

NUMERICAL SOLUTION OF VOLTERRA
DELAY-INTEGRO-DIFFERENTIAL EQUATIONS
VIA SPLINE/SPECTRAL METHODS

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Abstract: In this paper, we present a mixed spline/spectral method to solve Volterra delay-integro-differential equations (VDIDEs). This method is based on generating the sextic spline collocation methods in all subintervals. The approximation of the integration in these subintervals, can be calculated by the El-Gendi method. Convergence results of the present method are presented. Numerical results are given to illustrate the efficiency of the present method.

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1. Introduction

Volterra delay-integro-differential equations (VDIDEs) arise widely in scientific fields such as biology, ecology, medicine and physics (cf. [1, 2, 13]). This class of equations plays an important role in modelling diverse problems of engineering and natural science, and hence have come to intrigue researchers in numerical computation and analysis.

In this paper, we propose a numerical technique which is based on a mixed of sextic C^1 -spline collocation method [10] and El-Gendi method [8] to solve VDIDEs of the form

$$y'(t) = f\left(t, y(t), y(t - \tau), \int_{t-\tau}^t g(t, v, y(v))dv\right), \quad t \in [t_0, T], \quad (1a)$$

with the initial condition

$$y(t) = \phi(t) \quad t \in [-\tau, t_0], \quad (1b)$$

where the functions f, g are assumed to be sufficiently smooth with respect to their arguments, τ is a positive number, and ϕ is a given C^1 -function. Note that, when $\tau = 0$, Eq. (1) reduces to the standard initial value problem. One-step collocation and Runge-Kutta methods for solving VDIDEs were considered by many authors (see, e.g. [4, 11, 12, 14, 15] and the references cited therein). More related researches and their extensions refer to Brunner and van der Houwen [2], Brunner [3] and the references therein. Spline collocation methods for solving delay and neutral delay differential equations were studied in [8, 9, 10]. More detailed analysis for both the convergence and absolute stability were also given.

This paper is organized as follows. In Section 2, we give the basic idea of the El-Gendi method. The description of the spline/spectral method for the numerical solution of Eq. (1) are presented in Section 3. In Section 4, the convergence results of the present methods are presented. In Section 5, we illustrate our main results by numerical examples. The last section is conclusion.

2. El-Gendi Method

El-Gendi [6] proposes an integration matrix B to approximate the indefinite integral as follows:

$$\left[\int_{-1}^x (P_N f)(t) dt \right] = B[f], \quad x \in [-1, 1],$$

where $B = [b_{i,j}]$ is a square matrix of order $(N+1)$, $b_{i,j}$ are the elements of the matrix B , and the elements of the column matrix $[f]$ are given by $f_k = f(x_k)$, where x_k are the Gauss-Lobatto points

$$x_k = -\cos\left(\frac{k\pi}{N}\right), \quad k = 0(1)N \quad (2)$$

The main advantage of the El-Gendi method is that for a certain value of N the elements of the matrix B can be evaluated once and for all. Economisation in computation will be achieved if, for example, the matrix B is stored, for different values of N .

For the double integration

$$\int_{-1}^x \int_{-1}^x f(x) dx dx = \int_{-1}^x (x-s)f(s) ds$$

El-gendi [?] gives the following approximation

$$\left[\int_{-1}^x \int_{-1}^x f(x) dx dx \right] = B^2[f] = T[f]$$

where

$$t_{i,j} = (x_i - x_j)b_{i,j}$$

The integration matrix of the successive integration is given [7] as follows:

$$[I_n(x)] = \left[\int_{-1}^x \int_{-1}^{t_{n-1}} \int_{-1}^{t_{n-2}} \dots \int_{-1}^{t_2} \int_{-1}^{t_1} (P_N f)(t_0) dt_0 dt_1 \dots dt_{n-2} dt_{n-1} \right] = B^{(n)}[f],$$

where $B^n = [b_{i,j}^{(n)}], n \geq 1;$

$$b_{i,j}^{(n)} = \frac{(x_i - x_j)^{n-1}}{(n-1)!} b_{i,j}, \quad i, j = 0, 1, \dots, N.$$

3. Description of the Method

Consider the initial value problem for the VDIDEs (1). For a given positive integer n , the interval $[t_0, T]$ is partitioned into n equal subintervals $I_i = [t_{i-1}, t_i], i = 1(1)n$ with the stepsize $h = (T - t_0)/n$. A sextic spline collocation methods [10] $S \in C^1[t_0, T]$ at the interior knots $t_{i-4/5}, t_{i-3/5}, t_{i-2/5}, t_{i-1/5}$ as well as at t_i with $t_i = t_0 + ih, i = 1(1)n$, can be represented on each $I_i = [t_i, t_{i+1}]$ by

$$\begin{aligned} S_i(t) = & S_i + hA(\xi)S'_i + hB(\xi)S'_{i+1/5} + hC(\xi)S'_{i+2/5} + hD(\xi)S'_{i+3/5} \\ & + hE(\xi)S'_{i+4/5} + hF(\xi)S'_{i+1}, \end{aligned} \quad (3a)$$

with

$$\begin{aligned}
 A(\xi) &= \xi - \frac{137}{24}\xi^2 + \frac{125}{8}\xi^3 - \frac{2125}{96}\xi^4 + \frac{125}{8}\xi^5 - \frac{625}{144}\xi^6, \\
 B(\xi) &= \frac{25}{2}\xi^2 - \frac{1925}{36}\xi^3 + \frac{8875}{96}\xi^4 - \frac{875}{12}\xi^5 + \frac{3125}{144}\xi^6, \\
 C(\xi) &= -\frac{25}{2}\xi^2 + \frac{2675}{36}\xi^3 - \frac{7375}{48}\xi^4 + \frac{1625}{12}\xi^5 - \frac{6250}{144}\xi^6, \\
 D(\xi) &= \frac{25}{3}\xi^2 - \frac{325}{6}\xi^3 + \frac{6125}{48}\xi^4 - 125\xi^5 + \frac{6250}{144}\xi^6, \\
 E(\xi) &= -\frac{25}{8}\xi^2 + \frac{1525}{72}\xi^3 - \frac{5125}{96}\xi^4 + \frac{1375}{24}\xi^5 - \frac{3125}{144}\xi^6, \\
 F(\xi) &= \frac{1}{2}\xi^2 - \frac{125}{36}\xi^3 + \frac{875}{96}\xi^4 - \frac{125}{12}\xi^5 + \frac{625}{144}\xi^6,
 \end{aligned} \tag{3b}$$

and $t = t_i + \xi h$, $\xi \in [0, 1]$, with a similar expression for $S_i(t)$ in $[t_{i-1}, t_i]$. Since $S \in S_{n,6}^{(1)}$, then the approximate spline solution $S_i(t)$ defined by Eq. (3) satisfies Eq. (1), at the collocation points $t_j, j = i-1, i-4/5, i-3/5, i-2/5, i-1/5, i$, will be constructed as follows: for $i = 1(1)n$

$$\underline{S}_i = M_0 S_{i-1} + h M_1 \underline{S}'_{i-1} + h M_2 \underline{S}'_i \tag{4}$$

where $M_0 = (1, 1, 1, 1, 1)^T$, $M_1 = (\frac{19}{288}, \frac{14}{225}, \frac{51}{800}, \frac{14}{225}, \frac{19}{288})^T$,

$$M_2 = \begin{bmatrix} \frac{1427}{7200} & \frac{-133}{1200} & \frac{241}{3600} & \frac{-173}{7200} & \frac{3}{800} \\ \frac{43}{150} & \frac{7}{225} & \frac{7}{225} & \frac{-1}{75} & \frac{1}{450} \\ \frac{219}{800} & \frac{57}{400} & \frac{57}{400} & \frac{-21}{800} & \frac{3}{800} \\ \frac{64}{225} & \frac{8}{75} & \frac{64}{225} & \frac{14}{225} & 0 \\ \frac{25}{96} & \frac{25}{144} & \frac{25}{144} & \frac{25}{96} & \frac{19}{288} \end{bmatrix},$$

$$\underline{S}_i = (S_{i-4/5}, S_{i-3/5}, S_{i-2/5}, S_{i-1/5}, S_i)^T, \underline{S}'_i = (S'_{i-4/5}, S'_{i-3/5}, S'_{i-2/5}, S'_{i-1/5}, S'_i)^T, \\
 S'_j = f\left(t_j, S(t_j), S(t_j - \tau), \int_{t_j-\tau}^{t_j} g(t_j, v, S(v)) dv\right).$$

It is easy to observe that $S(t_j - \tau) = \phi(t_j - \tau)$ when $(t_j - \tau) \leq t_0$, and if $(t_j - \tau) \in [t_{k-1}, t_k], k = 1(1)i$, then $S(t_j - \tau)$ can be calculated from Eq. (3):

$$\begin{aligned}
 S(t_j - \tau) &= S_{k-1} + hA(\zeta)S'_{k-1} + hB(\zeta)S'_{k-4/5} + hC(\zeta)S'_{k-3/5} \\
 &\quad + hD(\zeta)S'_{k-2/5} + hE(\zeta)S'_{k-1/5} + hF(\zeta)S'_k,
 \end{aligned} \tag{5}$$

where

$$\zeta = \frac{(t_j - \tau) - t_{k-1}}{h} \in [0, 1].$$

and $A(\zeta), \dots, F(\zeta)$ be given in Eq. (3b).

To find an approximation to the integral $\int_{t_j-\tau}^{t_j} g(t, v, S(v))dv$, $j = i-4/5, i-3/5, i-2/5, i-1/5, i$, we subdivide its integration interval as follows: if $(t_j - \tau) \leq t_0$

$$\begin{aligned} \int_{t_j-\tau}^{t_j} g(t, v, S(v))dv &= \int_{t_j-\tau}^{t_0} g(t, v, S(v))dv + \int_{t_0}^{t_j} g(t, v, S(v))dv \\ &= \int_{t_j-\tau}^{t_0} g(t, v, \phi(v))dv + \sum_{m=0}^{i-2} \int_{t_m}^{t_{m+1}} g(t, v, S_{m+1}(v))dv \\ &\quad + \int_{t_{i-1}}^{t_j} g(t, v, S_i(v))dv, \end{aligned} \tag{6a}$$

when $t_0 < (t_j - \tau) \in [t_{k-1}, t_k], k = 1(1)i$

$$\begin{aligned} \int_{t_j-\tau}^{t_j} g(t, v, S(v))dv &= \int_{t_j-\tau}^{t_k} g(t, v, S_k(v))dv + \sum_{m=k}^{i-2} \int_{t_m}^{t_{m+1}} g(t, v, S_{m+1}(v))dv \\ &\quad + \int_{t_{i-1}}^{t_j} g(t, v, S_i(v))dv. \end{aligned} \tag{6b}$$

Each of the above integrals in Eq. (6) is approximated by applying El-Gendi method (see above Section 2). Since El-Gendi method is defined for the finite range $-1 \leq t \leq 1$, then all the integral subintervals in Eq. (6) must be converted to $[-1, 1]$ as:

$$\int_{t_{i-1}}^{t_i} g(t, v, S_i(v))dv = \int_{-1}^1 g(t, w, S_i(w))dw, \tag{7}$$

where

$$w = \frac{t_i - t_{i-1}}{2}v + \frac{t_i + t_{i-1}}{2}, \quad dw = \frac{t_i - t_{i-1}}{2}dv.$$

From Eq. (5)-(7), system (4) can be solved for $S_{i-4/5}, S_{i-3/5}, S_{i-2/5}, S_{i-1/5}, S_i$.

4. Convergence Results

In this section, we introduce the following theorem which shows the present method convergence with order six. For more details of error analysis and order of convergence see [10].

Theorem 1. *Let $f \in C^7([t_0, T] \times R \times R \times R)$, then for all $t \in [t_0, T]$, we have*

$$|S^{(k)}(t) - y^{(k)}(t)| < C_k h^6, \quad k = 0, 1, \quad (8)$$

where C_k denotes generic constants independent of h , but dependent on the order of the various derivatives.

For the proof of this theorem see [10].

5. Numerical Examples

In this section, the present method is implemented for tackling VDIDEs (1). To illustrate the effectiveness of this method we shall consider two test examples. We choose $N = 8$ in Eq. (2) for each of the following examples.

Example 5.1. *Consider the following VDIDE:*

$$y'(t) = (\lambda - 1)\exp(1 - t) - (\lambda + 1)y(t) + y(t - 1) - \lambda \int_{t-1}^t y(v)dv, \quad t \geq 0,$$

$$\phi(t) = \exp(-t), \quad t \leq 0.$$

The exact solution is given by

$$y(t) = \exp(-t).$$

In Table 1, we give the absolute errors between the exact solution and the numerical results by the present method with $\lambda = 3, h = 0.2$.

Example 5.2. [14] *Consider the VDIDE of partially variable coefficients*

$$y'(t) = -(6 + \sin(t))y(t) + y(t - \pi/4) - \int_{t-\pi/4}^t \sin(v)y(v)dv + 5\exp(\cos(t)),$$

$$t \geq 0,$$

$$\phi(t) = \exp(\cos(t)), \quad t \leq 0.$$

The exact solution is given by

$$y(t) = \exp(\cos(t)).$$

Table 1: Absolute errors for the solution of Example 5.1 with $h = 0.2$

t	Absolute errors
2	5.421496585000796E-13
4	6.515621375768887E-14
6	8.905202963926939E-15
8	1.222058478717036E-15
10	1.635112401379701E-16

Table 2 shows the absolute errors of the present method at various points of the interval $[0, 10]$ for $h = 0.1$.

Table 2: Absolute errors for the solution of Example 5.2 with $h = 0.1$

t	Absolute errors
2	3.233857626128156E-12
4	7.724931805341839E-13
6	1.082112177641648E-12
8	1.744271393988583E-12
10	3.766986722553156E-13

6. Conclusion

In this paper, we have presented a collocation method, based on a mixed spline/spectral methods, for the numerical solution of VDIDEs (1). From the illustrative examples, it can be seen that the presented method can obtain very accurate and satisfactory results. This method will be developing for solving the system of Volterra delay-integro-differential equations.

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