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ON THE MINIMALITY AND TOTAL DEVELOPABILITY OF THE TIME-LIKE RULED SURFACES WITH THE TIME-LIKE GENERATING SPACE IN THE MINKOWSKI SPACE IR_1^n

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Abstract: The purpose of this paper is, first, to introduce a summary of known results and the definition of the time-like ruled surface with the time-like generating space in the Minkowski space IR_1^n (Section 1); second, to present some characteristic results related with minimality and total developability of the ruled surface in the Minkowski space IR_1^n (Section 2).

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1. Introduction

We will assume throughout that this paper that all manifolds, maps, vector fields, etc. are differentiable of class C^∞.

First of all, we give some properties of a general submanifold M of the Minkowski $n-$space IR_1^n, [2]. Let $\overline{D}$ be a Levi-Civita connection of IR_1^n and D be a Levi-Civita connection of M. If $X, Y \in \chi(M)$ and V is the second fundamental tensor of M, we have by decomposing $\overline{D}_X Y$

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in a tangential and a normal component

$$\overline{D}_X Y = D_X Y + V(X, Y) \tag{1.1}$$

The equation (1.1) is called *Gauss Equation.*

If ζ is any normal vector field on M, we find the *Weingarten Equation* by decomposing $\overline{D}_X \zeta$ in a tangential component and a normal component as

$$\overline{D}_X \zeta = -A_\zeta(X) + D^{\perp}_X \zeta. \tag{1.2}$$

A_ζ determines at each point a self-adjoint linear map and $D^{\perp}$ is a metric connection in the normal bundle $\chi^{\perp}(M)$. We note that, in this paper, A_ζ will be used for the linear map and the corresponding matrix of the linear map.

If the metric tensor of IR^n_1 is denoted by $<,>$, from the equation (1.1) and (1.2), it follows

$$< V(X, Y), \zeta >=< A_\zeta(X), Y > . \tag{1.3}$$

If $\zeta_1, \zeta_2, ..., \zeta_{n-m}$ constitute an orthonormal basis of $\chi^{\perp}(M)$, then we set

$$V(X, Y) = \sum_{j=1}^{n-m} < A_\zeta(X), Y > \zeta_j. \tag{1.4}$$

The *mean curvature* H of M at the point P is given by

$$H = \sum_{j=1}^{n-m} \frac{\mathrm{tr} A_{\zeta_j}}{\dim M} \zeta_j. \tag{1.5}$$

For every $X_i \in \chi(M),\ 1 \leq i \leq 4$ the 4^{th} order covarient tensor field defined by R as

$$R(X_1, X_2, X_3, X_4) =< X_1, R(X_3, X_4)X_2 >$$

is called the *Riemannian curvature tensor field* and its value at a point $P \in M$ is called *Riemannian curvature* of M at the point P.

If V is the second fundamental tensor, then we have

$$< Y, R(X,Y)X >=< V(X,X), V(Y,Y) > - < V(X,Y), V(X,Y) > . \tag{1.6}$$

Let $\prod$ be a tangent plane of M at P. For all $X_P, Y_P \in \prod$, the real function K defined by

$$K(X_P, Y_P) = \frac{< R(X_P, Y_P)X_P, Y_P >}{< X_P, X_P >< Y_P, Y_P > - < X_P, Y_P >^2} \tag{1.7}$$

is called the *sectional curvature function*. $K(X_P, Y_P)$ is called the *sectional curvature* of M at P.

Let R be the Riemann curvature tensor and $\{e_1, e_2, ..., e_m\}$ be a system of orthonormal basis of $T_M(P)$. The tensor field S, defined in the form

$$S(X,Y) = \sum_{i=1}^{m} \varepsilon_i < R(X, e_i)Y, e_i > , \tag{1.8}$$

is called the *Ricci curvature tensor field* and the value of $S(X,Y)$ at $P \in M$ is called the *Ricci curvature*, where

$$\varepsilon_i =< e_i, e_i >, \varepsilon_i = \begin{cases} -1 & \text{if } e_i \text{ time-like}, \\ 1 & \text{if } e_i \text{ space-like}. \end{cases} .$$

The real number r_{sk}, defined in the form

$$r_{sk} = \sum_{i \neq j} K(e_i, e_j) = 2 \sum_{i<j} K(e_i, e_j), \tag{1.9}$$

is called the *scalar curvature tensor field* of M.

Let V be the second fundamental tensor of M. If

$$V(X,X) = 0 , \tag{1.10}$$

for $X \in \chi(M)$, then X called *asymptotic vector field* on M. If

$$V(X,Y) = 0 , \tag{1.11}$$

for all $X, Y \in \chi(M)$, then M is *totally geodesic.*

Let M be a $(k+1)-$dimensional ruled surface in IR_1^n. Then M can be locally represented by

$$\phi(s, u_1, u_2, ..., u_k) = \alpha(s) + \sum_{i=1}^{k} u_i e_i(s), \quad u_i \in IR,\ 1 \leq i \leq k. \tag{1.12}$$

If the generating space $E_k(s) = sp\{e_1, e_2, ..., e_k\}$ of M will be assumed a *time-like subspace* and *the base curve α is space-like*, then this surface is called *the* $(k+1)-$*dimensional time-like ruled surface* in IR_1^n, [1].

If

$$\operatorname{rank}[e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, ..., \overline{D}_{e_0} e_k] = 2k - m\,, \tag{1.13}$$

at each point P of M, then M will be called as $m-$*developable.* If $m = -1$, then the time-like generalized ruled surface M is called as *non-developable.* If $m = k-1$, M is called as *total developable*, where e_0 is the tangent vector of the base curve.

Suppose that $\{e_0, e_1, ..., e_k\}$ is an orthonormal base field of the tangential bundle $\chi(M)$ and $\{\zeta_1, \zeta_2, ..., \zeta_{n-k-1}\}$ an orthonormal base field of the normal bundle $\chi^{\perp}(M)$. Then an orthonormal base field of $\chi(IR_1^n)$ is $\{e_0, e_1, ..., e_k, \zeta_1, ..., \zeta_{n-k-1}\}$.

If we write the Weingarten derivative equation for this base vectors we have,

$$\overline{D}_{e_i} \zeta_j = -A_{\zeta_j}(e_i) + D^{\perp}_{e_i} \zeta_j\,, \tag{1.14}$$

or

$$\begin{aligned} \overline{D}_{e_0} \zeta_j &= a^j_{00} e_0 + \sum_{r=1}^{k} a^j_{0r} e_r + \sum_{s=1}^{n-k-1} b^j_{0s} \zeta_s, \\ &\quad 1 \leq j \leq n-k-1\,, \\ \overline{D}_{e_i} \zeta_j &= a^j_{i0} e_0 + \sum_{r=1}^{k} a^j_{ir} e_r + \sum_{s=1}^{n-k-1} b^j_{is} \zeta_s, \quad 1 \leq i \leq k\,. \end{aligned} \tag{1.15}$$

From the above derivative equation we have

$$A_{\zeta_j} = - \begin{bmatrix} a_{00}^j & a_{01}^j & \cdots & a_{0k}^j \\ \varepsilon_1 a_{01}^j & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_k a_{0k}^j & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} . \tag{1.16}$$

The Riemann curvature of the 2-dimensional section of M spanned by the vectors $(e_i)|_P,\ \ 1 \le i \le k$ and $(e_0)|_P$ can be given by

$$K(e_i, e_0) = -\varepsilon_i < \overline{D}_{e_i} e_0, \overline{D}_{e_i} e_0 >= \sum_{j=1}^{n-k-1} \varepsilon_i (a_{0i}^j)^2. \tag{1.17}$$

The mean curvature of M is

$$H = \frac{1}{k+1} V(e_0, e_0). \tag{1.18}$$

2. On the Minimality and Total Developability of the Time-Like Ruled Surfaces with the Time-Like Generating Space in the Minkowski Space

Theorem 1. *Let M be a $(k+1)-$dimensional time-like ruled surface and $\{e_1, e_2, ..., e_k\}$ be an orthonormal base field of the time-like generating space $E_k(s)$. The lines, which correspond to $\{e_1, e_2, ..., e_k\}$ are asymptotics and geodesics of M.*

Proof. Since the lines, corresponding to the orthonormal base field vectors $\{e_1, e_2, ..., e_k\}$ of the time-like generating space $E_k(s)$ are geodesics of IR_1^n, we have

$$\overline{D}_{e_i} e_i = 0,\ 1 \le i \le k.$$

From (1.1) we have

$$D_{e_i} e_i = -V(e_i, e_i)\,.$$

Since $D_{e_i} e_i \in \chi(M)$ and $V(e_i, e_i) \in \chi^{\perp}(M)$, we find

$$D_{e_i} e_i = 0, \quad V(e_i, e_i) = 0\,.$$

Therefore the lines, corresponding to $\{e_1, e_2, ..., e_k\}$ are asymptotics and geodesics of M. □

Theorem 2. *M is total developable iff $\overline{D}_{e_i} e_0 = 0, \quad 1 \leq i \leq k$.*

Proof. Let $\{e_0, e_1, ..., e_k\}$ be an orthonormal basis of M and M be total developable. Since the system $\{e_0, e_1, ..., e_k\}$ is linearly independent, $\overline{D}_{e_i} e_0$ has no component in the normal bundle $\chi^{\perp}(M)$, that is $V(e_i, e_0) = 0$.

We know that

$$\overline{D}_{e_0} e_i = V(e_0, e_i)\,. \tag{2.1}$$

Since V is symmetric, from (2.1) we have

$$\overline{D}_{e_i} e_0 = 0.$$

Conversly, assume that $\overline{D}_{e_i} e_0 = 0, \ 1 \leq i \leq k$. From the Gauss equation and (2.1) we have $V(e_i, e_0) = 0$. If we set this in the Gauss equation, we find

$$\overline{D}_{e_0} e_i = D_{e_0} e_i.$$

Therefore,

$$\overline{D}_{e_0} e_i \in sp\{e_0, e_1, ..., e_k\}$$

Thus we observe that

$$\operatorname{rank}\,[e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, \overline{D}_{e_0} e_2, ..., \overline{D}_{e_0} e_k] = k + 1.$$ □

Theorem 3. *M is total developable and minimal iff M is totally geodesic.*

Proof. If $X, Y \in \chi(M)$, we have

$$X = \sum_{i=1}^{k} a_i e_i + a e_0, \quad Y = \sum_{j=1}^{k} b_j e_j + b e_0\,.$$

Therefore we find

$$V(X, Y) = \sum_{i=1}^{k} (a_i b + b_i a) V(e_0, e_i) + ab V(e_0, e_0) + \sum_{i,j=1}^{k} a_i b_j V(e_i, e_j)\,.$$

Since $V(e_i, e_j) = 0$ and M is minimal and total developable we have

$$V(X,Y) = 0 \quad \text{for all} \quad X, Y \in \chi(M).$$

Conversly, let $V(X,Y) = 0$, for all $X, Y \in \chi(M)$. Then we have the following relations:

$$V(e_0, e_i) = 0, \quad V(e_0, e_0) = 0, \quad \text{and} \quad V(e_i, e_j) = 0, \quad 1 \leq i, j \leq k.$$

By using these equations and (2.1) we find $\overline{D}_{e_i} e_0 = 0$, $1 \leq i \leq k$, and so, M is total developable. Moreover, $V(e_0, e_0) = 0$ implies that $H = 0$. Therefore M is minimal. □

Let $\{e_0, e_1, ..., e_k\}$ an orthonormal basis of $\chi(M)$ and $\{\zeta_1, \zeta_2, ..., \zeta_{n-k-1}\}$ an orthonormal basis of $\chi^{\perp}(M)$. Moreover, we can give covariant derivative equations of the orthonormal basis $\{e_0, e_1, ..., e_k, \zeta_1, ..., \zeta_{n-k-1}\}$ of $\chi(IR_1^n)$, as follows:

$$\overline{D}_{e_0} e_r = \sum_{i=0}^{k} c_{ri} e_i + \sum_{m=1}^{n-k-1} c_{r(k+m)} \zeta_m, \quad 0 \leq r \leq k\,,$$

$$\overline{D}_{e_0} \zeta_j = \sum_{i=0}^{k} c_{(k+j)i} e_i + \sum_{m=1}^{n-k-1} c_{(k+j)(k+m)} \zeta_m, \qquad (2.2)$$
$$1 \leq j \leq n-k-1\,.$$

If we calculate the coefficient c_{st}, $0 \leq s, t \leq n-1$, and write the equation (2.2) in the matrix form we obtain:

$$\begin{bmatrix} \overline{D}_{e_0} e_0 \\ \overline{D}_{e_0} e_1 \\ \vdots \\ \overline{D}_{e_0} e_k \\ \overline{D}_{e_0} \zeta_1 \\ \vdots \\ \overline{D}_{e_0} \zeta_{n-k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & c_{01} & \cdots & c_{0k} & c_{0(k+1)} & \cdots & c_{0(n-1)} \\ -\varepsilon_1 c_{01} & 0 & \cdots & c_{1k} & c_{1(k+1)} & \cdots & c_{1(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_k c_{0k} & -\varepsilon_1\varepsilon_k c_{1k} & \cdots & 0 & c_{k(k+1)} & \cdots & c_{k(n-1)} \\ -c_{0(k+1)} & -\varepsilon_1 c_{1(k+1)} & \cdots & -\varepsilon_k c_{k(k+1)} & 0 & \cdots & c_{(k+1)(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{0(n-1)} & -\varepsilon_1 c_{1(n-1)} & \cdots & -\varepsilon_k c_{k(n-1)} & -c_{(k+1)(n-1)} & \cdots & 0 \end{bmatrix} \times \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_k \\ \zeta_1 \\ \vdots \\ \zeta_{n-k-1} \end{bmatrix}. \quad (2.3)$$

By using the equation (2.3) we can give the following theorem.

Theorem 4. *Let M be a $(k+1)-$dimensional time-like ruled surface in IR^n_1, $\{e_1, e_2, ..., e_k\}$ be an orthonormal base field of the time-like generating space $E_k(s)$ and let the base curve $\alpha(s)$ be an orthonormal trajectory of $E_k(s)$. Then the following propositions are equivalent:*

(i) M is total developable,

(ii) The Riemannian curvature $K(e_i, e_0)$ of M is zero, $1 \le i \le k$,

(iii) In the equation (2.3) $c_{rs} = 0, \quad 1 \le r \le k, \quad k+1 \le s \le n-1$,

(iv) $A_{\zeta_j}(e_i) = 0, \quad 1 \le i \le k, \quad 1 \le j \le n-k-1$,

(v) $\overline{D}_{e_0} e_i \in \chi(M)$.

Proof. (i) $\Rightarrow$ (ii): We assume that M is total developable. Then by the Theorem 2 and the equation (1.17) we find $K(e_i, e_0) = 0, \quad 1 \le i \le k$.

(ii) $\Rightarrow$ (iii): Let $K(e_i, e_0) = 0$. From (1.15) and (1.17) we find

$$< \overline{D}_{e_0} \zeta_j, e_i >= 0, \quad 1 \le i \le k, \quad 1 \le j \le n-k-1.$$

This equation shows that $\overline{D}_{e_0}\zeta_j$ has no component in the directions of $\{e_1, e_2, ..., e_k\}$. Hence we have

$$c_{rs} = 0, \quad 1 \leq r \leq k, \quad k+1 \leq s \leq n-1,$$

in the equation (2.3).

(iii) $\Rightarrow$ (iv): Now, we assume that $c_{rs} = 0$. Then from (2.2) we obtain

$$< \overline{D}_{e_0}\zeta_j, e_i >= -\varepsilon_i c_{is} = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n-k-1.$$

Therefore, from (1.17),

$$< \overline{D}_{e_0}\zeta_j, e_i >= \varepsilon_i a^j_{0i}\,,$$

then

$$a^j_{0i} = 0.$$

We know that from (1.17)

$$< \overline{D}_{e_i}\zeta_j, e_r >= 0.$$

Then, from last two equations, we obtain

$$A_{\zeta_j}(e_i) = 0.$$

(iv) $\Rightarrow$ (v): Let $A_{\zeta_j}(e_i) = 0$. Then from (1.17) we have

$$a^j_{0i} = 0\,,$$

and $\overline{D}_{e_0}\zeta_j$ has no component in the directions of $\{e_1, e_2, ..., e_k\}$, i.e.

$$c_{rs} = 0, \quad 1 \leq r \leq k, \quad k+1 \leq s \leq n-1.$$

Then from (2.3) we have

$$< \overline{D}_{e_0}\zeta_j, e_i >= 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq n-k-1.$$

Since

$$< \overline{D}_{e_0}\zeta_j, e_i >= - < \overline{D}_{e_0}e_i, \zeta_j >= 0, \text{ then } \overline{D}_{e_0}e_i \in \chi(M).$$

(v) ⇒ (i): Let $\overline{D}_{e_0} e_i \in \chi(M)$. This means that

$$\overline{D}_{e_0} e_i \in \mathrm{sp}\,\{e_0, e_1, ..., e_k\}.$$

Therefore,

$$\mathrm{rank}\,[e_0, e_1, ..., e_k, \overline{D}_{e_0} e_1, ..., \overline{D}_{e_0} e_k] = k + 1\,.$$

This means that M is total developable. □

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