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A NEW EXISTENCE THEOREM FOR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS ON A MEASURE CHAIN

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Abstract: In this paper we consider the following differential equation on a measure chain T

$$u^{\Delta\Delta}(t) + f(u(\sigma(t))) = 0, t \in [a, b] \cap T,$$

satisfying Sturm-Liouville boundary value conditions

$$\alpha u(a) - \beta u^{\Delta}(a) = 0,$$

$$\gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) = 0.$$

An existence result is obtained by using a Fixed Point Theorem due to Krasnoselskii and Zabreiko. Our conditions imposed on f are very easy to verify and our result is even new for the special cases of differential equations and difference equations, as well as in the general time scale setting.

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1. Introduction

Since Hilger initial paper [9] unifying continuous and discrete calaulus, attention is being given to differential equations on measure chains (time scales). To facilitate this, the calculus on measure chains has been developed by Agarwal and Bohner [1], Aulbach and Hilger [4], and Erbe and Hilger [5].

Recently, Erbe and Peterson [7] discussed the following differential equation on a measure chain

$$u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, t \in [a, b], \tag{1}$$

satisfying Sturm-Liouville boundary value conditions

$$\begin{aligned} \alpha u(a) - \beta u^{\Delta}(a) &= 0, \\ \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) &= 0, \end{aligned} \tag{2}$$

and obtained the existence of one positive solution under the assumption that f is either superlinear or sublinear by using Krasnoselskii Fixed Point Theorem in a cone [8, 11].

In 2001, Agarwal and O'Regan [3] considered the following boundary value problem on a measure chain

$$u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad t \in [a, b], \tag{3}$$

$$u(a) = 0 = u^{\Delta}(\sigma(b)), \tag{4}$$

and established the existence results of one or two positive solutions by using nonlinear alternative of Leray-Schauder type [2] and Krasnoselskii Fixed Point Theorem in a cone [8, 11].

In this paper, motivated by [3, 7], we shall establish a new existence result for the following differential equation on a measure chain T

$$u^{\Delta\Delta}(t) + f(u(\sigma(t))) = 0, t \in [a, b] \cap T, \tag{5}$$

satisfying Sturm-Liouville boundary value conditions

$$\begin{aligned} \alpha u(a) - \beta u^{\Delta}(a) &= 0, \\ \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) &= 0, \end{aligned} \tag{6}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $r = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(b) - a) > 0$. Our conditions imposed on f are very easy to be verified and our result is even new for the special cases of differential equations and difference equations, as well as in the general time scale setting.

The rest of the paper is organized as follows. In Section 2, we present some definitions and notation which are common to the recent literature. We also provide some background results and state a Fixed Point Theorem which is crutial to our proof. Criteria for the existence of the boundary value problem (5) and (6) is established in Section 3.

2. Background Definitions and a Fixed Point Theorem

In this section, we present some background material with regard to measure chains. In addition to Hilger unifying work [9] on measure chain calculus, the papers by Agarwal and Bohner [1], Aulbach and Hilger [4], and Kaymakcalan et al. [10] provide good references to this material. Yet, our sources for the background material are the two papers by Erbe and Peterson [7, 6].

Throughout, let T be a nonempty closed subset of R, with $a, b \in T$ and $a \leq b$, and let T have the subspace topology inherited from the Euclidean topology on R.

Definition 2.1. For $t < \sup T$, define the forward jump operator $\sigma : T \to T$ by

$$\sigma(t) = \inf\{\tau \in T : \tau > t\} ,$$

and for $t > \inf T$, define the backward jump operator $\rho : T \to T$ by

$$\rho(t) = \sup\{\tau \in T : \tau < t\} ,$$

for all $t \in T$. Higher-order jumps are defined inductively by

$$\sigma^j(t) = \sigma(\sigma^{j-1}(t)) \text{ and } \rho^j(t) = \rho(\rho^{j-1}(t)),$$

$j \geq 2$. If $\sigma(t) > t, t \in T$, we say t is right-scattered. If $\rho(t) < t, t \in T$, we say t is left-scattered. If $\sigma(t) = t, t \in T$, we say t is right-dense. If $\rho(t) = t, t \in T$, we say t is left-dense.

Definition 2.2. If $r, s \in T \cup \{-\infty, +\infty\}, r < s$, then an open interval (r, s) in T is defined by

$$(r, s) = \{t \in T : r < t < s\}.$$

Other types of intervals are defined similarly.

Definition 2.3. Assume that $x : T \to R$ and fix $t \in T$ (if $t = \sup T$, we assume t is not left-scattered).

Then x is called differential at $t \in T$ if there exists a $\theta \in R$ such that for any given $\epsilon > 0$, there is an open neighborhood U of t such that

$$|x(\sigma(t)) - x(s) - \theta\,|\sigma(t) - s|| \leq \epsilon\,|\sigma(t) - s|\,, s \in U.$$

In this case, θ is called the *delta derivative* of x at $t \in T$ and denote it by $\theta = x^{\Delta}(t)$. The *second delta derivative* of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t)$. If $F^{\Delta}(t) = f(t)$, then define the *integral* by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Next, we state the following well-known Fixed Point Theorem due to Krasnoselskii and Zabreiko [12], which is crucial to our proof.

Lemma 2.1. *Let X be a Banach space and $F : X \longrightarrow X$ be completely continuous. If there exists a bounded and linear operator $A : X \longrightarrow X$ such that 1 is not a eigenvalue of A and*

$$\lim_{\|u\| \longrightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0,$$

Then F has a fixed point in X.

3. Existence Result

To obtain a solution of the boundary value problem (5) and (6), we require a mapping whose kernel $G(t, s)$ is the Green function for

$$-u^{\Delta\Delta}(t) = 0, t \in [a, b],$$

satisfying (6). Further, it is known [7] that

$$G(t,s)=\frac{1}{r}\begin{cases}\{\alpha(t-a)+\beta\}\{\gamma[\sigma(b)-\sigma(s)]+\delta\}, t\le s,\\ \{\alpha[\sigma(s)-a]+\beta\}\{\gamma[\sigma(b)-t]+\delta\}, \sigma(s)\le t,\end{cases}$$

then it is easy to know [7] that $G(t,s)$ is nonnegative on $[a,\sigma^2(b)]\times[a,b]$.

We now state our main result.

Theorem 3.1. *Assume that $f: R\longrightarrow R$ is continuous and*

$$\lim_{s\longrightarrow\infty}\frac{f(s)}{s}=m.$$

If

$$|m|<d=[\sup_{t\in[a,\sigma^2(b)]}\int_a^{\sigma(b)}G(t,s)\Delta s]^{-1},$$

then the boundary value problem (5) and (6) has a solution u^*, and $u^*\neq 0$ when $f(0)\neq 0$.

Proof. Let the Banach space $X=C[a,\sigma^2(b)]$ be endowed with the norm

$$\|u\|=\sup_{t\in[a,\sigma^2(b)]}|u(t)|.$$

Define integral operator $F: X\longrightarrow X$ by

$$(Fu)(t)=\int_a^{\sigma(b)}G(t,s)f(u(\sigma(s)))\Delta s, t\in[a,\sigma^2(b)],$$

then it is well-known that F is completely continuous and that solutions of the boundary value problem (5) and (6) are fixed points of the operator F and conversely.

In order to apply Lemma 2.1 to establish the existence result of the boundary value problem (5) and (6), we consider the following boundary value problem

$$u^{\Delta\Delta}(t)+mu(\sigma(t))=0,\quad t\in[a,b]\cap T, \tag{7}$$

$$\begin{aligned} \alpha u(a) - \beta u^{\Delta}(a) &= 0, \\ \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) &= 0. \end{aligned} \tag{8}$$

Define $A : X \longrightarrow X$ by

$$(Au)(t) = m \int_a^{\sigma(b)} G(t,s)u(\sigma(s))\Delta s, t \in [a, \sigma^2(b)],$$

then it is easy to know that A is a completely continuous (so bounded) linear operator and that solutions of the boundary value problem (7) and (8) are fixed points of the operator A and conversely.

First, we claim that 1 is not a eigenvalue of A.

In fact, if $m = 0$, then it is obvious that the boundary value problem (7) and (8) has no nontrival solution.

If $m \neq 0$ and the boundary value problem (7) and (8) has a nontrival solution u, then $\|u\| > 0$, and so

$$\begin{aligned} \|u\| = \|Au\| &= \sup_{t \in [a,\sigma^2(b)]} \left| m \int_a^{\sigma(b)} G(t,s)u(\sigma(s))\Delta s \right| \\ &= |m| \sup_{t \in [a,\sigma^2(b)]} \left| \int_a^{\sigma(b)} G(t,s)u(\sigma(s))\Delta s \right| \\ &\leq |m| \sup_{t \in [a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s) \left| u(\sigma(s)) \right| \Delta s \\ &\leq |m| \, \|u\| \sup_{t \in [a,\sigma^2(b)]} \int_a^{\sigma(b)} G(t,s)\Delta s \\ &< d \, \|u\| \frac{1}{d} = \|u\| , \end{aligned}$$

which is impossible. So, 1 is not a eigenvalue of A.

Next, we can prove that

$$\lim_{\|u\| \longrightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0.$$

In fact, for any $\varepsilon > 0$, since $\lim_{s\longrightarrow\infty}\frac{f(s)}{s} = m$, there must exist a number $M_1 > 0$ such that

$$|f(s) - ms| < \varepsilon\,|s|\,, |s| > M_1. \tag{9}$$

Let

$$M = \max_{|s|\le M_1} |f(s)|\,,$$

and choose $L > M_1$ such that

$$\frac{M + |m|\,M_1}{L} < \varepsilon,$$

then for any $u \in X$ and $\|u\| > L$:

(i) when $s \in [a, \sigma(b)]$ and $|u(\sigma(s))| \le M_1$, we have

$$\begin{aligned} &|f(u(\sigma(s))) - mu(\sigma(s))| \\ &\quad \le |f(u(\sigma(s)))| + |m|\,|u(\sigma(s))| \le M + |m|\,M_1 < \varepsilon L \\ &\quad < \varepsilon\,\|u\|\,; \end{aligned} \tag{10}$$

(ii) when $s \in [a, \sigma(b)]$ and $|u(\sigma(s))| > M_1$, from (9), we know that

$$|f(u(\sigma(s))) - mu(\sigma(s))| < \varepsilon\,|u(\sigma(s)| \le \varepsilon\,\|u\|\,. \tag{11}$$

So, we can conclude form (10) and (11) that

$$|f(u(\sigma(s))) - mu(\sigma(s))| \le \varepsilon\,\|u\|\,, \forall s \in [a, \sigma(b)]. \tag{12}$$

From (12), we have

$$\begin{aligned} &\|F(u) - A(u)\| \\ = &\sup_{t\in[a,\sigma^2(b)]}\left|\int_a^{\sigma(b)} G(t,s)[f(u(\sigma(s))) - mu(\sigma(s))]\Delta s\right| \\ \le &\sup_{t\in[a,\sigma^2(b)]}\int_a^{\sigma(b)} G(t,s)\,|f(u(\sigma(s))) - mu(\sigma(s))|\,\Delta s \\ \le &\ \varepsilon\,\|u\|\sup_{t\in[a,\sigma^2(b)]}\int_a^{\sigma(b)} G(t,s)\Delta s = \frac{\varepsilon}{d}\,\|u\|\ , \end{aligned}$$

i.e.,

$$\lim_{\|u\| \longrightarrow \infty} \frac{\|F(u) - A(u)\|}{\|u\|} = 0.$$

Then, it follows from Lemma 2.1 that F has a fixed point $u^* \in X$, i.e., u^* is a solution of the boundary value problem (5) and (6). It is obvious that $u^* \neq 0$ when $f(0) \neq 0$. □

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References

[1] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, **35** (1999), 3-22.

[2] R.P. Agarwal, D. O'Regan, A coupled system of difference equations, *Appl. Math. Comput.*, **114** (2000), 39-49.

[3] R.P. Agarwal, D. O'Regan, Nonlinear boundary value problems on time scales, *Nonlinear Anal.*, **44** (2001), 527-535.

[4] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, In: *Nonlinear Dynamics and Quantum Dynamical Systems, Mathematic Research*, Volume **59**, Akademie Verlag, Berlin (1990).

[5] L. Erbe, S. Hilger, Sturmian theory on measure chains, *Differential Equation and Dynamical Systems*, **1** (1993), 223-246.

[6] L. Erbe, A. Peterson, Green's functions and comparison theorems for differential equation on measure chains, *Dynamics Contin. Discrete Impuls. Systems*, **6** (1999), 121-137.

[7] L. Erbe, A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, *Math. Comput. Modelling.*, **32**, No. 5-6 (2000), 571-585.

[8] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego (1988).

[9] S. Hilger, Analysis on measure chains—A unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18-56.

[10] B. Kaymakcalan, V. Lakshmikanthan, S. Sivasundaram, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston (1996).

[11] M.A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen (1964).

[12] M.A. Krasnoselskii, P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Spriger-Verlag, New York (1984).