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SUFFICIENT CONDITIONS FOR OSCILLATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: In this paper we study some sufficiant conditions for oscillation of second order nonlinear homogeneous differential equation. An example is included to illustrate the results.

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1. Introduction

The purpose of this paper is to establish some sufficient conditions for the oscillation of the second order nonlinear differential equation with variable coefficients

$$x''(t) + f(x(t))(x'(t))^2 + g(x(t)) = 0, \text{ for } t \geq \alpha, \tag{1.1}$$

where $\alpha \geq 0$ is a fixed real number and $f(x(t))$ and $g(x(t))$ are continuously differentiable functions on the interval $[\alpha, \infty)$.

Our attention is concentrated only to such solution $x(t)$ of the differential equation (1.1) which exists on some interval $[\beta, \infty)$, for $\beta > \alpha$.

Definition 1. A solution $x(t)$ of the differential equation (1.1) is said to be nontrivial if $x(t) \neq 0$ for at least one $t \in [\alpha, \infty)$.

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Definition 2. A nontrivial solution $x(t)$ of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeros on $[\beta, \infty)$, for $\beta > \alpha$ otherwise it said to be nonoscillatory.

The study of the oscillation of a second order nonlinear ordinary differential equations is of special interest. Many criteria have been found which involve the behavior of the integral of a combination of the coefficients of second order nonlinear differential equations. This approach has been motivated by authors (for example see [1] ,[2],[3],[4],[5], [6], [7],[8] and the authors therein).

The purpose here is to present some sufficient conditions for the oscillation of the differential equation (1.1).

2. Main Results

We prove the following theorem

Theorem 1. *If*

$$\lim_{t\to\infty}\left[\frac{1}{4}\int_{\alpha}^{t}\left\{\frac{4sg(x(s))}{f(x(s))}-\frac{1}{((f(x(s))^2+\frac{df(x(s))}{dx})}\right\}ds\right]=\infty \tag{2.1}$$

and

$$\lim_{t\to\infty}\int_{\alpha}^{t}\frac{(f(x(s))^2+\frac{df(x(s))}{dx})ds}{s}=\infty \tag{2.2}$$

Then any solution of the differential equation (1.1) is oscillatory on $[\alpha, \infty)$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on the interval $[\beta, \infty)$, where $\beta \geq \max\{\alpha, 1\}$, without loss of generality tis solution can be supposed such that $x(t) \geq 0$ on $[\beta, \infty)$.

We define

$$w(t) = -\frac{tx'(t)}{f(x(t))}.$$

Then $w(t)$ is well defined and satisfies the Riccati equation

$$w'(t) = \frac{(f(x(t))^2+\frac{df(x(t)))}{dx}}{t}\left\{(w(t))^2+\frac{1}{(f(x(t))^2+\frac{df(x(t)))}{dx})}w(t)\right\}+\frac{tg(x(t))}{f(x(t))}$$

$$w'(t) = \frac{(f(x(t))^2+\frac{df(x(t)))}{dx}}{t}\left\{\left(w(t)+\frac{1}{2(f(x(t))^2+\frac{df(x(t)))}{dx})}\right)^2\right\}$$

$$+\frac{tg(x(t))}{f(x(t))}-\frac{1}{4s\left(f(x(t))^2+\frac{df(x(t)))}{dx})\right)}.$$

Integrating both sides of the above equation from α to t we get

$$w(t)=$$
$$w(\beta)+\int_\alpha^t \frac{(f(x(s))^2+\frac{df(x(s))}{dx})}{s}\left\{\left(w(s)+\frac{1}{2((f(x(s))^2+\frac{df(x(s))}{dx})}\right)^2\right\}ds$$
$$+\int_\alpha^t \left(\frac{sg(x(s))}{f(x(s))}-\frac{1}{4\left((f(x(s))^2+\frac{df(x(s))}{dx})\right)}\right)ds.$$

By using the hypotheses (2) implies there exists $\beta>\alpha$ such that

$$w(t)\geq\int_\beta^t \frac{1}{s(f(x(s))^2+\frac{df(x(s))}{dx})}\left\{\left(w(s)+\frac{1}{2(f(x(s))^2+\frac{df(x(s))}{dx})}\right)^2\right\}ds.$$

Define a function $Q(t)$ for $t>\beta$ by

$$Q(t)=$$
$$\int_\beta^t \frac{(f(x(s))^2+\frac{df(x(s))}{dx})}{s}\left\{\left(w(s)+\frac{1}{2((f(x(s))^2+\frac{df(x(s))}{dx})}\right)^2\right\}ds, \quad (2.3)$$

then we have $w(t)\geq Q(t)>0$.

Differentiating (2.3) we get

$$\begin{aligned}Q'(t) &= \frac{(f(x(t))^2+\frac{df(x(t)))}{dx})}{t}\left(w(t)+\frac{1}{2((f(x(t))^2+\frac{df(x(t)))}{dx})}\right)^2\\ &\geq \frac{(f(x(t))^2+\frac{df(x(t)))}{dx})}{t}\left(Q(t)+\frac{1}{2((f(x(t))^2+\frac{df(x(t)))}{dx})}\right)^2\\ &\geq \frac{(f(x(t))^2+\frac{df(x(t)))}{dx})}{t}(Q(t))^2.\end{aligned}$$

Therefore

$$\frac{(f(x(t))^2+\frac{df(x(t)))}{dx})}{t}\leq\frac{Q'(t)}{(Q(t))^2}.$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for $t > \beta$ we get

$$\int_{\beta}^{t} \frac{((f(x(s))^2 + \frac{df(x(s)}{dx})ds}{s} \leq \frac{1}{Q(\beta)} - \frac{1}{Q(t)},$$

since $Q(t) > 0$. Therefore

$$\lim_{t\to\infty} \int_{\beta}^{t} \frac{((f(x(s))^2 + \frac{df(x(t)))}{dx})ds}{s} < \frac{1}{Q(\beta)}.$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory. □

3. An Example

Example 1. The following examples illustrate the applicability of both theorems.

$$x'' + \cot(x(t))(x')^2 + B\cot(x(t)) = 0, \tag{3.1}$$

for this differential equation we have $f(x) = \cot(x(t))$ and $g(x) = B\cot(x(t))$, where B is any real constant

To show the applicability of Theorem , the hypothesis is satisfied as follows

$$\begin{aligned}
&\lim_{t\to\infty}\left[\frac{1}{4}\int_{\alpha}^{t}\left\{\frac{4sg(x(s))}{f(x(s))} - \frac{1}{((f(x(s))^2 + \frac{df(x(s))}{dx})}\right\}ds\right] \\
= &\lim_{t\to\infty}\left[\frac{1}{4}\int_{\alpha}^{t}\left(\frac{4sB\cot(x(s)))}{\cot(x(s))} - \frac{1}{s((\cot(x(s))^2 - (\csc(x(s))^2)}\right)ds\right] \\
= &\lim_{t\to\infty}\left[\frac{1}{4}\int_{\alpha}^{t}\left((4Bs + \frac{1}{s}\right)ds\right] \\
= &\lim_{t\to\infty}\left[B\frac{s^2}{2} + \frac{1}{4}\ln s\right]_{\alpha}^{t} = \infty
\end{aligned}$$

and

$$\lim_{t\to\infty}\int_{\alpha}^{t}\frac{(f(x(s))^2+\frac{df(x(s))}{dx})ds}{s}$$

$$= \lim_{t\to\infty}\int_{\alpha}^{t}\frac{(\cot(x(s))^2-(\csc(x(s))^2}{s}ds = \lim_{t\to\infty}\int_{\alpha}^{\infty}\left(-\frac{1}{s}\right)ds$$

$$= \lim_{t\to\infty}\left[-\ln s\right]_{\alpha}^{t} = \infty.$$

Therefore the theorem implies that the differential equation is oscillatory.

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