International Journal of Differential Equations and Applications

Volume 7 No. 3 2003, 311-324

LIE GROUP ANALYSIS OF THE NAVIER-STOKES EQUATIONS IN THE POLAR CO-ORDINATES

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Abstract: The equations of motion of two dimensional unsteady Navier-Stokes equations for viscous incompressible flow are written in polar co-ordinates. By employing Lie theory, the full one-parameter infinitesimal transformation group leaving the equations of motion invariant is derived along with its associated Lie algebra. Subgroups of the full group are then used to obtain a reduction by one in the number of independent variable in the system. These reductions are continued until a system of ordinary differential equations is reached. An exact and a series type approximate solutions of these ordinary differential equations are obtained which lead to an exact and a series type approximate solutions in $R^2 \setminus \{0\}$ to momentum equations.

AMS Subject Classification: 22E65, 35Q30, 22E70

Key Words: full Lie group, infinitesimal operators, Navier-Stokes equations

1. Introduction

The analysis and classification of differential equations through the realm of group theory is associated with the name of Sophus Lie [7]. Using the

Received: June 3, 2003

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Lie theory, the exact solutions of Navier-Stokes equations in cartesian co-ordinates have been discussed by many authors (see Puhnachev [10], Bitev [1], Lloyd [8] and Biosvert [3]). Biosvert in particular, derived new exact solutions of the Navier-Stokes equations. He has shown how each of these solutions generates an infinite number of time-dependent solutions via three or four arbitrary functions of time. But, as we have known, little work has been done for handling unsteady Navier-Stokes equations in polar co-ordinates. Mohanty and Jain [6] studied the Navier-Stokes equations in cylindrical polar co-ordinates and proposed the numerical solutions to the system. Recently, Mohanty [9] have found numerical solutions for unsteady Navier-Stokes equations in polar co-ordinates.

In applications, the solution of unsteady Navier-Stokes equations in polar co-ordinates are of great importance in the problem of viscous fluid flow. Unsteady Navier-Stokes equations in polar co-ordinates often arises in the mathematical modelling used to solve problems in fluid dynamics involving turbulence. Therefore, any new solutions would be of value in testing of numerical algorithms and may provide further insight into the types of solutions admitted by the system.

In this work, we consider the two dimensional unsteady Navier-Stokes equations for viscous incompressible flow in polar co-ordinates. Lie group theory [2] is applied to the equations of motion. Here, instead of using the determining equations to obtain the infinitesimals for the equations of motion, we use a direct approach to calculate the infinitesimals, i.e., by using the following lemma.

Lemma A. (see Ibragimov [5]) *Consider the partial differential operator of the first order*

$$Q = \xi^1(\mathbf{x})\frac{\partial}{\partial x^1} + ... + \xi^n(\mathbf{x})\frac{\partial}{\partial x^n}, \tag{1.1}$$

where $\mathbf{x} = (x^1, ..., x^n)$. *Let* $x'^i = \varphi(\mathbf{x}), i = 1, ..., n$. *Then the operator (1.1) is written in the new variables in the form*

$$\bar{Q} = Q(\varphi^1)\frac{\partial}{\partial x'^1} + ... + Q(\varphi^n)\frac{\partial}{\partial x'^n},$$

where

$$Q(\varphi^i) = \xi^1(\mathbf{x})\frac{\partial \varphi^i}{\partial x^1} + ... + \xi^n(\mathbf{x})\frac{\partial \varphi^i}{\partial x^n}.$$

With these infinitesimals we obtained a full one-parameter infinitesimal transformation group which leave the equations of motion invariant. Then, by using different subgroups of full group, two different types of solutions are constructed: for the first type, translation in t and θ co-ordinates are used in transforming the partial differential equations into ordinary differential equations. Exact analytic solutions are found for this case. For the second type, translation in t and r co-ordinates are used to transform the partial differential system into an ordinary differential system. Since the outcoming ordinary differential equations are more involved for this case, only a series type of approximate solution is constructed.

2. Equations of Motion and Associated Full Lie Group and Algebra

The two dimensional un-steady Naiver-Stokes equations of motion for viscous incompressible flow in polar co-ordinates can be written in component form as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}(\frac{\partial u}{\partial \theta})-\frac{v^2}{r} = -\frac{\partial p}{\partial r}+\eta(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}-\frac{u}{r^2}), \quad (2.1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}(\frac{\partial v}{\partial \theta})-v\frac{u}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{v}{r^2}), \quad (2.2)$$

where $r \neq 0$, u, v are the velocity components and η is (constant) viscosity. To equations (2.1) and (2.2) must be added the continuity equation

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{\partial \theta}=0. \quad (2.3)$$

Lie group theory will be applied to equations (2.1)-(2.3) in search of exact solutions. One requires that the equations remain invariant under the infinitesimal Lie point transformations

$$
\begin{aligned}
t^* &= t + \varepsilon\eta_1(t, r, \theta, u, v, p) + O(\varepsilon)^2, \\
r^* &= r + \varepsilon\eta_2(t, r, \theta, u, v, p) + O(\varepsilon)^2, \\
\theta^* &= \theta + \varepsilon\eta_3(t, r, \theta, u, v, p) + O(\varepsilon)^2, \\
u^* &= u + \varepsilon\eta_4(t, r, \theta, u, v, p) + O(\varepsilon)^2, \\
v^* &= v + \varepsilon\eta_5(t, r, \theta, u, v, p) + O(\varepsilon)^2, \\
p^* &= p + \varepsilon\eta_6(t, r, \theta, u, v, p) + O(\varepsilon)^2.
\end{aligned}
\tag{2.4}
$$

In order to calculate the infinitesimals η_1, η_2, η_3, η_4, η_5 and η_6 directly, we use Lemma A. The procedure is, as follows:

The operator for Naiver-Stokes equations in Cartesian coordinates is (see [5])

$$Q = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + P\frac{\partial}{\partial p}, \tag{2.5}$$

where the infinitesimals T, X, Y, U, V and P are given by

$$
\begin{aligned}
T &= \alpha + 2\beta t, \\
X &= \beta x - \gamma y + f(t), \\
Y &= \beta y - \gamma x + g(t), \\
U &= -\beta u_1 - \gamma v_1 + f'(t), \\
V &= -\beta v_1 + \gamma u_1 + g'(t), \\
P &= -2\beta p + j(t) - x f''(t) - y g''(t),
\end{aligned}
$$

where u_1, v_1 are the velocity components in cartesian coordinates and α, β and γ are arbitrary parameters and $f(t)$, $g(t)$, $j(t)$ are arbitrary sufficiently smooth functions of time.

For polar coordinates, we define the transformations as $t = t$, $r = \sqrt{(}x^2 + y^2)$, $\theta = \arctan(\frac{y}{x})$, $u = u_1\cos(\theta) + v_1\sin(\theta)$, $v = v_1\cos(\theta) - u_1\sin(\theta)$, $p = p$.

By using Lemma A, the operator (2.5) in the new variables becomes

$$\bar{Q} = Q(t)\frac{\partial}{\partial t} + Q(r)\frac{\partial}{\partial r} + Q(\theta)\frac{\partial}{\partial \theta} + Q(u)\frac{\partial}{\partial u} + Q(v)\frac{\partial}{\partial v} + Q(p)\frac{\partial}{\partial p}. \quad (2.6)$$

Here Q(t)=η_1, Q(r)=η_2, Q(θ)=η_3, Q(u)=η_4, Q(v)=η_5 and Q(p)=η_6 are our required infinitesimals. These infinitesimals are calculated by using Lemma A and have the following forms

$$\eta_1 = \alpha + 2\beta t, \quad (2.7)$$

$$\eta_2 = \beta r + f(t)\cos(\theta) + g(t)\sin(\theta), \quad (2.8)$$

$$\eta_3 = \frac{1}{r}(\gamma r - f(t)\sin(\theta) + g(t)\cos(\theta)), \quad (2.9)$$

$$\eta_4 = ((v/r)g(t) + f'(t))\cos(\theta) + (-\frac{v}{r}f(t) + g'(t))\sin(\theta) - \beta u, \quad (2.10)$$

$$\eta_5 = (-(u/r)g(t) + g'(t))\cos(\theta) + (\frac{u}{r}f(t) - f'(t))\sin(\theta) - \beta v, \quad (2.11)$$

$$\eta_6 = -2\beta p + j(t) - r\cos(\theta)f''(t) - r\sin(\theta)g''(t). \quad (2.12)$$

Hence the operator in (2.6) and the group in (2.4) with infinitesimals η_1, η_2, η_3, η_4, η_5 and η_6 given in equations (2.7)-(2.12) is our required operator and full one-parameter infinitesimal transformation group for equations of motion. The infinitesimal operator (generator of Lie algebra) associated with each parameter is obtained from the operator (2.6) by setting the studied parameter equal to one while other parameters and arbitrary smooth functions equated to zero. The infinitesimal operator associated with each of the arbitrary functions $f(t)$, $g(t)$, $j(t)$ is obtained by setting the other arbitrary functions and all parameters identically equal to zero.

Let G_i, $i = 1, 2, 3, 4, 5, 6$ denotes the infinitesimal operator associated with the parameters α, β, γ and functions $f(t)$, $g(t)$, $j(t)$ respectively, then we have

$$G_1 = \frac{\partial}{\partial t},$$

$$G_2 = 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p},$$

$$G_3 = \frac{\partial}{\partial \theta},$$

$$\begin{aligned}G_4(f) &= f(t)\cos(\theta)\frac{\partial}{\partial r} - (1/r)f(t)\sin(\theta)\frac{\partial}{\partial \theta}\\ &+(f'(t)\cos(\theta) - (v/r)f(t)\sin(\theta))\frac{\partial}{\partial u}\\ &+((u/r)f(t) - f'(t))\sin(\theta)\frac{\partial}{\partial v} - r\cos(\theta)f''(t)\frac{\partial}{\partial p},\end{aligned}$$

$$\begin{aligned}G_5(g) &= g(t)\sin(\theta)\frac{\partial}{\partial r} + (1/r)g(t)\cos(\theta)\frac{\partial}{\partial \theta}\\ &+(g'(t)\sin(\theta) + (u/r)g(t)\sin(\theta))\frac{\partial}{\partial u}\\ &+(-(u/r)g(t) + g'(t))\cos(\theta)\frac{\partial}{\partial v} - r\sin(\theta)g''(t)\frac{\partial}{\partial p},\end{aligned}$$

$$G_6(j) = j(t)\frac{\partial}{\partial p}.$$

These infinitesimal operators will generate a Lie algebra over the field of real or complex numbers. The commutator table of Lie algebra for (2.1)-(2.3) is given below, where the entry in the i-th row and j-th column is defined as $[G_i, G_j] = G_iG_j - G_jG_i$.

	G_1	G_2	G_3
G_1	0	$2G_1$	0
G_2	-$2G_1$	0	0
G_3	0	0	0
G_4	-$G_4(f')$	-$G_4(2tf' - f)$	$G_5(f)$
G_5	-$G_5(g')$	-$G_5(2tg' - g)$	-$G_4(g)$
G_6	-$G_6(j')$	-$2G_6(tj' + j)$	0

	G_4	G_5	G_6
G_1	$G_4(f')$	$G_5(g')$	$G_6(j')$
G_2	$G_4(2tf' - f)$	$G_5(2tg' - g)$	$2G_6(tj' + j)$
G_3	-$G_5(f)$	$G_4(g)$	0
G_4	0	0	0
G_5	0	0	0
G_6	0	0	0

3. Two-Dimensional Solutions

In this section, we first consider the subgroup of (2.7)-(2.12) with $\alpha = 1$ and $j(t)$ be any arbitrary smooth function while other parameters and arbitrary functions identically equal to zero. This subgroup, $\eta_1 = 1, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0, \eta_5 = 0$ and $\eta_6 = j(t)$, has the associated operator

$$\bar{Q} = \frac{\partial}{\partial t} + 0\frac{\partial}{\partial r} + 0\frac{\partial}{\partial \theta} + 0\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v} + j(t)\frac{\partial}{\partial p}.$$

The characteristic equations for finding the invariant transformations would then be

$$\frac{dt}{1} = \frac{dr}{0} = \frac{d\theta}{0} = \frac{du}{0} = \frac{dv}{0} = \frac{dp}{j(t)}.$$

The invariant variables and functions are

$$\begin{aligned} &r = \overline{r},\ \theta = \overline{\theta}, \\ &u = \overline{u}(\overline{r}, \overline{\theta}),\ v = \overline{v}(\overline{r}, \overline{\theta}),\ p = \overline{p}(\overline{r}, \overline{\theta}) + k(t), \end{aligned} \tag{3.1}$$

where

$$k(t) = \int j(t)dt.$$

One now substitutes the invariant variables and functions into the original equations of motion and obtains

$$\overline{u}\frac{\partial \overline{u}}{\partial \overline{r}} + \frac{\overline{v}}{\overline{r}}(\frac{\partial \overline{u}}{\partial \overline{\theta}}) - \frac{\overline{v^2}}{\overline{r}} = -\frac{\partial \overline{p}}{\partial \overline{r}} + \eta(\frac{\partial^2 \overline{u}}{\partial \overline{\overline{r}^2}} + \frac{1}{\overline{\overline{r^2}}}\frac{\partial^2 \overline{u}}{\partial \overline{\overline{\theta}^2}} + \frac{1}{\overline{r}}\frac{\partial \overline{u}}{\partial \overline{r}} - \frac{2}{\overline{\overline{r^2}}}\frac{\partial \overline{v}}{\partial \overline{\overline{\theta}}} - \frac{\overline{u}}{\overline{\overline{r^2}}}), \tag{3.2}$$

$$\overline{u}\frac{\partial \overline{v}}{\partial \overline{r}} + \frac{\overline{u}}{\overline{r}}(\frac{\partial \overline{v}}{\partial \overline{\overline{\theta}}}) - \overline{v}\frac{\overline{u}}{\overline{r}} = -\frac{1}{\overline{r}}\frac{\partial \overline{p}}{\partial \overline{\theta}} + \eta(\frac{\partial^2 \overline{v}}{\partial \overline{\overline{r}^2}} + \frac{1}{\overline{\overline{r^2}}}\frac{\partial^2 \overline{v}}{\partial \overline{\overline{\theta}^2}} + \frac{1}{\overline{r}}\frac{\partial \overline{v}}{\partial \overline{r}} + \frac{2}{\overline{\overline{r^2}}}\frac{\partial \overline{u}}{\partial \overline{\overline{\theta}}} - \frac{\overline{v}}{\overline{\overline{r^2}}}), \tag{3.3}$$

$$\frac{\partial \overline{u}}{\partial \overline{r}} + \frac{\overline{u}}{\overline{r}} + \frac{\partial \overline{v}}{\partial \overline{\overline{\theta}}} = 0. \tag{3.4}$$

We note that (3.2)-(3.4) are in steady-state. Therefore, by using (3.1), any steady-state solution of (3.2)-(3.4) can be transformed into a time-dependent solution involving an arbitrary function of time variable. Another interesting consequence of the transformation is that different subgroups of the reduced (time-independent) full group may now be used to transform (3.2)-(3.4) into a system of ordinary differential equations. Thus the solution of (2.1)-(2.3) is obtained by solving these ordinary differential equations and a process of back substitution. The infinitesimals corresponding to reduced (time-independent) full group for (3.2)-(3.4) are

$$\tilde{\eta_2} = \beta\overline{r}, \quad \tilde{\eta_3} = \gamma, \quad \tilde{\eta_4} = -\beta\overline{u}, \quad \tilde{\eta_5} = -\beta\overline{v}, \quad \tilde{\eta_6} = -2\beta\overline{p}. \tag{3.5}$$

3.1. Solution via Theta-Direction

Subgroup of (3.5) obtained by taking $\gamma = 1$ and other parameters equal to zero has the values

$$\tilde{\eta_2} = 0, \tilde{\eta_3} = 1, \tilde{\eta_4} = 0, \tilde{\eta_5} = 0, \tilde{\eta_6} = 0. \tag{3.6}$$

The characteristic equations associated with (3.6) are

$$\frac{d\overline{r}}{0} = \frac{d\overline{\theta}}{1} = \frac{d\overline{u}}{0} = \frac{d\overline{v}}{0} = \frac{d\overline{p}}{0}.$$

The invariant variables and functions are

$$\begin{aligned} &\overline{r} = \tau, \\ &\overline{u} = \hat{u}(\tau), \ \overline{v} = \hat{v}(\tau), \ \overline{p} = \hat{p}(\tau). \end{aligned} \tag{3.7}$$

In these new variables, (3.2)-(3.4) become the ordinary differential equations

$$\hat{u}\frac{d\hat{u}}{d\tau} - \frac{\hat{v^2}}{\tau} = -\frac{d\hat{p}}{d\tau} + \eta(\frac{d^2\hat{u}}{d\tau^2} + \frac{1}{\tau}\frac{d\hat{u}}{d\tau} - \frac{\hat{u}}{\tau^2}), \tag{3.8}$$

$$\hat{u}\frac{d\hat{v}}{d\tau} - \hat{v}\frac{\hat{u}}{\tau} = \eta(\frac{d^2\hat{v}}{d\tau^2} + \frac{1}{\tau}\frac{d\hat{v}}{d\tau} - \frac{\hat{v}}{\tau^2}), \tag{3.9}$$

$$\frac{d\hat{u}}{d\tau} + \frac{\hat{u}}{\tau} = 0. \tag{3.10}$$

Integrating (3.10), we have

$$\hat{u} = \frac{c}{\tau}, \tag{3.11}$$

where c is a constant of integration. Using (3.11) in (3.8) and (3.9), we obtain

$$\frac{d\hat{p}}{d\tau} = \frac{c^2}{\tau^3} + \frac{\hat{v^2}}{\tau}, \tag{3.12}$$

$$\frac{c}{\tau}\frac{d\hat{v}}{d\tau} - \frac{c}{\tau^2}\hat{v} = \eta(\frac{d^2\hat{v}}{d\tau^2} + \frac{1}{\tau}\frac{d\hat{v}}{d\tau} - \frac{\hat{v}}{\tau^2}). \tag{3.13}$$

By arranging (3.13), we have

$$\eta\tau^2\frac{d^2\hat{v}}{d\tau^2} + (\eta - c)\tau\frac{d\hat{v}}{d\tau} + (c-\eta)\hat{v} = 0. \tag{3.14}$$

The above equation is Cauchy-Euler equation. Its general solution is

$$\hat{v} = c_1\tau^{\frac{(c-\eta)}{\eta}} + c_2\tau, \tag{3.15}$$

where c_1, c_2 are arbitrary constants. After using (3.15) in (3.12) and then integrating, we obtain

$$\hat{p} = -\frac{c^2}{2\tau^2} + \frac{\eta c_1^2}{2(c-\eta)}\tau^{\frac{2(c-\eta)}{\eta}} + \frac{c_2^2}{2}\tau^2 + \frac{2\eta c_1 c_2}{c}\tau^{\frac{c}{\eta}} + c_3, \tag{3.16}$$

where c_3 is a constant of integration. Retracing our steps back to the original variables leads us to the time-dependent solution.

$$u = \frac{c}{r}, \tag{3.17}$$

$$v = c_1 r^{\frac{(c-\eta)}{\eta}} + c_2 r, \tag{3.18}$$

$$p = k(t) + \frac{c_2^2}{2}r^2 + \frac{\eta c_1^2}{2(c-\eta)}r^{\frac{2(c-\eta)}{\eta}} + \frac{2\eta c_1 c_2}{c}r^{\frac{c}{\eta}} - \frac{c^2}{2r^2} + c_3. \tag{3.19}$$

3.2. Solution via r-Direction

The subgroup of (3.5) in r-direction is obtained by taking $\beta = 1$ and $\gamma = 0$. For this subgroup the infinitesimals take the following form

$$\tilde{\eta_2} = \overline{r}, \tilde{\eta_3} = 0, \tilde{\eta_4} = -\overline{u}, \tilde{\eta_5} = -\overline{v}, \tilde{\eta_6} = -2\overline{p}.$$

The characteristic equations are

$$\frac{d\overline{r}}{\overline{r}} = \frac{d\overline{\theta}}{0} = \frac{d\overline{u}}{-\overline{u}} = \frac{d\overline{v}}{-\overline{v}} = \frac{d\overline{p}}{-2\overline{p}}.$$

The invariant variables and functions associated with this subgroup are

$$\begin{aligned} &\overline{\theta} = \xi, \\ &\overline{u} = \frac{1}{\overline{r}}\breve{u}(\xi),\ \overline{v} = \frac{1}{\overline{r}}\breve{v}(\xi),\ \overline{p} = \frac{1}{\overline{r}^2}\breve{p}(\xi). \end{aligned} \tag{3.20}$$

Substituting the new variables into the original system yields the following ordinary differential equation system

$$-\breve{u}^2 + \breve{v}\frac{d\breve{u}}{d\xi} - \breve{v}^2 = 2\breve{p} + \eta(\frac{d^2\breve{u}}{d\xi^2} - 2\frac{d\breve{v}}{d\xi}), \tag{3.21}$$

$$-2\breve{u}\breve{v} + \breve{u}\frac{d\breve{v}}{d\xi} = -\frac{d\breve{p}}{d\xi} + \eta(\frac{d^2\breve{v}}{d\xi^2} + 2\frac{d\breve{u}}{d\xi}), \tag{3.22}$$

$$\frac{1}{\overline{r}^2}\frac{d\breve{v}}{d\xi} = 0. \tag{3.23}$$

Integrating (3.23), we have

$$\breve{v} = m, \tag{3.24}$$

where m is a constant of integration. Using (3.24) in (3.21) and (3.22), we obtain

$$\eta\frac{d^2\breve{u}}{d\xi^2} - m\frac{d\breve{u}}{d\xi} + \breve{u}^2 + m^2 + 2\breve{p} = 0, \tag{3.25}$$

$$\frac{d\breve{p}}{d\xi} = 2m\breve{u} + 2\eta\frac{d\breve{u}}{d\xi}. \tag{3.26}$$

Since the system (3.25)-(3.26) is analytic. Therefore by using Theorem 8.1 [10, p. 34], the solutions of system (3.25)-(3.26) can be expressed into series in the form:

$$\begin{aligned} &\breve{u} = \Sigma_{k=0}^{\infty} a_k \xi^k, \\ &\breve{p} = \Sigma_{k=0}^{\infty} b_k \xi^k. \end{aligned} \tag{3.27}$$

Substituting the series into (3.26), one obtains the recursion relation

$$b_{k+1} = \frac{2}{k+1}(\eta(k+1)a_{k+1} + ma_k),\ \ k = 0, 1, 2\cdots . \qquad (3.28)$$

Similarly from (3.25), one obtains the sets of equations

$$2\eta a_2 - ma_1 + a_0^2 + m^2 + 2b_0 = 0\ (k = 0), \qquad (3.29)$$

$$6\eta a_3 - 2ma_2 + 2a_1 a_0 + 2b_1 = 0\ (k = 1), \qquad (3.30)$$

$$12\eta a_4 - 3ma_3 + (a_1^2 + 2a_0 a_2) + 2b_2 = 0\ (k = 2), \qquad (3.31)$$

and

$$\begin{aligned} &\eta(k+1)(k+2)a_{k+2} - m(k+1)a_{k+1} + \Sigma_{i+j=k} a_i a_j + 2b_k = 0 \\ &(k = 3, 4, 5, ...). \end{aligned} \qquad (3.32)$$

a_k and b_k coefficients can be determined by solving the above equations. From (3.29), we have

$$a_2 = \frac{ma_1 - a_0^2 - m^2 - 2b_0}{2\eta}. \qquad (3.33)$$

From (3.30), we have

$$a_3 = \frac{m(ma_1 - (a_0 + 4\eta)a_0 - m^2 - 2b_0) - 2\eta(a_0 + 2\eta)a_1}{6\eta^2}. \qquad (3.34)$$

Similarly by using (3.28)

$$b_1 = 2(\eta a_1 + ma_0), \qquad (3.35)$$

$$b_2 = 2ma_1 - a_0^2 - m^2 - 2b_0, \qquad (3.36)$$

$$b_3 = \frac{2}{3\eta}((m^2 - \eta a_0 - 2\eta^2)a_1 - m(a_0 + 2\eta)a_0 - m(m^2 + 2b_0)). \qquad (3.37)$$

By using the values of the coefficients in (3.37), we have

$$\breve{u} = a_0 + a_1\xi + \frac{1}{2\eta}(ma_1 - a_0^2 - m^2 - 2b_0)\xi^2 + \frac{1}{6\eta^2}(m(ma_1 - (a_0 + 4\eta)a_0 - m^2 - 2b_0) - 2\eta(a_0 + 2\eta)a_1)\xi^3 + \cdots, \tag{3.38}$$

$$\breve{p} = b_0 + 2(\eta a_1 + ma_0)\xi + (2ma_1 - a_0^2 - m^2 - 2b_0)\xi^2 + \frac{2}{3\eta}((m^2 - \eta a_0 - 2\eta^2)a_1 - m(a_0 + 2\eta)a_0 - m(m^2 + 2b_0))\xi^3 + \cdots .$$

In terms of original variables, (3.38) becomes

$$u = \frac{1}{r}a_0 + \frac{1}{r}a_1\theta + \frac{1}{2r\eta}(ma_1 - a_0^2 - m^2 - 2b_0)\theta^2 + \frac{1}{6r\eta^2}(m(ma_1 - (a_0 + 4\eta)a_0 - m^2 - 2b_0) - 2\eta(a_0 + 2\eta)a_1)\theta^3 + \cdots ,$$

$$v = m, \tag{3.39}$$

$$p = k(t) + \frac{1}{r^2}b_0 + \frac{2}{r^2}(\eta a_1 + ma_0)\theta + \frac{1}{r^2}(2ma_1 - a_0^2 - m^2 - 2b_0)\theta^2 + \frac{2}{3r^2\eta}((m^2 - \eta a_0 - 2\eta^2)a_1 - m(a_0 + 2\eta)a_0 - m(m^2 + 2b_0))\theta^3 + \cdots .$$

4. Concluding Remarks

We have derived the full one-parameter infinitesimal group which leaves the equation of continuity and equations of motion of unsteady Navier-Stokes equations for viscous incompressible flow invariant. Commutator table represents the Lie algebra associated with this group. Then by using different subgroups of the full group an exact and a series type approximated solutions in $R^2 \setminus \{0\}$ are constructed. In these solutions the velocity components u and v are independent of time whereas the generalized pressure p depends upon time.

Acknowledgement

We are very thankful to Jiang Jifa, Professor at University of Science and Technology of China and A.H. Kara, Professor at School of Mathematics, Wits University, P Bag 3, Wits 2050, South Africa for their valuable suggestion and careful reading of the manuscript.

The project supported by the NNSF of China (No. 10071080).

References

[1] V.O. Bitev, *Group Properties of the Equations of Navier-Stokes*, In Collection: Numerical Methods of the Mechanics of Fluids, Novosibirsk, Academy of Sciences of USSR, **3**, No. 5 (1972), 13-17, In Russian.

[2] G.W. Bluman, S. Kumei, *Symetries and Differential Equations*, Springer, New York (1989).

[3] R.E. Boisvert, *Group Analysis of the Navier-Stokes Equations*, Ph.D. Dissertation, Georgia Institute of Technology, Atlanta Georgia (1982).

[4] E.A. Coddington, S. Levison, *Theory of Ordinary Differential Equations*, Mc Graw-Hill, New York (1955).

[5] Nail H. Ibragimov, *Elementrary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons, New York, (1999).

[6] M.K. Jain, R.K. Jain, R.K. Mohanty, The numerical solutions of the two dimenssional unsteady Navier-Stokes equations using fourth difference method, *Internat. J. Comput. Math.*, **39** (1991), 125-134.

[7] S. Lie, Uber die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichungen, *Arch, für Math.*, **6**, No. 3 (1881), 328.

[8] S.P. Lloyd, The infinitesimal group of the Navier-Stokes equations, *Acta Mechanica*, **38** (1981), 85-98.

[9] R.K. Mohanty, High accuracy shemes for a class of three space dimensional singular parabolic equations with variable co-efficients,

Journel of Computational and Applied Mathematics, **89** (1997), 39-51.

[10] V.V. Puhnachev, Group properties of the equations of Navier-Stokes in the plane, *Journal of Appl. Mech. and Tech. Phys.*, **1** (1960), 83-90, In Russian.