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INTEGRABILITY OF A RATE-TYPE MODEL

M.D. Kolev[1], V. Radeva[2] §, V. Todorov[3]

[1]Department of Applied Mechanics
University of Chemical Technology and Metallurgy (UCTM)
8, St. Kl. Ohridski Blvd., Sofia, 1756, BULGARIA

[2]Department of Mathematics
University of Chemical Technology and Metallurgy (UCTM)
8, St. Kl. Ohridski Blvd., Sofia, 1756, BULGARIA

[3]Department of Electrotechnics and Electronics
University of Chemical Technology and Metallurgy (UCTM)
8, St. Kl. Ohridski Blvd., Sofia, 1756, BULGARIA

Abstract: This paper concerns a mathematical model of rate-type materials which are inelastic and have nonlinear behavior under some stress. We consider a geometrical method for integration of the considered model as for this purpose use involutive distributions in order to obtain certain solutions known as simple states.

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1. Introduction

In this note we study the integrability of a mathematical model describing rate-type materials which are subjected to one-dimensional deformation processes.

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§Correspondence author

It is a nonhomogeneous hyperbolic quasilinear system of PDEs having the form

$$\left| \begin{array}{l} u_t + u_x = 0 \\ v_t + h(v)u_x = f(v), \end{array} \right. \tag{1}$$

where t and x are time and space, respectively; $u_t \equiv \partial u/\partial t$, $u_x \equiv \partial u/\partial x$, u is the Lagrangian velocity, v is the stress; $h(v) > 0$, $f(v) > 0$ ($v \in \mathbb{R}^1$) are material response functions, which measure out, respectively, the instantaneous and noninstantaneous response of the material to an increment of the stress. Existence and uniqueness result obtained by a classical approach can be seen in [2]. Some applications in gas-dynamics, electrodynamics (see, e.g. [4], [14]) and other branches of science exist in [10], [11], [13], [17], [18]. In the present paper we consider a geometrical approach to solve such systems, that was established in [15] and applied to different mathematical models by several authors (see, e.g. [3]-[6], [8]-[14]). Classification of waves of geometrical point of view can be found in [19]. A model with time dependent coefficients was considered in [16].

Furthermore, consider a quasilinear nonhomogeneous system of general type

$$\left| \begin{array}{l} u_t + u_x = 0 \\ v_t + a(u,v)u_x + b(u,v)v_x = g(u,v), \end{array} \right. \tag{2}$$

where a, b, g are sufficiently smooth functions of their arguments.

Now we intend to sketch the method. First, we require the system (2) to be strictly hyperbolic. Then transform it in Pfaffian differential system of two equations in five variables, which define a completely integrable involutive two-dimensional distribution $\theta_1(z)$ ($z \in \mathbb{R}^5$). Hence for each two linearly independent vector fields $\eta_1, \eta_2 \in \theta(z)$ we are able to define the system

$$\eta_1 \ \Phi \ = \ 0, \qquad \eta_2 \ \Phi \ = \ 0, \tag{3}$$

where $\Phi \ = \ \Phi(z)$ is unknown scalar function. Since the system (3) is involutive then in accordance with the theory of the integrable systems, there exist three functionally independent integrals $\Phi^i(z) = c^i$ ($i = 1, 2, 3$). Here c^i are constants depending on the initial data, $z \equiv (t, x, y, u, v)$, y is either u_x or v_x. We obtain by the classical implicit function theorem an existence and uniqueness result for the simple state to the system (2) (cf. for instance [15]-[19]).

2. Riemann Invariants and Existence Analysis

The theory of Riemann invariants is well known so we use only the known results for (2) without focusing on the details (see, e.g. [2], [7]). Consider

the model (1). Denote by $T > 0$ the time beyond which the solution can not be continued. This period of time can be given asymptotically by the smaller of two numbers $[-\lambda_r(0)\max r_x(0)]^{-1}$ and $[-\mu_s(0)\max s_x(0)]^{-1}$, where r and s are Riemann invariants, i.e. $r = u + \int [h(v)]^{-1/2} dv$, $s = u - \int [h(v)]^{-1/2} dv$, and $\lambda \neq \mu$. Here λ and μ, where $\lambda = -\mu = \sqrt{h(v)}$, are eigenvalues of the considered system for $t \in [0, T]$. Denote by $l \equiv (l_1, l_2)$ the left eigenvector for (1) corresponding to λ. It is well known that the characteristic equation has the form

$$l_1 u^{'} + l_2 v^{'} = l_2 h(v), \tag{4}$$

where we follow the Lax notation $(\ ')\equiv \partial_t + \lambda\partial_x$, and $(\ `\ ) \equiv \partial_t + \mu\partial_x$. After introducing the integrating factor into (4) we obtain $r^{'} = F$, where $F \equiv f(v) r_v$ and differentiating it w.r.t. x, obtain $r_{tx} + \lambda r_{xx} + \lambda_r r_x^2 + \lambda_s s_x r_x = F_r r_x + F_s s_x$. By letting $w \equiv r_x$ we get $w_t + \lambda w_x + \lambda_r w^2 + \lambda_s s_x w = F_r w + F_s s_x$, and taking into account the derivatives $s^{'} = s_t + \lambda s_x$, $s^{`} = s_t + \mu s_x$ through the characteristics $x = X(t)$, $x = \tilde{X}(t)$, respectively, obtain the Ricatti equation

$$w^{'} = -\lambda_r w^2 + p(t) w + q(t), \tag{5}$$

where $p(t) \equiv F_r - \lambda_s s_x$, $q(t) \equiv F_s s_x$. Furthermore, we use the transformation $w = E(t) z$, where $E(t) \equiv e^{\int p(t) dt}$, $w^{'} \equiv w_t + \lambda w_x$, $z^{'} = a(t) z^2 + b(t)$, and $a \equiv -\lambda_r E(t)$, $b \equiv q(t)/E(t)$. Since $a(t)$ and $b(t)$ are continuous functions w.r.t. $t \in [0, T]$, then $a_1 < a(t) < a_2$, $b_1 < b(t) < b_2$ (for some $a_i, b_i \in \mathbb{R}$, $i = 1, 2$) and we are able to construct the comparison equations

$$\tilde{z}^{'} = a_i \tilde{z}^2 + b_i \quad (i = 1, 2), \tag{6}$$

possessing solutions $z_i = [\tilde{z}_0 k_i + b_i \tanh k_i t][k_i + a_i \tilde{z}_0 \tanh k_i t]^{-1}$ ($a_i b_i > 0$, $k_i \equiv \sqrt{a_i b_i}$), under initial data $z_0 = z(0) = \tilde{z}(0) = \tilde{z}_0$. Thus having in mind the above stated notations we are able to formulate the following statement:

Theorem 1. *Assume that $a_1 z_0 < 0$. If $\tilde{z}_1^{'}(0) \equiv a_1 \tilde{z}_{i0}^2 + b_1 < a(0) z_0^2 + b(0) \equiv z^{'}(0)$, then it follows that $\tilde{z}_1(t)$ is a lower bound for $z(t)$ in $[0, T]$, i.e. $z(t) \geq \tilde{z}_1(t)$, and $|z(t)| \to +\infty$, as $t \to -k_1/(a_1 z_0 \tanh k_1)$.*

3. Involutive Distributions and Simple States

It is well known that Riemann invariants exist if and only if the gradients ∇u and ∇v are collinear, and in accordance with the conception suggested in [15] obtain that

$$u_t v_x - u_x v_t = 0. \tag{7}$$

So we look for solutions satisfying (7). Combining (2) and (7) we obtain an overdetermined system,

$$\left|\begin{array}{l} u_t + v_x = 0 \\ v_t + au_x + bv_x = g \\ u_t v_x - u_x v_t = 0 \end{array}\right. \tag{8}$$

Furthermore, letting $y \equiv u_x$, $K \equiv v_x/u_x$ into the third equation of (8) obtain $K^2 - bK + y^{-1} - a = 0$, hence $K_{1,2} = 0.5(b \pm \sqrt{D(y,u,v)})$, $D(y,u,v) \equiv b^2 + 4(a - gy^{-1})$. Thus we get from (8)

$$u_t = -yK, \quad v_t = -yK^2, \quad v_x = yK. \tag{9}$$

Suppose that the following hypothesis hold:

H. The functions $a, b, g \in C^2(\Omega)$ $(\Omega \subset \mathbb{R} \times \mathbb{R})$, $D(p,q,r) > 0$, $(p,q,r) \in \Omega_1 \subset \mathbb{R} \setminus \{0\} \times \mathbb{R} \times \mathbb{R}$.

Furthermore, consider the corresponding to (9) Pfaffian differential system

$$\left|\begin{array}{l} \omega^1 \equiv du + yKdt - ydx = 0 \\ \omega^2 \equiv dv + yK^2dt - yKdx = 0 \end{array}\right. \tag{10}$$

Proposition 1. *There exist three linearly independent vector fields*

$$\xi_1 = (0,0,1,0,0)^t, \quad \xi_2 = (0,1,0,y,yK)^t, \quad \xi_3 = (1,K,0,0,0)^t. \tag{11}$$

annulling the differential forms (10)

The proof follows by direct calculation.

Proposition 2. *There exists a three dimensional distribution $\theta(z)$ as a linear hull of the fields (11).*

The proof is similar to that in [15].

Theorem 2. *(cf. [15]) There exist vector field $\eta = \eta_k \in \theta(z)$ $(k = 1, 2)$ so that*

$$\omega^i(\eta) = 0, \quad \partial\omega^k(\xi_j, \eta) \quad (i = 1,2;\ j = 1,2,3). \tag{12}$$

Proof of Theorem 2. We sketch the proof omitting the technical details. Define the vector fields

$$\begin{aligned} &\eta_1 \equiv -y^2(K_u + KK_v)\xi_1 + yK_y\xi_2 + \xi_3 = \\ &= (1, yK_y + K, -y^2(K_u + KK_v), y^2K_y, y^2KK_y)^t, \\ &\eta_2 \equiv q\xi_3 = (1,K,0,0,0)^t. \end{aligned}$$

Hence it is clear that the above stated vector fields satisfy (10) and (12). □

Furthermore, define the distribution $\theta_1(z) \subset \theta(z)$. In order to prove involutivity of $\theta_1(z)$ we need to show that the rank of matrix $\{\eta_1, \eta_2, [\eta_1, \eta_2]\}$ is two. Here it is clear that $[\eta_1, \eta_2] = 0$, which implies that the system (10) possesses characteristic vector field.

Theorem 3. *The system (2) has unique simple state derived from*

$$\begin{array}{l} \eta_1\Phi \equiv \Phi_t \ + \ (yK_y + K)\Phi_x \ - \\ - \ y^2(K_u + KK_v)\Phi_y \ + \ y^2K_y\Phi_u \ + \ y^2KK_y\Phi_v \ = \ 0 \\ \eta_2\Phi \equiv \Phi_t \ + \ K\Phi_x \ = \ 0 \end{array} \tag{13}$$

under initial data $u(0, x^0) = u^0$, $v(0, x^0) = v^0$, $y(0, x^0) = y^0$.

Proof of Theorem 3. Define the vector fields

$$\tilde{\eta}_1 = \eta_1 - \eta_2, \qquad \tilde{\eta}_2 = yK_yK^{-1}\eta_2 \quad (\tilde{\eta}_1, \tilde{\eta}_2 \in \theta(z)).$$

Then from the system $\tilde{\eta}_i\Phi = 0$ $(i = 1, 2)$ obtain:

$$\begin{array}{l} dx/dt = K, \quad dy/dt = -yKK_y^{-1}(K_u + KK_v), \\ du/dt = yK, \quad dv/dt = yK^2. \end{array} \tag{14}$$

Integrating the equations in (14) we get the existence of unique simple state, that is, $u = u(t, x)$, $v = v(t, x)$, satisfying the initial data; as well $y \equiv u_x = y(t, x)$. □

The following statement concerns the rate-type model (1).

Corollary 1. *Let* $a = h(v)$, $b \equiv 0$, $g = f(v)$, *then there exists a unique simple state (in explicit form)*

$$\begin{array}{l} a)\ t = \int\limits_{v^0}^{v} f^{-1}(s)[h(s) - D^0]ds, \quad b)\ u = Av + B, \\ c)\ x = \epsilon\sqrt{D^0}t + x^0, \quad d)\ y = f(v)[h(s) - D^0]^{-1}, \end{array} \tag{15}$$

where $D^0 = D^0(y^0, v^0)$, $A = \epsilon/\sqrt{D^0}$, $B = u^0 - \epsilon v^0/\sqrt{D^0}$, $\epsilon = \pm 1$.

Consider the equality (15) (a). There are two different cases:

Case (i). $h(s) > D^0 \in \mathbb{R}_+$, $\forall s \in [v^0, v]$. Then from the equation $dv/dt = D^0 f(v)[h(v) - D^0]^{-1} > 0$, it follows that v is a strictly monotone increasing function in t, i.e. $v > v^0 > 0$.

Case (ii). $h(s) < D^0 \in \mathbb{R}_+$, $\forall s \in [v, v^0]$. Here is a strictly monotone decreasing in t, i.e. $0 < v < v^0$.

Corollary 2. *There exists geometrical blow up (according to the terminology in [1]), i.e. the gradients ∇u and ∇v tend to infinity in finite time.*

Remark. Note that the integral in (15) (a), the right hand side, should be bounded from above by $T = -k_1/(a_1 z_0 \tanh k_1)$.

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