential equations in normalized form

$$\frac{d^2 x^i}{dt^2} + 2G^i(\mathbf{x}, \mathbf{y}) = 0, \quad i = 1, 2, \dots, n, \quad (4)$$

where $G^i(\mathbf{x}, \mathbf{y})$ are smooth functions defined in a local system of coordinates on $T\mathcal{M}$. In fact the system of (4) is motivated by Euler-Lagrange equations of classical dynamics

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} &= F_i \\ y^i &= \frac{dx^i}{dt}, \end{aligned} \quad i = 1, 2, \dots, n, \quad (5)$$

where L is the Lagrangian of $\mathcal{M}$, and F_i are the external forces [7].

In order to find the KCC differential invariants of the system (4) under the non-fixed coordinate transformation

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, x^2, \dots, x^n) \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \end{aligned} \quad i = 1, 2, \dots, n. \quad (6)$$

We now define the *KCC-covariant differential* of a vector field $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ in an open subset $\Omega \subseteq \mathbf{R}^n \times \mathbf{R}^n$ as [4, 5, 8, 17]

$$\frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N_j^i \xi^j. \quad (7)$$

where $N_j^i = \frac{\partial G^i}{\partial y^j}$ defines the coefficients of a non-linear connection N on the tangent bundle $T\mathcal{M}$. For $\xi^i = y^i$, one obtains

$$\frac{Dy^i}{dt} = N_j^i y^j - 2G^i = -\epsilon^i. \quad (8)$$

The contravariant vector field ϵ^i on Ω is called the *first KCC invariant* and plays the role of an external force. The terms ϵ^i has geometrical character since with respect to a coordinate transformation (6), we have

$$\tilde{\epsilon}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \epsilon^j. \quad (9)$$

Furthermore, if one varies the paths $x^i(t)$ of the system (4) into nearby ones prescribed by

$$\tilde{x}^i(t) = x^i(t) + \eta \xi^i(t), \quad (10)$$

where $|\eta|$ is a small parameter and $\xi^i(t)$ are the components of some contravariant vector field defined along the path $x^i(t)$. We derive the following variational equations after substituting Eqs.(10) into Eqs.(4) and taking the limit as $\eta \rightarrow 0$ [3, 4, 15]

$$\frac{d^2 \xi^i}{dt^2} + 2N_j^i \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j} \xi^j = 0. \quad (11)$$

We can write Eqs.(11) in terms of the KCC covariant differential (7) in the covariant form

$$\frac{D^2 \xi^i}{dt^2} = P_j^i \xi^j, \quad (12)$$

where the right hand side (1,1)-tensor

$$P_j^i = -2\frac{\partial G^i}{\partial x^j} - 2G^l G_{jl}^i + y^l \frac{\partial N_j^i}{\partial x^l} + N_l^i N_j^l, \quad (13)$$

and $G_{jl}^i \equiv \partial N_j^i / \partial y^l$ is called the Berwald connection [3, 15]. Eq. (12) is called the *Jacobi equations*, or *the variation equations* associated with the system of second order differential equations (4), and P_j^i is called *the second KCC-invariant*.

The third, fourth and fifth invariants of the system (4) are given by [6]

$$P_{jk}^i \equiv \frac{1}{3} \left(\frac{\partial P_j^i}{\partial y^k} - \frac{\partial P_k^i}{\partial y^j} \right), \quad P_{jkl}^i \equiv \frac{\partial P_{jk}^i}{\partial y^l}, \quad D_{jkl}^i \equiv \frac{\partial G_{jk}^i}{\partial y^l}. \quad (14)$$

The third invariant represents a torsion tensor, while the fourth and fifth invariants are interpreted as the Riemann-Christoffel curvature tensor, and the Douglas tensor, respectively [4, 15]. These tensors which describe the geometrical properties of a system of second-order differential equations always exist [4, 15].

The term Jacobi stability within the KCC theory is justified by the fact that, when (4) represents the second order differential equations for the geodesic equations in Finsler or Riemannian geometry [9], then (12) is the Jacobi field equations for the geodesic deviation. The Jacobi equation (12) of the Finsler manifold $(M; F)$ can be written in the scalar form [9]:

$$\frac{d^2 v}{ds^2} + K \cdot v = 0,$$

where $\xi^i = v(s)\eta^i$ is a Jacobi field along the geodesic $x^i(s)$, η^i is the unit normal vector field, and K is the flag curvature of $(M; F)$. The sign of K affects the geodesic rays: if $K > 0$, then the geodesics bunch together (are Jacobi stable), and if $K < 0$, then they disperse (are Jacobi unstable). Therefore, positive/negative flag curvature is equivalent to negative/positive eigenvalues of P_j^i .

The following result is known [17] :

Theorem. *The trajectories of (4) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P_j^i are strictly negative everywhere, and Jacobi unstable, otherwise.*

The geometrical meaning of the above theorem in the context of an Euclidean, Riemannian or Finslerian structure is discussed in [17].

3.2. The Relation between Linear Stability and Jacobi Stability

The correlation between linear stability and Jacobi stability has been completely discussed in [17] for the 2-dimensional dynamical systems case. The result of this discussion is crucial for the present paper. Here one wants to compare the signs of the eigenvalues of the Jacobian matrix J at a fixed point with the signs of the eigenvalues of the deviation curvature tensor P_i^j evaluated at the same point.

Let us consider the following autonomous system of ODE:

$$\begin{cases} \dot{u} = f(u, v) \\ \dot{v} = g(u, v) \end{cases}, \quad (15)$$

such that the point $(0, 0)$ is an equilibrium, i.e., $f(0, 0) = g(0, 0) = 0$. One can always make variable changes $\bar{u} = u - u_0$ and $\bar{v} = v - v_0$ to translate a general fixed point (u_0, v_0) to the origin $(0, 0)$. We denote by J the Jacobian matrix of (15), i.e.

$$J(u, v) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} \quad (16)$$

where the subscripts are partial derivatives with respect to u and v . The characteristic equation is given by

$$\lambda^2 - (\text{tr}A)\lambda + \det A = 0, \quad (17)$$

where $\text{tr}A$ and $\det A$ are the trace and the determinant of the matrix $A := J|_{(0,0)}$, respectively. The signs of the trace, determinant of A , and of the discriminant $\Delta = (f_u - g_v)^2 + 4f_v g_u = (\text{tr}A)^2 - 4\det A$ give the linear stability of the fixed point $(0, 0)$ as described in the previous section.

In applying the KCC theory to the ODE system (15), by elimination of one of the variables, one can transform (15) into a system of differential equations of the form (4) and compute the deviation curvature. Boehmer *et al.* [17] obtain the following theorem:

Theorem. *Let us consider the ODE (15) with the fixed point $P = (0, 0)$ such that $g_u|_{(0,0)} \neq 0$. Then,*

$$4P_1^1|_{(0,0)} = -4g_{,1}^1|_{(0,0)} + (g_{,1}^1)^2|_{(0,0)} = \Delta := (\text{tr}A)^2 - 4\det A. \quad (18)$$

where $g^1(x, y) \equiv -g_u(u(x, y), x)f(u(x, y), x) - g_v(u(x, y), x)y$ and $y \equiv g(u, v)$ and $g_{,1}^1$ is simply $\frac{\partial g^1}{\partial x}$. In this case, the trajectory $v = v(t)$ is Jacobi stable if and only if $\Delta < 0$.

More generally, we can conclude that in the case $g_u|_{(0,0)} \neq 0$ and $f_v|_{(0,0)} \neq 0$ both trajectories $u = u(t)$ and $v = v(t)$ are Jacobi stable if and only if $\Delta < 0$. We emphasize on this theorem greatly because our main Jacobi analysis in this paper is based on this theorem. Boehmer *et al.* [17] also present the following corollary:

Corollary. *Let us consider the ODE (15) with the fixed point $P = (0, 0)$ such that $g_u|_{(0,0)} \neq 0$. Then, the Jacobian J estimated at the point P has complex eigenvalues if and only if P is a Jacobi stable point.*

A combination of the linear stability as depicted in the Fig. 1, with the definition of Jacobi stability with the conditions stated in the theorem above, one obtains the following relations:

1. Region I	$\Delta > 0$	Jacobi unstable
	$\text{tr} A > 0$	Unstable node
	$\det A > 0$	
2. Region II	$\Delta < 0$	Jacobi stable
	$\text{tr} A > 0$	Unstable focus
	$\det A > 0$	
3. Region III	$\Delta < 0$	Jacobi stable
	$\text{tr} A < 0$	Stable focus
	$\det A > 0$	
4. Region IV	$\Delta > 0$	Jacobi unstable
	$\text{tr} A < 0$	Stable node
	$\det A > 0$	
5. Region V	$\Delta > 0$	Jacobi unstable
	$\det A < 0$	Saddle point

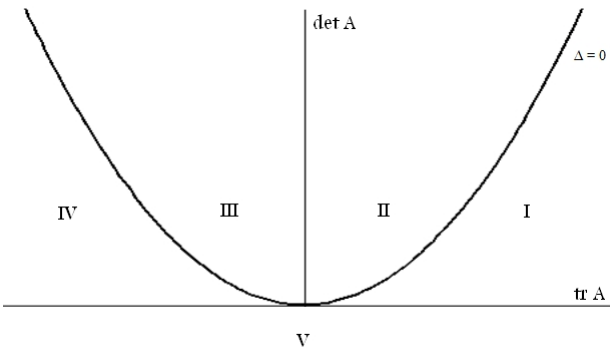


Figure 1: Linear stability

The above description of all possible cases in 2-dimension suggests that linear stability and Jacobi stability do not have to agree in all cases. In fact, the stability in Jacobi sense refers to a linear stability type of the trajectories in the

curved space endowed with a nonlinear connection and a curvature tensor. Here the role of usual partial derivative is played by the covariant derivative along the flow. This suggests the difference in the meaning of linear stability and Jacobi stability. Nevertheless, in the present paper, we have found a known physical case, namely, the stability of circular orbits in the Schwarzschild spacetime, where these two stability concepts would match.

4. Particle Orbits in the Schwarzschild Spacetime Revisited[18]

The Schwarzschild metric, describing a spherical body of mass M , has the following line element in standard (spherical polar) Schwarzschild coordinates:

$$ds^2 = g_{ab}dx^a dx^b = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (19)$$

where g_{ab} is the metric tensor, and we use the Einstein summation convention where repeat indices are summed over. Because of spherical symmetry of the metric, motion is always confined to a single plane, and we can choose that plane to be the equatorial plane $\dot{\theta} = 0$. For timelike geodesics, we choose τ to be the proper time; and for null geodesics, we choose τ to be an affine parameter. Thus the metric (19), recalling that $(\theta = \pi/2)$, takes the form

$$-\kappa = g_{ab}dx^a dx^b = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \quad (20)$$

where $\cdot \equiv \frac{d}{d\tau}$ and

$$\kappa = \begin{cases} 1, & \text{timelike geodesics} \\ 0, & \text{null geodesics} \end{cases}. \quad (21)$$

Time translational symmetry represented by the Killing vector $\xi^a = (\partial/\partial t)^a$ introduces a constant of motion, $\tilde{E}$,

$$\tilde{E} = -g_{ab}\xi^a \dot{x}^b = (1 - 2M/r)\dot{t}. \quad (22)$$

In general, one may interpret $\tilde{E}$ for timelike geodesics as representing the total energy (including gravitational potential energy) per unit rest mass of a particle following the geodesic in question, relative to a static observer at infinity. Similarly, in the null case, $\hbar\tilde{E}$ represents the total energy of a photon.

The rotational Killing vector $\psi^a = (\partial/\partial\phi)^a$ also yields a constant of the motion, L ,

$$L = g_{ab}\psi^a \dot{x}^b = r^2\dot{\phi}. \quad (23)$$

One may interpret L as the angular momentum per unit rest mass of a particle in the timelike case, and we may interpret $\hbar L$ as the angular momentum of a photon in the null case. Substituting equations (22) and (23) in the metric (20), one obtains the equation for geodesics,

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\kappa + \frac{L^2}{r^2}\right) = \frac{1}{2}\tilde{E}^2. \quad (24)$$

Note that the equation (24) above for $\dot{r}$ can be written as a Newtonian central force problem

$$E = \frac{1}{2}\dot{r}^2 + V(r), \quad (25)$$

where $E \equiv \tilde{E}^2/2$ and the effective potential $V(r)$ is given by

$$V(r) \equiv \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\kappa + \frac{L^2}{r^2}\right) = \frac{1}{2}\kappa - \kappa\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}. \quad (26)$$

It should be noted that the crucial new feature provided by general relativity is that in equation (26) in addition to the "Newtonian term," $-\kappa M/r$, and the "centrifugal barrier term," $L^2/2r^2$, we have the new term, $-ML^2/r^3$, which dominates over the centrifugal barrier term at small r .

Differentiating with respect to τ and canceling $\dot{r}$ from both sides of (25) gives the following one-dimensional geodesics equation

$$\ddot{r} = -dV/dr, \quad (27)$$

which is our main starting point for dynamical systems approach to our problem.

5. Qualitative Analysis

5.1. Liapunov Stability Analysis

We present here the qualitative solution of particle motion following a geodesics in Schwarzschild spacetime using standard application of the fixed point and stability analysis to the one-dimensional radial equation (27). We present a dynamical systems analysis because firstly, it is an elegant approach and secondly, our Jacobi stability analysis follows the same approach. There has been rarely a full dynamical systems treatment of Schwarzschild geodesy presented in the general relativity literature. The 2nd-order differential equation $\ddot{r} = -V'(r)$

can always be transformed to a corresponding 1st-order system in (r, p) space defined by

$$\begin{cases} \dot{r} = p \\ \dot{p} = -V'(r) \end{cases}, \quad (28)$$

where (r, p) space is called phase space in Hamiltonian mechanics. Because of the energy conservation (25), the integral curves $t \rightarrow (r(t), p(t))$ of the 1st-order systems lie on one of the curves in the (r, p) plane with equation

$$\frac{p^2}{2} + V(r) = E = \text{constant}.$$

To apply linearization procedure, now assume that $V : (0, \infty) \rightarrow \mathbf{R}$ is twice continuously differentiable, and has only *finitely* many critical points i.e., $V'(r) = 0$. Let $\mathbf{f} : (0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}^2$ be the vector field

$$\mathbf{f}(r, p) = (p, -V'(r)), \quad (29)$$

for the system $\dot{r} = p, \dot{p} = -V'(r)$. Clearly the fixed points for $\mathbf{f}$ are the points $(r_*, 0)$, where r_* is the critical point of V where the effective potential energy is an extremum. The Jacobian matrix of $\mathbf{f}$ at any (r, p) is easily seen to be

$$J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(r, p) = \begin{bmatrix} 0 & 1 \\ -V''(r) & 0 \end{bmatrix} \quad (30)$$

and has characteristic equation: $\lambda^2 + V''(r) = 0$. Thus its eigenvalues are

$$\lambda = \pm \sqrt{-V''(r)}. \quad (31)$$

Here we consider only fixed points that are simple (which only happens when $V''(r_*) \neq 0$). In that case (31) shows that $(r_*, 0)$ is a saddle point when $V''(r_*) < 0$ (that is, when V has a local maximum at r_*) and is a *possible* center when $V''(r_*) > 0$ (i.e., when V has a local minimum at r_*). In the latter case, we are assured that the fixed point is actually a center, because the energy function

$$E(r, p) \equiv \frac{p^2}{2} + V(r) \quad (32)$$

is in fact a Liapunov function for the system. To see this, note that

$$\nabla E(r, p) = (V'(r), p), \quad (33)$$

and so $\nabla E(r, p) \cdot \mathbf{f}(r, p) = 0$, for all (r, p) . Thus, the orthogonality condition between the gradient field and the level surface for a Liapunov function is

satisfied. The second condition on the Liapunov function E requires that the fixed point be a local minimum of E . This can be checked by looking at its Hessian, $\frac{\partial^2 E}{\partial r \partial p}$, which in this case is the 2×2 matrix

$$\mathcal{H}_E = \begin{bmatrix} V''(r) & 0 \\ 0 & 1 \end{bmatrix}. \quad (34)$$

Clearly this matrix is positive definite when $V''(r) > 0$, and thus at a possible center $(r_*, 0)$, the Liapunov function E has a local minimum. Hence the possible center is an actual center. To summarize, the fixed point $(r_*, 0)$ is

$$\begin{cases} \text{Liapunov stable if} & V''(r_*) > 0 \\ \text{Liapunov unstable if} & V''(r_*) < 0. \end{cases} \quad (35)$$

One should realize that within the setting of the particle motion in the Schwarzschild spacetime, the fixed points of the radial equation, i.e., $r = r_* = \text{constant}$ correspond to *circular* orbits of a particle about the origin $r = 0$. Fig. 2 depicts the phase portrait for stable and unstable circular orbits for both timelike and null geodesics in the Schwarzschild spacetime.

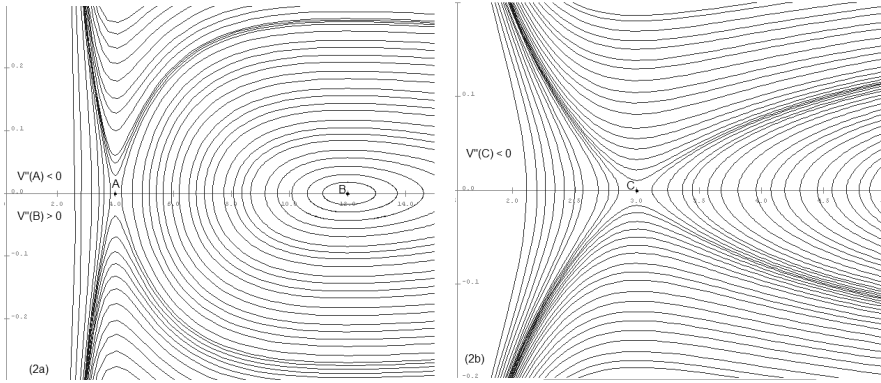


Figure 2: The $r - p$ phase portrait or the energy curves $p^2/2 + V(r) = E$, for various values of E for (2a) Schwarzschild spacetime timelike geodesics where $V''(A) < 0$ and $V''(B) > 0$ hence fixed point B , a circular orbit, is a stable center and fixed point A , another circular orbit, is an unstable saddle point, (2b) Schwarzschild spacetime null geodesics where $V''(C) < 0$ and hence fixed point C , a circular orbit, is an unstable saddle point.

One can actually find these fixed points easily. First, in timelike geodesics case where $\kappa = 1$, The fixed points are the solutions of

$$0 = V'(r) = r^{-4}[Mr^2 - L^2r + 3ML^2] \quad (36)$$

which has the roots

$$R_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M} \quad (37)$$

Thus, if $L^2 < 12M^2$, there are no fixed points. A particle moving toward the center of attraction $\dot{r} \leq 0$ with $L^2 < 12M^2$ will fall directly to the $r = 2M$ surface and will continue its fall into the spacetime singularity at $r = 0$. For $L^2 > 12M^2$, it is easy to check that $V''(r_* = R_+) < 0$ so the extremum R_+ is a minimum of V , while R_- is a maximum. Thus, stable circular orbits ($\dot{r} = 0$) exist at the radius $r_* = R_+$, and unstable circular orbits exist at $r_* = R_-$. One should note that according to equation (37), R_+ is restricted to the range

$$R_+ > 6M. \quad (38)$$

Therefore, in general relativity, no stable circular orbits exist at radii smaller than $6M$. Furthermore, the unstable circular orbits are restricted to the range

$$3M < R_- < 6M. \quad (39)$$

Thus, no circular orbits at all exist at radii less than $3M$.

We now consider the null geodesics. Setting $\kappa = 0$ in equation (26), we find the effective potential for null geodesics to be simply

$$V = \frac{L^2}{2r^3}(r - 2M). \quad (40)$$

Therefore, the only extremum of V is a maximum occurring at $r = 3M$. Thus, in general relativity, unstable circular orbits of photons exist at radius $3M$, so that, physically, gravity has a very significant effect on the propagation of light rays in the strong field regime. It's important to remember that these are only the geodesics; there is nothing to stop an accelerating particle from dipping below $r = 3M$ and emerging, as long as it stays beyond $r = 2M$.

In the next subsection, we will show that in order for the circular orbits to be Jacobi stable, one would arrive at the same condition stated in (35). In other word, the Jacobi stability and Liapunov stability condition (35) are in complete agreement for the stability of circular orbits.

5.2. Jacobi Stability Analysis

Following the discussion in section 3.2 and the relation (18), we are now prepared to perform Jacobi stability analysis for the vector field (29). Recalling

$$\Delta := \text{tr}(J)^2 - 4\det(J) = \begin{cases} < 0 & \text{Jacobi Stable} \\ \geq 0 & \text{Jacobi Unstable.} \end{cases}$$

where J is the Jacobian of the vector field at the fixed point. The two eigenvalues of the Jacobian of the vector field (29) are given by $\lambda = \pm\sqrt{-V''(r)}$ which determines $\text{tr}J = \lambda_1 + \lambda_2 = 0$ since the eigenvalues are identical with opposite signs and $\det J = \lambda_1\lambda_2 = V''(r)$. Therefore we have

$$\Delta \Big|_{r=r_*} = 0 - 4V''(r) = -4V''(r) \Rightarrow \begin{cases} \text{Jacobi stable if} & V''(r_*) > 0 \\ \text{Jacobi unstable if} & V''(r_*) < 0. \end{cases}$$

which is in complete agreement with the condition derived from Liapunov stability condition (35).

6. Discussions and Final Remarks

Two main stability analysis methods, namely Liapunov stability analysis and the Jacobi stability analysis are reviewed in the present paper. Boehmer *et al.* [17] have considered the stability properties of several dynamical systems that play important roles in gravitation and cosmology where both methods of Liapunov linear stability analysis and Jacobi stability analysis, or the KCC theory, were used. Their study of stability has been done by analyzing the behavior of steady states of the respective dynamical system. Linear stability analysis is performed by the linearization of the dynamical system via the Jacobian matrix of a non-linear system at the fixed points, while the KCC theory involves stability of a whole trajectory in a tubular region [15], where Jacobi stability (or instability) means that the trajectories of the structure equations will bunch together (or disperse) when approaching the fixed point.

Boehmer *et al.* [17] have shown that there is a good correlation (but not perfect) between the linear stability of the fixed points, and the robustness of the corresponding trajectory to a small perturbation. In fact, there are cases where a fixed point is Liapunov stable but Jacobi unstable or vice versa and hence some discrepancies in the results of these two stability methods. In the present paper, we have explicitly shown that the condition for the (in)stability

of circular orbits in the Schwarzschild spacetime derived from Liapunov stability analysis is exactly the same as criteria for Jacobi (un)stable circular orbits. To do this task, we have also given a full dynamical systems analysis of motion of a particle in the Schwarzschild spacetime, which is rarely discussed in classical general relativity or dynamical systems textbooks.

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