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δ-CONTROLLABILITY OF IMPULSIVE SYSTEMS AND APPLICATIONS TO SOME PHYSICAL AND BIOLOGICAL CONTROL SYSTEMS

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Abstract: In this paper the conditions for $\delta-$controllability of nonlinear impulsive control systems are investigated using Morales fixed point theorem for strongly accretive maps. The results obtained are applied to impulsive automatic controlled heating and cooling compartments and impulsive control hematopoiesis model.

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1. Introduction

Impulsive systems are systems found to be exhibiting some jumps, shock etc., during the process of evolution (see [1], [13]). These systems, of late are now constituting the core of many investigations in the ordinary differential equations and the control systems (see [7], [17]). Control systems in an area where impulsive system found a lot of applications (see [14]-[17]), even through, this area is experiencing gradual growth of late as per impulsive system theory is concerned. More work is expected to be recorded in the next two decades.

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In control system, there is the system whose behavioral activity is being governed by some equations, for which the fundamental problem is to regulate the system a prescribed manner. The state of the system may be impulsive and even, in some cases, the control variables that even regulate the system may also exhibit some impulsive tendency during the evolutionary process.

In control theory many techniques have been developed to solve some specific problems or in some cases generalized problems. In impulsive control system there are a lot of open problems. The most outstanding ones are how to construct an impulsive control variable in a concrete term and develop models that are of impulsive family.

Many processes characterized by impulses that are of control type are found in nature. Hence, it is natural to expect extension of some of the findings in the literature to control systems which are of impulsive family. It must be emphasized that, the last fifteen years have witnessed increase in the studies on impulsive control systems (see [9], [14]-[18]).

Fixed point technique for studying controllability for systems represented with nonlinear evolution equations have extensively studies (see [17]). However, the use of most celebrated accretive operators which are known to be vital tool in nonlinear operator theory has not yet attracted much needed applications in impulsive control systems.

It is a well known fact (see Browder [5], [6]; Kartsatos [7], Kato [8], Oyelami and Ale [12]) that accretive operators provide generalized settings for studying systems behaviors. Hence, if elaborated on them, there shall be a rich perspective for investigation into the theory of nonlinear impulsive control systems. In this paper, It is against this background that the investigation on impulsive control system, is carried out using the advantages of generality and flexibility of accretive operators.

It is worthy of note to mention that, an impulsive control systems can be treated as a typical nonlinear fixed point problem and δ-controllability criteria can be fashioned out from them by exploiting a fixed point due to Morales.

δ-controllability is about finding the $\delta-$neighborhood about the time for which the system is controllable. It is useful in biomedicine where control is strictly needed to regulate biochemical substance in the cells and tissues.

An application of this kind of system is useful in the design of an automatic temperature controlled swimming pool, incubator, nuclear reactors, or heating and cooling system in biological and physical systems that heat or temperature is required to be impulsive. We also considers hematopoiesis non-delay Saker and Alzebut model (see [16]) to measure the replacement of the blood by new blood cells as a result of use of drug, or food supplement and obtained controllability criterion for the model.

2. Preliminary Notes on Accretive Maps and Impulsive Control Systems

Throughout this paper we will make use of the following notations:

1. $C(R^+, R^n)$ will represent the space of continuous functions on $R^+ = [0, \infty)$ taking values in R^n,where $R = (-\infty, +\infty)$;
2. $C^1(R^+, R^n)$ will denote the space of continuously differentiable function on R^+ and taking values in R^n;

$PC(R^+, R^n) = \{y(t); y(t) \in C(R^+ - \{t_k\}, R^n),\ k = 1, 2, \cdots$ and $\lim\limits_{t \to t_k + 0} y(t)$ exists and it is equal to $y(t_k)\}$. It is worthy to say that $PC(R^+, R^n)$ together with the sup norm,

$$|x|_{PC(R^+,R^n)} = Sup|x(t)|$$

is a Banach space.

Define function of class K as $K = \{a(r) \in C([0, r], R^+) : a(r)$ is increasing in r, $a(0) = 0$, and $\lim\limits_{r\to\infty} a(r) = +\infty\}$.

For the Banach space E, denote by J the normalized duality mapping from E to 2^E and it is given as

$$J(x) = \{f \in E^* : |f^*|^2 = \langle x, f^* \rangle\}$$

where

E^* denotes the dual space of E and $\langle \cdot \rangle$ is the generalized duality pairing.

Accretive operators were introduced independently by (see Browder [5], Kato [8]).

$M_n(\cdot)$ will represent $n \times n$ matrix on $(\cdot)$

Definition 1. (Accretive Operators) Attractiveness of an operator is define as follows:

An operator $T : D(T) \subset E$ is said to be accretive if for each $x, y \in D(T)$ there exists $j \in J(x - y)$ such that

$$\langle Tx - Ty, j \rangle \geq 0.$$

where $D(\cdot)$ is the domain of the operator T.

Definition 2. (Due to Kato's Lemma, see Kato [6]) The following characterization of accretive operator can be made: T is accretive if and only if for each $x, y \in D(T)$ and $\gamma > 0$

$$|x - y| \leq |x - y + \gamma(Tx - Ty)|.$$

Closely related to attractiveness are contraction and nonexpansivity of operators and are define as: **contractive map**

Let D be subset of X. A mapping $T : D \longrightarrow E$ is to be a contraction map if there exists $k \in (0, 1)$ such that

$$|Tx - Ty| \leq k|x - y| \text{ for each } x, y \in D.$$

for the case, when $k = 1, T$ is called a nonexpansive operator and are intimately related to the family of accretive operators in the following way:

T is accretive if and only if $I + \gamma T$ is expansive and $(I + \gamma T)^{-1}$ exists as a mapping from $R(I + \gamma T)$ into $D(T)$, where $R(T)$ is range of T.

Definition 3. An accretive operator T is said to be M-accretive if the range $I + \gamma T$ is all in E for some $\gamma > 0$. That is

$$R(I + \gamma T) = E.$$

Definition 4. An operator $T : D(T) \subset E \longrightarrow E$ is called strongly accretive if there exists some $k > 0$ such that for each $x, y \in D(T)$

$$\langle Tx - Ty,\ j >\geq k|x - y|^2$$

for some $j \in J(x - y)$.

3. Formulation of a Nonlinear Control System

Consider a non linear impulse control system (ICS)

$$\dot{x}(t) = f(t, x(t), u(t)) + B(t)u(t), \quad t \neq t_k, k = 1, 2, 3, \dots, \tag{1}$$

$$\Delta x(t_k) = I_1(x(t_k))$$

$$\Delta u(t_k) = I_2(u(t_k))$$

for $0 < t_0 < t_1 < \cdots < t_k; \lim_{k\to\infty} t_k = +\infty$. Here $f : R^+ \times R^n \times U \longrightarrow R^n$ is non-linear function which is continuous in all its arguments ;$I_1 : R^n \longrightarrow R^n$ and $I_2 : U \longrightarrow R^n$ are continuous in their respective domains of definitions.

$B(t)$ is $n \times n$ continuous matrix function with respect to t. The geometry of the trajectory of (ICS) can be characterized just like that of impulsive differential equations (e.g. [1]&[11]). The solution space associated with the impulsive control system (ICS) in equation (1) is a Banach space with its norm defined as

$$|(x(t), u(t)| = \sup_{\tau \in R^+} |x(\tau)| + ess \sup_{\tau \in R^+} |u(\tau)| \tag{2}$$

for $x(t) \in A$ and $u(t) \in L_\infty(R^+)$,where $L_\infty(R^+)$ is the Banach space of essentially bounded function on R^+ equipped with essential supnorm

$$|x|_{L_\infty(R^+)} = ess \sup_{t \in R^+} |x(t)| \tag{3}$$

In addition to the preliminaries in the previous section, consider following definitions and notations:

$x(t) = x(t, t_0, x_0, u)$ will be said to be the solution of equation (1) i.e. the solution of (ICS) existing for $t \geq t_0$ if the following conditions are satisfied:

(i) $x(t) \in PC(D, R^n)$ such that $\dot{x}(t) = f(t, x(t), u(t)) + B(t)u(t)$ for $t \neq t_k$, $k = 0, 1, 2, 3, ...$

(ii) For $t = t_k, k = 0, 1, 2, .., x(t)$ instantaneously jump from t_k to new positions

$$\Delta x(t_k) = x(t_k + 0) - x(t_k) = I_1(x(t_k))$$
$$\Delta u(t_k) = I_2(u(t_k))$$

And satisfies the following equation.

$$\left.\begin{array}{l} \dfrac{dz(t)}{dt} = f(t, z(t), u(t)) + B(t)u(t), t \neq t_k, k = 0, 1, 2, .. \\ \\ \Delta z(t = t_k) = I_1(z(t_k)) \\ \Delta u(t = t_k) = I_2(u(t_k)) \\ 0 < t_0 < t_1 < t_2 < ... < t_k, \lim\limits_{k \to \infty} t_k = +\infty \end{array}\right\} \tag{4}$$

where

$$z = x(t) \in PC(R^+, R^n) \text{ for } u(t) \in C.$$

Definition 5. Let $x(t)$ be the solution of an impulsive control system (ICS) passing through $(t_0, x(t_0+0) = x_0).(ICS)$ is said to be completely controllable if there exists a control variable $u(t) \in U$ which steers x_0 to $x(t_1^*) = x_1$ in the finite time interval $(t_0,\ t_1^*)$ where U is the control space.
The question of δ-controllability of an impulsive control system (ICS) is that of selecting a control function $u(t)$ among admissible control in
$U \subseteq L_\infty(R^+)$ to steer the solution $x(t) = x(t,u)$ of (ICS) to $x(t_k+\delta, u) = x_\delta$, from the origin to x_δ, in the interval of length $|t_k + \delta - t_0|$, for $\delta > 0$ in a finite period of time t if such solution exists.

In this paper, without loss of generality, E will be taken to be $E = R^n$ and equip it with the usual supnorm $|\cdot|$.

Definition 6. The set

$$\begin{aligned} A = \{x(t,u) \in PC(R^+, R^n) :& \\ u(t) \in U, x(t,u) = x(t, t_0+0, x_0, u) &\text{ such that} \\ x(t_0, t_0+0, x_0, u) = x_0 \text{ for } t \in R^+\} & \end{aligned} \tag{5}$$

is called Attainable set. Any two points $x(t), y(t) \in A$ is said to be exponentially equivalent, if given any real number $\epsilon > 0$, there exists a continuous function $\delta(\epsilon, \delta) > 0, \exists\delta > 0$ such that

$$|x(t) - y(t)| \leq \epsilon q(t-\delta) exp(t-\delta)$$

whenever $|x_r - y_r| < \delta$
where

$$x_\delta = x(\delta, u), y_\delta = y(\delta, u) \ \text{ and } \ q \in K.$$

Remark 1.

$$\text{if } \ t : 0 < t < \delta - \log(q(t-\delta)) \ \text{ and } \ \lim_{t\to\infty} q(t-\delta) = 0$$

for every finite δ.

Then the concept of exponential equivalence is equivalent to the concept of asymptotic stability.
The control system in equation (1) is equivalent to the following operator equation

$$\mathcal{L}x = g \tag{6}$$

$$\mathcal{L} := \left(\frac{d}{dt}, \Delta_1, \Delta\right), g_1 := (f, I_1, I_2), g_2 := (B(t)u(t), 0, 0)$$

and $g := g_1 + g_2$.

Then the solution of this operator equation can be determine as a typical fixed point problem.

Let

$$M = \int_{\delta}^{t} B^*(s)X^*(t,s)X(t,s)B(s)ds \tag{7}$$

and take

$$\begin{aligned} u(t) &= -B^*(t)X^*(t,t_0)M^{-1}[x^{\delta} - \sum_{t_0<t_k<t} X(t_k,s)(I_1(x(t_k)) \\ &\quad + I_2(u(t_k))], \quad k = 1,2,\cdots \end{aligned} \tag{8}$$

and

$$\begin{aligned} x(t) &= X(t,t_0)x^{\delta} + \int_{t_0}^{t} X(s,t)B(s)u(s)ds \\ &\quad + \sum_{s_k \le t_k < t} X(t_k,s_k)(I_1(x(t_k)) + I_2(u(t_k)). \end{aligned} \tag{9}$$

Here $X(t,t_0)$ is the fundamental solution of equation (1) for the case when $B \equiv 0$. It is in fact the BKZ function (see Lemma 1 below).

Now define

$$A_{\epsilon}x(t) = x(t) - \epsilon(\mathcal{L}\, x(y) - x(t)) \tag{10}$$

where $\epsilon > 0$ is an arbitrary small real number for $x(t) \in D(A_{\epsilon})$. Clearly, if $x(t)$ is the fixed point of operator $\mathcal{L}$ then it is also fixed point of A_{ϵ}. Therefore, controllability properties of equation (1) is equivalent to that of finding the fixed point of A_{ϵ}.

4. Auxiliaries Results

Consider the impulsive system

$$\dot{x}(t) = A(t)x(t) + f(t,x(t), \quad t \neq t_k \tag{11}$$

$$x(t_k + 0) = Q_k x(t_k - 0), k = 1,2,\cdots$$

$$x(t_0 + 0) = 0$$

$$A(t) \in M_n(C(R^+,R)), \quad f(t,x(t) \in C(R^+ \times R^n), x(t) \in R^n$$

$Q_k \in C_0(R^+, R), \{t_k\}$ for $k = \{1, 2, \cdots\}$ are the fixed moments for which impulses take place such that the following conditions satisfied:

$$0 < t_0 < t_1 < t_2 < \cdots < t_k, \quad \lim_{k\to\infty} t_k = \infty.$$

In order to start our studies on equation (11), first all, consider the unperturbed situation of the equation (12) such that $f \equiv 0$ and assume that $x(t) = U(t)x_0$ is a solution of the equation, then

$$\begin{cases} \dot{x}(t) = A(t)x(t), \quad t \neq t_k, k = 0, 1, 2, \cdots \\ x(t_k + 0) = Q_k x(t_k - 0) \end{cases} \tag{12}$$

for $0 < t_0 < t_1 < \cdots < t_k, \lim_{k\to\infty} t_k = \infty$.

Let $U(t, s)$ be the Cauchy matrix associated with the solution of equation (12) when the impulses are absent.

Then the relationship that exists between $U(t, s)$ and the Cauchy matrix $W(t, s)$ of equation (11) in the presence of impulses will be established. Before embarking on this, some known standard results on semigroup properties of $U(t, s)$ that will be needed in obtaining subsequent results can be stated as:

a $U(t, s) = U(t, \tau)U(\tau, s)$

b $U(t + s, \tau) = U(t, \tau)U(s, \tau)$;

c $U(t, t) = I(t, 0) = I =$ identity operator

d $\dfrac{dU(t, \tau)}{dt} = AU(t, \tau)$

For all $t, s, \tau, t_0 \in R^+$.,

A natural candidate for $U(t, \tau)$ when $A(t)$ is autonomous is $U(t, s) = e^{A(t-a)}$. Following lemma establishes the relationship that exists between $U(t, s)$ and $W(t, s)$. It should be noted that the result had appeared in (see Bainov et al [2]-[3]) though state therein without proof. The structure of the proof is in fact not as trivial as one will think of.

Lemma 1. *For $t_k < t < t_{t+1}$, $(k = 0, 1, 2, \cdots)$ then the following relationship holds:*

$$u(t) = U(t, t_k)Q_k U(t_k, t_{k-1})Q_{k-1} \cdots Q_1 U(t_1, t_0)$$

and

$$U(t, s), t_n < s \leq t \leq t_{n-1}$$

$$u(t,s) = U(t,t_n)Q_nU(t_n,s),\ t_n < s < t_n < t < t_{n+1}$$

$$U(t,t_n)\left(\prod_{j=n}^{k+1} Q_jU(t_j,t_{j-1})Q_kU(t_k,s)\right),\quad t_{k-1} < s \leq t_k < t \leq t_{n+1}$$

Furthermore, $W(t,s)$ satisfies similar semigroup properties as $U(t,s)$.

Proof. The solution of equation (11) is

$$u(t) = U(t,t_0+0)K, K = \text{ constant vector in } R^n$$

hence

$$u(t_k+0) = U(t_k+0,t_0+0)K = U(t_k,t_0)K = Q_kU(t_{k-1},t_0)K.$$

Thus

$u(t_k+0) = Q_kU(t_{k-1},t_0)$

by prop (1) of $U(s,t)$, it follows that

$$U(t,t_0) = U(t,t_k)U(t_k,t_0)$$

by induction on K, one obtain

$$U(t,t_0) = U(t,t_k)Q_kU(t_{k-1},t_{k-2})Q_{k-1}U(t_{k-2},t_{k-3})\cdots Q_1U(t_1,t_2).$$

This establishes the first part of the proof. Now for $t_n < \tau < t < t_{n+1}$; it is trivial to show that

$$W(t,s) = U(t,s) \tag{14}$$

next if

$t_{n-1} < s \leq t_n < t < t_{n+1}$

then by equation (14)

$$U(t_{n-1},s) = U(t_{k-1},t_n)Q_nU(t_n,s) = W(t_n,s)$$

but

$$U(t,s) = U(t,t_n)Q_nU(t_n,t_{n-1})Q_{n-1}U(t_{n-1},t_{n=1})$$

$$= U(t,t_n)\left(\prod_{j=1}^{k+1} Q_jU(t_,t_{j-1})\right)Q_kU(t_k,s)$$

Thus the second part of the proof also follows. On the proof of semigroup properties of $W(t,s)$ this is straight forward since they are inherited from the of $U(t,s)$(see Oyelami & Ale[13]).

Lemma 2. (Morales' Theorem) *If E is a Banach space and $T : E \longrightarrow E$ is continuous and strongly accretive then T is surjective and the equation $Tx = f$ for a given f in E has a solution in E.*

The impulsive control system in equation (1) has a solution in $PC(R^+, E) \cap C'(R^+, E)$ which is given by

$$x(t) = X(t,t_0)x^\delta + \int_{t_0}^{t} X(s,t)B(s)u(s)ds \tag{15}$$

$$+ \sum_{s_k \leq t_k < t} X(t_k, s_k)(I_1(x(t_k)) + I_2(u(t_k)).$$

$$x(t_k + \delta) := x^\delta \text{ for some } \delta > 0.$$

Proof. By direct differentiation, it is not difficult to see that

$$x(t) \in PC(R^+, R^n) \cap C'(R^+, R^n)$$

and it even satisfies the impulsive system in equation (13).

5. Main Results

Theorem 1. *Suppose that the following conditions are satisfied:*

1. *M is in equation (7) is invertible.*
2. *The control variable $u(t)$ is defined in equation (8) and let*

   $$\phi^\delta = \phi(t_k + \delta, \delta), \phi^{-\delta} = \phi(\delta, t_k + \delta).$$

   Then $\phi^{-\delta} \cdot \phi^\delta = I$ and

   $$x_\delta = \phi^\delta \left[x^\delta - \sum_{t_0 < t_k < s} \phi(t_k s)(I_1(x(t_k))) + I_2(u(t_k)) \right]$$

   for $x(t) \in A$ and $u(t) \in U$.
   B. (ICS) is completely controllable if and only if it is δ-controllable.

Proof. (A) is straight forward, it follows, therefore, that

$$\begin{aligned} x(t_k+\delta) &= \phi^\delta x_\delta - \left[x^\delta - \sum_{s_k \le t_k < s} \phi(t_k s_i)(I_1(x(t_i)) + I_2(u(t_i))) \right] \\ &= \phi^\delta \cdot \phi^\delta (x^\delta - \sum_{t_0 < t_i < t_k+\delta} \phi(s_i, t_i)(I_1(x(t_i)) + I_2(u(t_i))) \\ &= x^\delta - x_\delta. \end{aligned}$$

Thus

$$\langle (z \cdot x^\delta)e,\ j(z)\rangle > \langle z \cdot (\phi^\delta x_\delta + x^\delta)e,\ j(z)\rangle$$
$$> \langle (z \cdot x\delta)e,\ j(z)\rangle$$

A contradiction, hence, if $x_\delta \equiv 0$ then (ICS) is δ-controllable.

Introduce the conditions (A) and (B) as follows: condition (A) is said to have been satisfied if:

A1: The function $f : R^+ R^n \times U \longrightarrow R^n$ is continuous in $R^+ \times R^n \times U$, $f(t, 0, 0) = 0$ for $t \in R^+$ and there exists a constant $C_1 > 0$ such that

$$|f(t, x_1(t), u_1(t)) - f(t, x_2(t), u_2(t)|$$
$$\le C_1[|x_1(t) - x_2(t)| + |u_1(t) - u_2(t)|]$$
$$\text{for } \ t \in R^+ \ \text{ and } \ x_1, x_2 \in R^n, u_1, u_2 \in U.$$

A2 :The function $I_1 : R^n \longrightarrow R^n$, is a continuous in R^n and $I_i(0) = 0$ and there exists constant C_2 such that

$$|I_1(x(t_k)) - I_1(y(t_k))| \le C_2 |x(t_k) - y(t_k)| \tag{14}$$

for $x(t), y(t) \in R^n$.

A3: There exists $x(t) \in PC(R^+, R^n) \cap A$ such that

$$x(t_k + 0) = x(t_k) + \Delta x(t_k)$$

$$x(t_k - 0) = x(t_k).$$

Condition (B) is said to have been satisfied if the following conditions are satisfied:

B1: The function $I_2 : \mathcal{L}_\infty(R^+) \longrightarrow R^n$ is continuous in R^n and $I_2(0) = 0$ such that there exists a constant C_2 such that

$$\begin{aligned} |I_2(u(t_k)) \quad - \quad I_2(u_2(t_k))| \leq C_2|u_1(t_k) - u_2(t_k)| \\ u_1 u_2 \in \mathcal{L}_\infty(R^+). \end{aligned} \tag{15}$$

B2: There exists $u(t) \in \mathcal{L}_\infty(R^+)$ such that

$$u(t_k + 0) = u(t_k) + \Delta u(t_k)$$

$$u(t_k - 0) = u(t_k).$$

Theorem 2. *Let the following conditions be satisfied:*

1. *Properties (A) and (B);*
2. *$x(t), y(t) \in A$ are exponentially equivalent*

Then (ICS) is completely controllable and $A = R^n$.
Proof. Let $x(t), y(t) \in A$ and $j \in J(x(t) - y(t))$ then

$$A_\xi x(t) - A_\xi y(t)|^2 = (1 - \xi)\langle x(t) - y(t), j\rangle - \xi\langle Lx(t) - Ly(t), j\rangle \tag{16}$$

Now let

$$M_0 = \sup_{t,s \in R^+} |\phi(t,s)|, B_0 = \sup_{s \in R^+} |B(s)|, \alpha = \max_k |t - t_k|, k = 1, 2, 3 \cdots$$

Then the following estimates can be made:

$$\begin{aligned} |Lx(t) \quad - \quad & Ly(t)| \leq m_0|x_\delta - y_\delta| + \textstyle\int_{t_0}^t |\phi(t,s)||B(s)||u_1(s) - u_2(s)|ds \\ & + \sum_{t_0 < t_k < t} M_0[K_1|x(t_k) - y(t_k)| + K_2|u_1(t_k) - u_2(t_k)|]. \end{aligned}$$

Hence,

$$\begin{aligned} |Lx(t) \quad - \quad & Ly(t)| \leq m_0(1 + B_0 N\alpha)|x_\delta - y_\delta + \textstyle\int_{t_0}^t M_0 K_1|x(s) - y(s)|ds \\ & +2\textstyle\sum_{t_0 < t_k < t} M_0[K_2|x(t_k) - y(t_k)| + |u_1(t_k) - u_2(t_k)|]. \end{aligned}$$

Hence,

$$|Lx(t) - Ly(t)| \leq m_0(1+B_0N\alpha)|x_\delta - y_\delta + \int_{t_0}^t M_0K_1|x(s)-y(s)|ds$$
$$+2\sum_{t_0<t_k<t} M_0[K_2|x(t_k)-y(t_k)| + |u_1(t_k)-u_2(t_k)|].$$

Hence, $x(t), y(t)$ are fixed points of L and by Lemma 1, hence, for $z(t) = x(t)-y(t)$,

$$|z(t)| \leq C(1+M_0K_1)^{i(\delta,t)} \cdot \exp M_0K_1(t-\delta) = \Gamma \exp K^*(t-\delta), \tag{17}$$

where

$$\begin{aligned} C &= M_0(1+\beta_0N\alpha)|Z^\delta| + \sum_{t_0<t_k<t} K_2|u_1(t_k)-u_2(t_k)| \\ K^* &= M_0K_1 \\ \Gamma &= C(1+M_0K_1)^{i(\delta,t)} \end{aligned}$$

$i(\delta,t)$ is the counting number, that is, the number of impulses present in the interval $[\delta,t]$. Since $x(t)$ is assumed to be exponential equivalent to $y(t)$ in A. Then

$$\begin{aligned} |A_\xi X(t) - A_\xi y(t)|^2 &\geq (1-\xi)|x-y|^2 = \xi\Gamma\exp(K^*(t-\delta))|x(t)-y(t)| \\ &\geq (1-\xi)|x(t)-y(t)|^2 - \frac{\xi\Gamma}{\epsilon_0}|x(t)-y(t)|^2 \\ &= \left(1-\xi-\frac{\xi\Gamma}{\xi_0}\right)|x(t)-y(t)|^2. \end{aligned}$$

Choose $\xi \in \left(0, \left(1+\frac{\Gamma}{\xi_0}\right)^{-1}\right)$ then $A_\xi : R^n \longrightarrow R^n$ is strongly accretive. Next we show that is continuous. Suppose on the contrary that A_ξ is not continuous, then, for every $\epsilon > 0$ there exists $n_0 > n$ such that

$$|A_\xi x_n(t) - A_\xi x(t)| < \epsilon \text{ whenever } |x_n(t)-x(t)| > \frac{\epsilon}{2}$$

for some $n_0 > n$.

Let ϵ be in (0,1) then

$$\epsilon \geq \epsilon^2 \geq |A_\xi x_n(t) \quad - \quad A_\xi x(t)|^2 \geq \left(1-\xi-\tfrac{\xi\Gamma}{\epsilon_0}\right)|x_n(t)-x(t)|^2$$
$$> \quad \frac{\epsilon^2}{4}\left(1-\xi-\frac{\xi\Gamma}{\epsilon_0}\right).$$

it follows that $\epsilon \in \left(-3\left(1+\dfrac{1}{\epsilon_0}\right)^{-1}, 0\right)$ which contradicts strong accretativity of $A_\epsilon(\epsilon \in (0,1))$. Hence A_ϵ is continuous and by Morales' Theorem there exists a fixed point of A_ϵ which is the solution of the (ICS). Now, invoking Theorem 1, it implies that the system is δ-controllable.

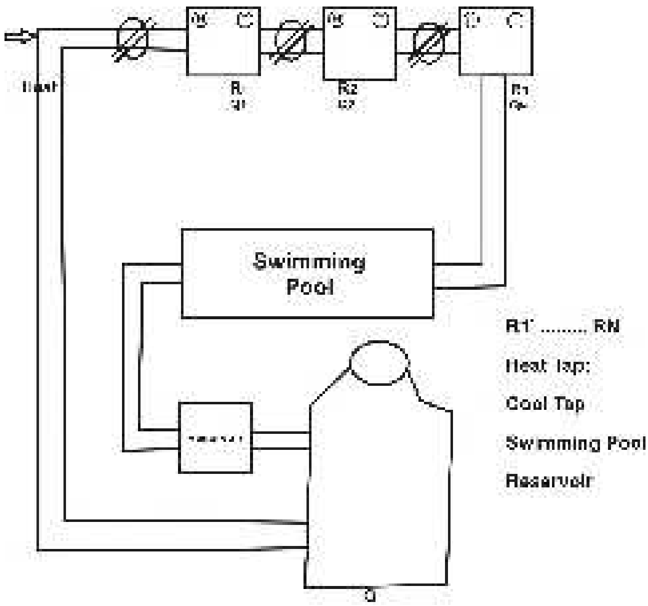
6. Applications

Example 1. Consider the application of a non-linear control system with impulsive action. The impulsive control system consist of the three compartments A, B and C. The purpose of the system is to design an automatic control system in which the temperature of the substance in B for example water, solvent or air and so on are maintained at a particular temperature.

The compartment B is heated up and the compartments A and C contained cooling substances which allows flow of cooling substances (e.g. water, solvent or air) from A or C to the compartment B to lower the temperature of the substance in it if it is above a given threshold value. An application of this kind of system is useful in the design of an automatic temperature controlled swimming pool, incubator, nuclear reactors, or heating and cooling in biological and physical systems that heat or temperature is required to be impulsive.

We will illustrate how the impulsive automatic control system works. In the figure 1 there is an automatic control system constituting of a big tank,a hot water reservoir, room water reservoirs,a swimming pool, a cool water reservoir and an automatic temperature controller (heat sensor). The the connection of the component of system is as shown in the diagram. The system operates as follows:

- Whenever the heat of the rooms and swimming pool go up above the threshold value (equilibrium temperature), the cooling systems are activated by allowing cool water into system to lower the temperature of water in the reservoir in the rooms and the swimming pool.



- Whenever the heat of the rooms and swimming pool go below the threshold value(equilibrium temperature), the heating systems are activated by allowing warm water into system to increase the temperature of water in the reservoir in the rooms and the swimming pool.

6.1. Heat Transfer

Temperature change can occur in the three ways through radiation, conviction and conduction. The net heat flow at time t in the system is $Q(t) = Q_2(t) - Q_1(t) - Q_3(t)$.The automatic controller switch on the heater if $Q_2(t) - Q_1(t) - Q_3(t) \geq \Phi(t)$ and off the heater

if $Q_2(t) - Q_1(t) - Q_3(t) < \Phi(t)$.$\Phi(t)$ is the threshold heat required to be maintained in the system.

Therefore the net heat per unit area A per time is

$$Q(t) = Q_1(t) - Q_2(t) - Q_3(t) = \Phi(t)$$

$$\text{Therefore} \qquad \frac{\partial Q(t)}{\partial t} = k\left(\frac{\partial^2 Q(t)}{\partial^2 x} + \frac{\partial^2 Q(t)}{\partial^2 y}\right) + G(Q, T)$$

$$\text{And} \qquad \frac{dT}{dx} = \frac{Q}{A} - \delta T^4 + h(T - T_\infty) - \int_{t_0}^{t} Q(s)u(s)ds$$
$$\Delta T(t = t_k) = I(T(t_k)), \Delta u(t_k) = 0$$
$$0 < t_0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty$$

Here:

σ =coefficient of radiation

h =thermal conductivity

T =absolute temperature of the compartment B which is assumed to be impulsive across section of the compartment B.

T_∞ is amount predictable temperature of the compartment B

Now take $T_\infty = \frac{Q}{Ah}$ and define

$U = \{u \in R^+ : |u| = \int_0^t \sup\{|w(s)||u(s)|ds < +\infty\}$,

$Q_0 = \max_{t\in R^+} |Q(t)|$ and let $g(t, u(t), T_\infty) = \frac{Q}{A} - hT_\infty - \int_0^t Q(s)u(s)ds$.

Then we investigate whether properties (A) and (B) are satisfied.

clearly,

$$f(t, 0, 0) = 0$$
$$|f(t, T_1, u_1) - f(t, T_2, u_2)| \leq (\delta^* + |h|)|T_2 - T_1| + Q_0|u_2 - u_1|$$
$$\text{for } T_i \in PC(R^+, R), u_i \in U, i = 1, 2.$$

and also

$$|I(T_2) - I(T_1)| \leq \gamma|T_2 - T_1|$$
$$\text{for } \gamma = |\beta_k|, k = 0, 1, 2, ...$$

And

$$\frac{dT}{dx} = -\delta T^4 - hT - g$$
$$\Delta T(t_k) = \beta_k I(T(t_k))$$
$$0 < t_0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty$$

The above equation is an impulsive Bernoulli (hyperlogistic) equation of order 4 and the fundamental solution $g = 0$ (see Oyelami & Ale[13] and Bainov et. al.[1]).

$$\Psi(t, T_0) = \left[\frac{T_0}{\Pi_{0<t_k<t}(\frac{1}{\beta_k}) \exp -3\delta t - 3h \int_0^t \Pi_{0<t_k<t}(\frac{1}{\beta_k}) \exp -\delta(t-s)ds}\right]^{1/3}$$

Therefore the general solution using variation of constant parameter is

$$T(t) = T = \Psi(t, T_0)T_0 + \int_{t_0}^{t} \Psi(s, T_0)g(T(s))ds$$

Therefore we have the following estimation

$$|T(x) - T(y)| \leq \frac{T_0^{1/3}}{\left(\Pi_{0<t_k<t}(\frac{1}{\beta_k}) \exp -3\delta x)\right)^{1/3}} - \frac{T_0^{1/3}}{\left(\Pi_{0<t_k<t}(\frac{1}{\beta_k}) \exp -3\delta y)\right)^{1/3}} + |\int_{x}^{y} \Psi(s, T_0)g(T(s))ds|$$

Hence

$$|T(x) - T(y)| \leq \left[\frac{T_0}{\Pi_{0<t_k<t}(\frac{1}{\beta_k})}\right]^{1/3} \left[\sigma \exp -(\delta x + \delta y) + e^{-pt}|u|\right] |x - y|$$

Let $q(\sigma, x, y) = \sigma \exp -\sigma(x + y)$ and $Q_0 = e^{-pt}$ then $\lim_{\substack{x\to\infty \\ y\to\infty}} q(\sigma, x, y) = 0$.
Hence, by Theorem1, the system is $\delta-$controllable. The control that regulates this is $u(t)$ and it is defined in equation (7) when $M = I =$identity matrix and $X(t, T_0) = \Psi(t, T_0)$ in equation (8) and $I(u(t_k)) = 0$.

Example 2. Consider an impulsive control hematopoiesis model(see Saker and Alzebut[16])

$$\begin{aligned} p^{/}(t) &= \frac{p(t)}{1 + p^n(t)} - \gamma p(t) + \int_{t_0}^{t} k(s, t)u^2(s)ds \\ \Delta p(t_k) &= (1 + \beta_k)p(t_k) \\ & 0 < t_0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty \end{aligned}$$

Where: $p(t)$ is production of blood cells from the marrow. The rate of blood lost from circulation being γ and we have introduced $\int_{t_0}^{t} k(s, t)u^2(s)ds$ into the hematopoiesis model in Saker and Alzebut model[16] to measure the replacement of the blood by new blood cells as a result of use of drug, or food supplement, where $k(s, t)$ is the production function of the blood cells and the quadratic controller regulate the production level of the blood cells under the influence of the blood enhancing drugs or food supplements.

If

$$f(t, p(t)) = \frac{p(t)}{1 + p^n(t)} - \gamma p(t) + \int_{t_0}^{t} K(s, t)u^2(s)ds$$

Then for $p, q \in (0,1)$, $r(t) = ke^{-\alpha t}, k$ andα are positive constants. Take $t_k = \delta - \log r(t_k - \delta), \delta > 0$ such that $\lim_{t\to\infty} \int_{t_0}^{t} |u_1(s) - u_2(s)|ds = 0$.

Here $p_\delta = p(\delta, u_1)$ and $q_\delta = q(\delta, u_2)$ such that $\delta = |p_0 - q_0|$ and p and q are solutions to the model.

Therefore

$$
\begin{aligned}
&|f(t,p,u_1(t)) - f(t,q,u_2(t))| \\
&\leq \frac{|(1+q^n)p - (1+p^n)q|}{(1+p^n)(1+q^n)} + \gamma|p-q| \\
&+ \int_{t_0}^{t} |K(s,t)||u_1^2(s) - u_2^2(s)|ds \\
&\leq (1+\gamma)|p-q| + 2kk^* \int_{t_0}^{t} |u_1(s) - u_2(s)|ds
\end{aligned}
$$

Since $p^n, q^n \to 0$ as $n \to \infty$ where $k^* = \max(|u_1(s)|, |u_2(s)|)$and $k = \max_{(t,s)\in R^+\times R^+} |K(t,s)|$.

Then by Theorem 2, the system is $\delta-$controllable which means that we can find a $\delta-$neighborhoods about the time t for which the blood production can be enhanced through drugs or food supplements which serves as artificial control for blood production.

Example 3. Consider the impulsive control system

$$
\dot{x}(t) = ax(t) - bu(t)x^2(t) + g(x(t)) + \int_0^\infty e^{-ps}u(s)ds, t \neq t_k, k = 0,1,2,\cdots
$$

$$\Delta x(t = t_k^+) = \gamma_k x(t_k)$$

$$x(t_0 + 0) = x_0$$

$$0 < t_0 < t_1 < t_2 < ... < t_k, \lim_{k\to\infty} t_k = +\infty$$

Here: $x(t) \in PC(R^+, R^+), g \in C(R^+, R^+), u \in U = \{u : |u| \leq 1\}, a, b, \gamma_k,$ are positive constants for $k = 0, 1, 2$

From the results in (see Oyelami and Ale [13]), the fundamental solution the system for $g = 0, u = 0$ is

$$
\Phi(t,t_0) = \frac{1}{\Pi(1+\frac{1}{\sqrt{\gamma_k}})\exp -a(t-t_0) - bN\Pi(1+\frac{1}{\sqrt{\gamma_k}})}
$$

Hence by variation of constant parameter the solution to the problem is

$$
x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)g(x(s))ds
$$

$$+\int_{t_0}^{t}\int_{0}^{\infty}\Phi(t,s)e^{-ps}u^2(s)dsdt$$

If $g(x(t))$ is lipschitz with respect to $x(t)$ and

$$\lim_{t\to\infty}\int_{t_0}^{t}\int_{0}^{\infty}\Phi(t,s)e^{-ps}|u_1(s)-u_2(s)|dsdt = 0, u_i(t)\in U, \quad i=1,2.$$

We can show that the system is $\delta-$controllable in R^+.

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