

COMPARISON AND CONVERGENCE THEOREMS FOR INITIAL-BOUNDARY VALUE PROBLEM TO ONE NONLINEAR PARABOLIC EQUATION

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Abstract: The difference scheme of the initial-boundary value problem for one nonlinear parabolic equation is considered. The convergence of the solution of the difference scheme to the solution of source problem is proved. For the same difference scheme the comparison theorem is proved and the uniqueness of the solution is obtained. The iteration process for finding the solution of difference scheme is constructed and its convergence is obtained.

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1. Introduction

In some mathematical models of diffusion process the system may be described as:

$$U_t = a(x, t) U_{xx} + b(x, t) (U_x)^2 + f(x, t), \quad (x, t) \in \Omega \times (0, T], \quad (1)$$

$$U(x, 0) = \varphi(x), \quad x \in \bar{\Omega}, \quad (2)$$

$$U(0, t) = \phi_0(t), \quad U(1, t) = \phi_1(t), \quad t \in (0, T], \quad (3)$$

where T is a positive constant, $U = U(x, t)$ is solution, a, b, f, φ, ϕ_0 and ϕ_1 are given functions, $\Omega = (0, 1)$.

We assume that functions $a = a(x, t)$, $b = b(x, t)$, $f = f(x, t)$ are continuous on $\bar{\Omega} \times [0, T]$, φ is continuous on $\bar{\Omega}$, ϕ_0, ϕ_1 are continuous on $[0, T]$ and

$$a(x, t) > 0, \quad (x, t) \in \bar{\Omega} \times [0, T]. \quad (4)$$

The initial-boundary problem for the equation like (1) was considered in article [5] by means of finite-element approximation. The difference schemes to nonlinear parabolic equations were considered in a number of works (see, for example, [1], [2], [3], [4], [8]). This author received the same results as mentioned above for problem containing another kind of parabolic equation in articles [7], [9].

In the present work we investigate questions of approximate solution of the problem (1)-(3) using the difference scheme. In certain conditions the convergence of the solution of the difference scheme to the solution of source problem is proved. For the same difference scheme the comparison theorem is proved and the uniqueness of the solution is obtained. The iteration process for finding of the solution of difference scheme is constructed and in certain conditions its convergence is obtained.

2. Problem Formulation

Define a grid $\omega_{h\tau}$ on the domain $\bar{\Omega} \times [0, T]$ as follows:

$$\omega_h = \{x_i = ih, \quad h > 0, \quad i = 0, 1, \dots, M; \quad hM = 1\},$$

$$\omega_\tau = \{t_j = j\tau, \quad \tau > 0, \quad j = 0, 1, \dots, N; \quad \tau N = T\},$$

$$\omega_{h\tau} = \omega_h \times \omega_\tau.$$

Let y_i^j be a function defined on the grid $\omega_{h\tau}$. Enter the following notations:

$$\|y^j\|_{C(\omega_h)} = \max_{i=0,1,\dots,N} |y_i^j|,$$

$$\|y\|_{C(\omega_{h\tau})} = \max_{j=0,1,\dots,N} \|y^j\|_{C(\omega_h)}.$$

Let

$$g(u) = \frac{1}{2} \left(1 - \frac{1}{1+u^2} \right). \quad (5)$$

Note that

$$0 \leq g(u) \leq \frac{1}{2}, \quad -1 < g'(u) < 1. \quad (6)$$

Make the proper approximation of the term $\left(\frac{\partial U}{\partial x}\right)^2$:

$$\left(\frac{\partial U}{\partial x}\right)^2 = \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} [U(x_{i+1}, t^j) - U(x_{i-1}, t^j)] \right) + O(h^2),$$

where the value of λ can be chosen arbitrarily, therefore we will specify its value later.

Construct the difference scheme to the problem (1)-(3) as follows:

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{\tau} &= a(x_i, t^{j+1}) \frac{1}{h^2} [u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}] + \\ &+ b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} [u_{i+1}^{j+1} - u_{i-1}^{j+1}] \right) + f(x_i, t^{j+1}), \\ i &= 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N-1, \end{aligned} \quad (7)$$

$$u_i^0 = \varphi_i, \quad i = 0, 1, \dots, M, \quad (8)$$

$$u_0^j = \phi_0^j, \quad u_M^j = \phi_1^j, \quad j = 1, 2, \dots, N, \quad (9)$$

where $\varphi_i = \varphi(x_i)$, $i = 0, 1, \dots, M$, $\phi_0^j = \phi_0(t^j)$, $\phi_1^j = \phi_1(t^j)$, $j = 1, 2, \dots, N$.

Further in this work we investigate the difference scheme (7)-(9). We prove the theorem of convergence of the solution of scheme (7)-(9) to the solution of the problem (1)-(3). We also prove the comparison theorem for the scheme (7)-(9), which implies the uniqueness of the solution of the scheme (7)-(9). Also, we construct the iteration process for finding of the solution of the scheme (7)-(9) and show its convergence to the solution of the scheme (7)-(9).

3. Problem Solution

In this section we will prove the theorems of comparison, uniqueness and convergence for the solution of the difference scheme (7)-(9). First we will refer to the two important theorems from book [6], pages 15-17.

Consider the grid function y_i defined on the grid ω_h . Define the maximum norm on the grid ω_h as follows

$$\|y\|_{C(\omega_h)} = \max_{i=0,1,\dots,M} |y_i|.$$

Let the grid function y_i satisfy the following conditions:

$$\Lambda[y_i] = A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \dots, N-1, \quad (10)$$

$$y_0 = \mu_1, \quad y_N = \mu_2, \quad (11)$$

where A_i, B_i, C_i, F_i are constants.

Theorem 1. (see [6]) *Let the conditions*

$$A_i > 0, \quad B_i > 0, \quad D_i = C_i - A_i - B_i \geq 0,$$

be fulfilled for all $i = 1, 2, \dots, M-1$ and the grid function y_i , which is not constant identically, satisfy at all the inner nodes $i = 1, 2, \dots, M-1$ the condition $\Lambda[y_i] \leq 0$. Then y_i cannot take the minimal negative value at the inner points, that is, for $i = 1, 2, \dots, M-1$. In particular, if $\mu_1 \geq 0$ and $\mu_2 \geq 0$, then $y_i \geq 0$ for all $i = 0, 1, \dots, M$.

Theorem 2. (see [6]) *Let the conditions*

$$|A_i| > 0, \quad |B_i| > 0, \quad D_i = |C_i| - |A_i| - |B_i| > 0$$

hold for all $i = 1, 2, \dots, M-1$ and $\mu_1 = \mu_2 = 0$. A solution of the problem (10), (11) admits the estimate

$$\|y\|_{C(\omega_h)} \leq \max_{i=1,2,\dots,M-1} \left| \frac{F_i}{D_i} \right|.$$

With Theorem 1 and Theorem 2 we can then prove our result:

Theorem 3. (Comparison Theorem) *Let $\bar{u}$ and $\bar{\bar{u}}$ be functions defined on the grid $\omega_{h\tau}$, satisfying the following inequalities:*

$$\begin{aligned} & \frac{\bar{\bar{u}}_i^{j+1} - \bar{\bar{u}}_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} [\bar{\bar{u}}_{i+1}^{j+1} - 2\bar{\bar{u}}_i^{j+1} + \bar{\bar{u}}_{i-1}^{j+1}] - \\ & - b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} [\bar{\bar{u}}_{i+1}^{j+1} - \bar{\bar{u}}_{i-1}^{j+1}] \right) \geq \\ & \geq \frac{\bar{u}_i^{j+1} - \bar{u}_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} [\bar{u}_{i+1}^{j+1} - 2\bar{u}_i^{j+1} + \bar{u}_{i-1}^{j+1}] - \\ & - b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} [\bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1}] \right), \\ & i = 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N-1, \end{aligned} \tag{12}$$

$$\bar{\bar{u}}_i^0 \geq \bar{u}_i^0, \quad i = 0, 1, \dots, M, \tag{13}$$

$$\bar{\bar{u}}_0^j \geq \bar{u}_0^j, \quad \bar{\bar{u}}_M^j \geq \bar{u}_M^j, \quad j = 1, 2, \dots, N. \tag{14}$$

If λ satisfies the following condition

$$0 < \lambda < \frac{\min_{(x,t) \in \bar{\Omega} \times [0,T]} \{a(x,t)\}}{\max_{(x,t) \in \bar{\Omega} \times [0,T]} \{|b(x,t)|\}}, \tag{15}$$

then $\bar{\bar{u}}_i^j \geq \bar{u}_i^j$ on the grid $\omega_{h\tau}$.

Proof. Enter the following notations

$$y_i^j = \bar{u}_i^j - \bar{u}_i^j, \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N.$$

Then the inequalities (12) can be written as follows:

$$\begin{aligned} & \frac{y_i^{j+1} - y_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} \left[y_{i+1}^{j+1} - 2y_i^{j+1} + y_{i-1}^{j+1} \right] \geq \\ & \geq b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[g \left(\frac{1}{2\lambda} \left[\bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right) - g \left(\frac{1}{2\lambda} \left[\bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right) \right], \\ & i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \end{aligned} \quad (16)$$

Transform the right side of the inequality (16) using the multidimensional scheme of Lagrange Mean Value Theorem:

$$\begin{aligned} & b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[g \left(\frac{1}{2\lambda} \left[\bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right) - g \left(\frac{1}{2\lambda} \left[\bar{u}_{i+1}^{j+1} - \bar{u}_{i-1}^{j+1} \right] \right) \right] = \\ & = \frac{\lambda}{h^2} b(x_i, t^{j+1}) \left(y_{i+1}^{j+1} - y_{i-1}^{j+1} \right) \times \\ & \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1} \bar{u}_{i+1}^{j+1} + \left(1 - \theta_i^{j+1} \right) \bar{u}_{i+1}^{j+1} - \theta_i^{j+1} \bar{u}_{i-1}^{j+1} - \left(1 - \theta_i^{j+1} \right) \bar{u}_{i-1}^{j+1} \right] \right), \end{aligned}$$

where θ_i^{j+1} are numbers from the interval (0, 1) according to the multidimensional scheme of the Lagrange Mean Value Theorem.

Denote by A_i^j , and B_i^j the following quantities:

$$\begin{aligned} A_i^{j+1} &= \frac{1}{h^2} \left[a(x_i, t^{j+1}) + \lambda b(x_i, t^{j+1}) \times \right. \\ & \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1} \bar{u}_{i+1}^{j+1} + \left(1 - \theta_i^{j+1} \right) \bar{u}_{i+1}^{j+1} - \theta_i^{j+1} \bar{u}_{i-1}^{j+1} - \left(1 - \theta_i^{j+1} \right) \bar{u}_{i-1}^{j+1} \right] \right) \Big], \\ B_i^{j+1} &= \frac{1}{h^2} \left[a(x_i, t^{j+1}) - \lambda b(x_i, t^{j+1}) \times \right. \\ & \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1} \bar{u}_{i+1}^{j+1} + \left(1 - \theta_i^{j+1} \right) \bar{u}_{i+1}^{j+1} - \theta_i^{j+1} \bar{u}_{i-1}^{j+1} - \left(1 - \theta_i^{j+1} \right) \bar{u}_{i-1}^{j+1} \right] \right) \Big], \end{aligned}$$

Then (16) can be written as follows:

$$A_i^{j+1} y_{i+1}^{j+1} - \left(A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right) y_i^{j+1} + B_i^{j+1} y_{i-1}^{j+1} \leq -\frac{y_i^j}{\tau},$$

$$i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \quad (17)$$

Applying the method of mathematical induction on index j we prove, that

$$y_i^j \geq 0, \quad i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \quad (18)$$

From condition (13) when $j = 0$, we have

$$y_i^0 \geq 0, \quad i = 0, 1, \dots, M.$$

Suppose, that (18) is valid when $j = l - 1$ and then we will show its validity for $j = l$. Write (17) as follows:

$$A_i^l y_{i+1}^l - \left(A_i^l + B_i^l + \frac{1}{\tau} \right) y_i^l + B_i^l y_{i-1}^l \leq -\frac{y_i^{l-1}}{\tau}, \quad i = 0, 1, \dots, M. \quad (19)$$

According to the conditions (4), (6), (15) $A_i^l > 0$, $B_i^l > 0$, $A_i^l + B_i^l + \frac{1}{\tau} > 0$. Taking into account inequalities (14), we have $y_0^l \geq 0$, $y_M^l \geq 0$. As for the right side of (19), it is nonpositive. Therefore we can apply the Theorem 1 to y_i^l , and we obtain $y_i^l \geq 0$, $i = 0, 1, \dots, M$. The Theorem 3 is proved. $\square$

Theorem 4. (Uniqueness) *Let λ satisfy the following condition*

$$0 < \lambda < \frac{\min_{(x,t) \in \bar{\Omega} \times [0, T]} \{a(x, t)\}}{\max_{(x,t) \in \bar{\Omega} \times [0, T]} \{|b(x, t)|\}}.$$

If the solution of the difference scheme (7)-(9) exists, it is unique.

The proof is simply obtained from the Theorem 3.

Theorem 5. (Convergence of Solution) *Let there exist a solution U of the problem (1)-(3) such that $\left| \frac{\partial^4 U}{\partial x^4} \right| \leq V$, $\left| \frac{\partial^2 U}{\partial t^2} \right| \leq V$ for some positive constant V . If λ satisfies the following condition*

$$0 < \lambda < \frac{\min_{(x,t) \in \bar{\Omega} \times [0, T]} \{a(x, t)\}}{\max_{(x,t) \in \bar{\Omega} \times [0, T]} \{|b(x, t)|\}} \quad (20)$$

and the solution u of the difference scheme (7)-(9) exists for small values of τ and h then the following estimate takes place

$$\|u - U\|_{C(\omega_{h\tau})} \leq TW (\tau + h^2),$$

where W is positive constant independent from τ and h .

Proof. For the difference equations (7) we have:

$$\begin{aligned} \frac{U_i^{j+1} - U_i^j}{\tau} &= a(x_i, t^{j+1}) \frac{1}{h^2} [U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}] + \\ &+ b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g\left(\frac{1}{2\lambda} [U_{i+1}^{j+1} - U_{i-1}^{j+1}]\right) + f(x_i, t^{j+1}) + \underline{Q}(\tau + h^2), \\ i &= 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N-1, \end{aligned}$$

where $\underline{Q}(\tau + h^2)$ is a quantity, which can be expressed as follows

$$\underline{Q}(\tau + h^2) = (\tau + h^2) q(\tau, h)$$

for some bounded function q .

The grid function $Y_i^j = u_i^j - U_i^j$, is the solution of the following equations:

$$\begin{aligned} \frac{Y_i^{j+1} - Y_i^j}{\tau} - a(x_i, t^{j+1}) \frac{1}{h^2} [Y_{i+1}^{j+1} - 2Y_i^{j+1} + Y_{i-1}^{j+1}] + \underline{Q}(\tau + h^2) &= \\ = b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[g\left(\frac{1}{2\lambda} [u_{i+1}^{j+1} - u_{i-1}^{j+1}]\right) - g\left(\frac{1}{2\lambda} [U_{i+1}^{j+1} - U_{i-1}^{j+1}]\right) \right] \\ i &= 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N-1. \end{aligned} \quad (21)$$

Transform the right side of the inequality (21) using Lagrange Mean Value Theorem:

$$\begin{aligned} b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[g\left(\frac{1}{2\lambda} [u_{i+1}^{j+1} - u_{i-1}^{j+1}]\right) - g\left(\frac{1}{2\lambda} [U_{i+1}^{j+1} - U_{i-1}^{j+1}]\right) \right] &= \\ = \frac{\lambda}{h^2} b(x_i, t^{j+1}) (Y_{i+1}^{j+1} - Y_{i-1}^{j+1}) \times \\ \times g' \left(\frac{1}{2\lambda} [\theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - (1 - \theta_i^{j+1}) U_{i-1}^{j+1}] \right). \end{aligned}$$

where θ_i^{j+1} are numbers from the interval $(0, 1)$ according to the multidimensional scheme of Lagrange Mean Value Theorem.

Denote by A_i^j , and B_i^j the following quantities:

$$\begin{aligned} A_i^{j+1} &= \frac{1}{h^2} [a(x_i, t^{j+1}) + \lambda b(x_i, t^{j+1}) \times \\ &\times g' \left(\frac{1}{2\lambda} [\theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - (1 - \theta_i^{j+1}) U_{i-1}^{j+1}] \right)], \end{aligned}$$

$$B_i^{j+1} = \frac{1}{h^2} [a(x_i, t^{j+1}) - \lambda b(x_i, t^{j+1}) \times \\ \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1} u_{i+1}^{j+1} + (1 - \theta_i^{j+1}) U_{i+1}^{j+1} - \theta_i^{j+1} u_{i-1}^{j+1} - (1 - \theta_i^{j+1}) U_{i-1}^{j+1} \right] \right)],$$

Then equation (21) can be written as follows

$$A_i^{j+1} Y_{i+1}^{j+1} - \left(A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} \right) Y_i^{j+1} + B_i^{j+1} Y_{i-1}^{j+1} = -\frac{Y_i^j}{\tau} + \underline{O}(\tau + h^2), \\ i = 0, 1, \dots, M, \quad j = 0, 1, \dots, N. \quad (22)$$

According to the conditions (4),(6),(20) $A_i^{j+1} \geq 0$, $B_i^{j+1} \geq 0$, $A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau} > 0$, $|A_i^{j+1} + B_i^{j+1} + \frac{1}{\tau}| - |A_i^{j+1}| - |B_i^{j+1}| > 0$. Therefore we can apply the theorem 2 to (22). We obtain

$$\|Y^{j+1}\|_{C(\omega_h)} \leq \tau \left(\frac{1}{\tau} \|Y^j\|_{C(\omega_h)} + W(\tau + h^2) \right), \quad j = 0, 1, \dots, N-1.$$

Taking into account that $\|Y^0\|_{C(\omega_h)} = 0$, we have

$$\|Y^{j+1}\|_{C(\omega_h)} \leq \|Y^j\|_{C(\omega_h)} + \tau W(\tau + h^2) \leq \|Y^{j-1}\|_{C(\omega_h)} + 2\tau W(\tau + h^2) \leq \\ \leq \|Y^{j-2}\|_{C(\omega_h)} + 3\tau W(\tau + h^2) \leq \|Y^0\|_{C(\omega_h)} + (j+1)\tau W(\tau + h^2) \leq \\ \leq TW(\tau + h^2), \quad j = 0, 1, \dots, N-1.$$

The theorem 5 is proved.

4. Convergence of Iteration Scheme

In this section we construct the iteration process for finding of the solution of the difference scheme (7)-(9).

$$u_i^{j+1,l+1} = (1 - \sigma) u_i^{j+1,l} + \sigma \left\{ \tau \left[a(x_i, t^{j+1}) \frac{1}{h^2} [u_{i+1}^{j+1,l} - 2u_i^{j+1,l} + u_{i-1}^{j+1,l}] + \right. \right. \\ \left. \left. + b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} g \left(\frac{1}{2\lambda} [u_{i+1}^{j+1,l} - u_{i-1}^{j+1,l}] \right) + f(x_i, t^{j+1}) \right] + u_i^j \right\}, \\ l = 0, 1, \dots; \quad i = 1, 2, \dots, M-1, \quad j = 0, 1, \dots, N-1, \quad (23)$$

$$u_i^{j+1,0} = u_i^j, \quad i = 0, 1, \dots, M, \quad (24)$$

$$u_1^{j+1,l} = \phi_0^{j+1}, \quad u_M^{j+1,l} = \phi_1^{j+1}, \quad l = 1, 2, \dots; \quad j = 0, 1, \dots, N-1. \quad (25)$$

□

Theorem 6. (Convergence of Iteration Scheme) *Let λ satisfies the following condition*

$$0 < \lambda < \frac{\min_{(x, t) \in \bar{\Omega} \times [0, T]} \{a(x, t)\}}{\max_{(x, t) \in \bar{\Omega} \times [0, T]} \{|b(x, t)|\}}, \quad (26)$$

and σ satisfies the following condition

$$0 < \sigma \leq \frac{h^2}{h^2 + 2\tau \max_{(x, t) \in \bar{\Omega} \times [0, T]} \{a(x, t)\}} \quad (27)$$

then iteration process (23)-(25) converges for each j and the limit is the solution of the difference scheme (7)-(9).

Proof. It is clear, that if the $\lim_{l \rightarrow \infty} (u_0^{j+1, l}, u_1^{j+1, l}, \dots, u_M^{j+1, l})$ exists then it will be the solution of the difference scheme (7)-(9).

Consider $u_i^{j+1, l+1} - u_i^{j+1, l}$. Taking into account the method, used in the proof of theorem 3, we have:

$$\begin{aligned} u_i^{j+1, l+1} - u_i^{j+1, l} &= (1 - \sigma) (u_i^{j+1, l} - u_i^{j+1, l-1}) \\ &+ \sigma \tau \left\{ a(x_i, t^{j+1}) \frac{1}{h^2} \left[(u_{i+1}^{j+1, l} - u_{i+1}^{j+1, l-1}) - 2(u_i^{j+1, l} - u_i^{j+1, l-1}) \right. \right. \\ &\quad \left. \left. + (u_{i-1}^{j+1, l} - u_{i-1}^{j+1, l-1}) \right] \right. \\ &+ b(x_i, t^{j+1}) \frac{2\lambda^2}{h^2} \left[g \left(\frac{1}{2\lambda} [u_{i+1}^{j+1, l} - u_{i-1}^{j+1, l}] \right) - g \left(\frac{1}{2\lambda} [u_{i+1}^{j+1, l-1} - u_{i-1}^{j+1, l-1}] \right) \right] \left. \right\} \\ &= (1 - \sigma) (u_i^{j+1, l} - u_i^{j+1, l-1}) \\ &+ \frac{\sigma \tau}{h^2} \left\{ a(x_i, t^{j+1}) \left[(u_{i+1}^{j+1, l} - u_{i+1}^{j+1, l-1}) - 2(u_i^{j+1, l} - u_i^{j+1, l-1}) \right. \right. \\ &\quad \left. \left. + (u_{i-1}^{j+1, l} - u_{i-1}^{j+1, l-1}) \right] \right. \\ &+ \lambda b(x_i, t^{j+1}) \left[(u_{i+1}^{j+1, l} - u_{i+1}^{j+1, l-1}) - (u_{i-1}^{j+1, l} - u_{i-1}^{j+1, l-1}) \right] \left. \right\} \\ &\times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1, l} u_{i+1}^{j+1, l} + (1 - \theta_i^{j+1, l}) u_{i+1}^{j+1, l-1} - \theta_i^{j+1, l} u_{i-1}^{j+1, l} \right. \right. \\ &\quad \left. \left. - (1 + \theta_i^{j+1, l}) u_{i-1}^{j+1, l-1} \right] \right) \\ &= \left\{ 1 - \sigma - \frac{\sigma \tau}{h^2} [A_i^{j+1, l} + B_i^{j+1, l}] \right\} (u_i^{j+1, l} - u_i^{j+1, l-1}) \end{aligned}$$

$$+ \frac{\sigma\tau A_i^{j+1,l}}{h^2} \left(u_{i+1}^{j+1,l} - u_{i+1}^{j+1,l-1} \right) + \frac{\sigma\tau B_i^{j+1,l}}{h^2} \left(u_{i-1}^{j+1,l} - u_{i-1}^{j+1,l-1} \right),$$

where

$$\begin{aligned} A_i^{j+1,l} &= a(x_i, t^{j+1}) + \lambda b(x_i, t^{j+1}) \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1,l} u_{i+1}^{j+1,l} \right. \right. \\ &\quad \left. \left. + \left(1 - \theta_i^{j+1,l} \right) u_{i+1}^{j+1,l-1} - \theta_i^{j+1,l} u_{i-1}^{j+1,l} - \left(1 + \theta_i^{j+1,l} \right) u_{i-1}^{j+1,l-1} \right] \right), \end{aligned}$$

$$\begin{aligned} B_i^{j+1,l} &= a(x_i, t^{j+1}) - \lambda b(x_i, t^{j+1}) \times g' \left(\frac{1}{2\lambda} \left[\theta_i^{j+1,l} u_{i+1}^{j+1,l} \right. \right. \\ &\quad \left. \left. + \left(1 - \theta_i^{j+1,l} \right) u_{i+1}^{j+1,l-1} - \theta_i^{j+1,l} u_{i-1}^{j+1,l} - \left(1 + \theta_i^{j+1,l} \right) u_{i-1}^{j+1,l-1} \right] \right), \end{aligned}$$

and $\theta_i^{j+1,l}$ is a number from the interval $(0, 1)$ according to the multidimensional scheme of Lagrange Mean Value Theorem.

Apply the maximum norm to the both sides of the last equation:

$$\begin{aligned} \left\| u^{j+1,l+1} - u^{j+1,l} \right\|_{C(\omega_h)} &= \max_{i=1,2,\dots,M-1} \left| \frac{\sigma\tau A_i^{j+1,l}}{h^2} \left(u_{i+1}^{j+1,l} - u_{i+1}^{j+1,l-1} \right) + \right. \\ &\quad \left. + \left\{ 1 - \sigma - \frac{\sigma\tau}{h^2} \left[A_i^{j+1,l} + B_i^{j+1,l} \right] \right\} \left(u_i^{j+1,l} - u_i^{j+1,l-1} \right) + \right. \\ &\quad \left. + \frac{\tau\sigma B_i^{j+1,l}}{h^2} \left(u_{i-1}^{j+1,l} - u_{i-1}^{j+1,l-1} \right) \right| \leq \\ &\leq \max_{i=1,2,\dots,M-1} \left(\left| 1 - \sigma - \frac{\sigma\tau}{h^2} \left[A_i^{j+1,l} + B_i^{j+1,l} \right] \right| + \right. \\ &\quad \left. + \left| \frac{\sigma\tau A_i^{j+1,l}}{h^2} \right| + \left| \frac{\sigma\tau B_i^{j+1,l}}{h^2} \right| \right) \left\| u^{j+1,l} - u^{j+1,l-1} \right\|_{C(\omega_h)}. \end{aligned} \quad (28)$$

(4), (6), (26) imply that

$$0 < A_i^{j+1,l} \leq 2 \max_{(x,t) \in \overline{\Omega} \times [0, T]} \{a(x, t)\},$$

$$0 < B_i^{j+1,l} \leq 2 \max_{(x,t) \in \overline{\Omega} \times [0, T]} \{a(x, t)\}.$$

According to (27) we have

$$1 - \sigma - \frac{\sigma\tau}{h^2} \left[A_i^{j+1,l} + B_i^{j+1,l} \right] \geq 0,$$

therefore we can remove the module in the right part of (28) and we have:

$$\left\| u^{j+1,l+1} - u^{j+1,l} \right\|_{C(\omega_h)} \leq (1 - \sigma) \left\| u^{j+1,l} - u^{j+1,l-1} \right\|_{C(\omega_h)}.$$

It is clear that $0 < (1 - \sigma) < 1$, therefore the sequence $\left(u_i^{j+1,l} \right)_{l=1}^{\infty}$ converges and its limit is the solution of the difference scheme (7)-(9). $\square$

5. Conclusion

The results obtained in this work allow to get numerical solution of the problem like (1)-(3). Numerical experiments were conducted which confirm the validity of the theorems proved above.

References

- [1] R. Buckmire, K. McMurtry, R.E. Mickens, Numerical studies of a nonlinear heat equation with square root reaction term, *Numer. Methods Partial Differ. Equations*, **25**, No. 3 (2009), 598-609.
- [2] M.M. Butt, M.S.A. Taj, Numerical methods for heat equation with variable coefficients, *Int. J. Comput. Math.*, **86**, No. 9 (2009), 1612-1623.
- [3] R. Horvath, On the sign-stability of finite difference solutions of semilinear parabolic problems, *Lecture Notes in Computer Science*, Springer, Berlin, **5434** (2009), 305-313.
- [4] J. Luo, N.V. Krylov, On the rate of convergence of the finite-difference approximations for parabolic Bellman equations with constant coefficients, *Appl. Math. Optim.*, **58**, No. 3 (2008), 315-344.
- [5] B. Neta, Numerical solution of a nonlinear integro-differential equation, *Journal of Mathematical Analysis and Applications*, **48**, No. 2 (1982), 598-611.
- [6] A. Samarski, *The Theory of Difference Schemes*, Marcell Dekker, New York (2001).
- [7] M. Tutberidze, The comparison properties and convergence of the difference scheme for one nonlinear system of partial differential equations, *Proc. I. Vekua Inst. Appl. Math.*, **54-55** (2004-2005), 74-84.

- [8] C.-R. Ye, Z.-Z. Sun, A linearized compact difference scheme for an one-dimensional parabolic inverse problem, *Appl. Math. Modelling*, **33**, No. 3 (2009), 1521-1528.
- [9] M. Tutberidze, On the Numerical Integration of One Nonlinear Parabolic Equation, *International Journal of Pure and Applied Mathematics*, **66**, No. 2 (2011), 209-220.