International Journal of Differential Equations and Applications
Volume 12 No. 4 2013, 185-195
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v12i4.1345

acadpubl.eu

A SIMPLE ANALYTICAL TECHNIQUE TO SOLVE HIGHER-ORDER APPROXIMATIONS OF A NONLINEAR OSCILLATOR WITH DISCONTINUITIES

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Abstract: In this paper, a simple analytical technique has been developed to determine higher-order approximate periodic solutions of a nonlinear oscillator with discontinuities for which the elastic force term is proportional to $sgn(x)$. The classical harmonic balance method cannot be applied directly for such nonlinear problems. It is very difficult to solve nonlinear problems and in general, it is often more difficult to get an analytic approximation than a numerical one for such nonlinear problems. Analytical solutions of algebraic equations are not always possible, especially in the case of a large oscillation. In this article different parameters of the same nonlinear problems are found, for which the simple analytical solution produces desired results even for the large oscillation. We have been found out that a simple analytical principle which works very well for the excellent agreement of the approximate frequencies and period with the exact ones.

AMS Subject Classification: 34A34, 34A45
Key Words: nonlinear oscillator, higher-order approximate periodic solutions

1. Introduction

Many complex real world problems in nature are due to nonlinear phenomena.

Received: March 27, 2013

Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of approximate solutions of nonlinear differential equations (NDEs) play a crucial role to understand the internal mechanism of nonlinear phenomena. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving differential equations, dynamical systems and related areas. In recent years, both the mathematicians and physicists have made significant improvement in finding a new mathematical tool would be related to nonlinear differential equations and dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. They establish many effective and powerful methods to handle the NDEs.

The study of given nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essential nonlinear and are described by nonlinear equations. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. There are many analytical approaches to solve nonlinear differential equations. One of the widely used techniques is perturbation [1]-[4], whereby the solution is expanded in powers of a small parameter. However, for the nonlinear conservative systems, generalizations of some of the standard perturbation techniques overcome this limitation. In particular, generalization of LP method and He's homotopy perturbation method yield desired results for strongly nonlinear oscillators [5]-[12].

The harmonic balance method (HBM) [13]-[22] is another technique for solving strongly nonlinear systems. Usually, a set of difficult nonlinear algebraic equations appears when HBM is formulated. In article [22], such nonlinear algebraic equations are solved in powers of a small parameter. The solutions derived (in [22]) for *Duffing* equation agree with numerical solutions when $[x(0) = a_0,\ \dot{x}(0) = 0]$, $a_0 = \mathrm{O}(1)$. Sometimes, higher approximations also fail to measure the desired results when $a_0 >> 1$. In this article this limitation is removed. Approximate solutions of the same equations are found in which the nonlinear algebraic equations are solved by a new parameter. The higher order approximations (mainly third approximation) have been obtained for the mentioned nonlinear oscillator.

2. The Method

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad [x(0) = a_0,\ \dot{x}(0) = 0], \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, $\omega_0 \geq 0$ and ε is a constant.

Consider a periodic solution of Eq. (1) is in the form

$$x = a_0(\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5\omega t) + w \cos(7\omega t) + z \cos(9\omega t) \cdots), \tag{2}$$

where a_0, ρ and ω^2 are constants. If $\rho = 1-u-v-\cdots$ and the initial phase $\varphi_0 = 0$, solution Eq. (2) readily satisfies the initial conditions $[x(0) = a_0,\ \dot{x}(0) = 0]$.

Substituting Eq. (2) into Eq. (1) and expanding $f(x, \dot{x})$ in a Fourier series, it converts to an algebraic identity

$$a_0[\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots] = -\varepsilon[F_1(a_0, u, \cdots)\cos(\omega t) + F_3(a_0, u, \cdots)\cos(3\omega t) + \cdots] \tag{3}$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \tag{4}$$

With help of the first equation, ω is eliminated from all the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1), \quad 24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \cdots \tag{5}$$

Substitution $\rho = 1-u-v-\cdots$, and simplification, second-, third- equations of Eq. (5) take the following form

$$u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \cdots, \tag{6}$$

where $G_1,\ G_2,\ \cdots$ exclude respectively the linear terms of $u,\ v,\ \cdots$.

Whatever the values of ω_0 and a_0, there exists a parameter $\mu_0(\omega_0, \varepsilon, a_0) << 1$, such that $u,\ v,\ \cdots$ are expandable in following power series in terms of λ_0 as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where $U_1,\ U_2, \cdots, V_1,\ V_2, \cdots$ are constants.

Finally, substituting the values of $u,\ v, \cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. (2).

3. Examples

3.1. Nonlinear Oscillator with Discontinuities

Let us consider an anti-symmetric, piecewise constant force oscillator, is governed by the following differential equation which has been considered by Belendez *et al.*[25]

$$\ddot{x} + sgn(x) = 0. \tag{8}$$

with initial conditions

$$x(0) = a_0 and \dot{x}(0) = 0 \tag{9}$$

and $sgn(x)$ is defined as

$$\begin{aligned} sgn(x) &= -1, x < 0 \\ sgn(x) &= 1, x > 0 \end{aligned} \tag{10}$$

Eq. (8) models the motion of a punctual ball rolling in a "V" shape trough in a constant gravitational field. The arms of the "V" make equal angles with horizontal plane and the origin of the (horizontal) x coordinate is taken to the point of interaction of the two arms Mickens, R.E., [13]. In a suitable set if units, the equation of motion can be written as Eq. (8).

All the solutions of Eq. (8) are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(a_0)$, i.e. the functional behavior of ω as a function of the initial amplitude a_0.

Firstly, we consider a first-order approximate solution of Eq. (8) is

$$x = a_0 \cos\varphi \tag{11}$$

Where $\varphi_1 = \omega_1 t$ and ω_1 represent the angular frequency. Let us consider $sgn(x) = 1$ can be expanded in a Fourier series as

$$sgn(x) = \sum_{n=0}^{\infty} b_{2n+1} sgn(x) = b_1 \cos\varphi + b_3 \cos 3\varphi + \cdots . \tag{12}$$

Herein $b_1, b_3, \cdots$ are evaluated as

$$b_{2n+1} = \frac{4}{\pi} \int_0^{\pi/2} sgn(x) \cos[(2n+1)\varphi] d\varphi \tag{13}$$

From Eq. (13) we obtained the following coefficient is

$$b_1 = \frac{4}{\pi} \tag{14}$$

Now using Eq. (12)-(14) into Eq. (8) and then equating the coefficient of $\cos\varphi_1$ we obtained the following equation is

$$-a_0\omega_1^2 + \frac{4}{\pi} = 0 \tag{15}$$

Here, we can easily found the first-order approximate frequency and period are

$$\dot{\varphi} = \sqrt{\frac{4}{a_0\pi}} = \frac{1.128379}{\sqrt{a_0}} \tag{16}$$

$$T_1 = \frac{2\pi\sqrt{a_0}}{1.128379} = 5.568328\sqrt{a_0} \tag{17}$$

We use the solution of the form of Eq. (2) a second-order approximate solution of Eq. (8) is

$$x(t) = a_0(\cos\varphi_2 + u(\cos 3\varphi_2 - \cos\varphi_2), \tag{18}$$

where $\varphi_2 = \omega_2 t$. Herein $b_1, b_3, \cdots$ are evaluated as

$$\begin{aligned} b_1 &= \tfrac{4}{\pi}\ , \\ b_3 &= -\tfrac{4}{3\pi}\ , \end{aligned} \tag{19}$$

and so on.

Now substituting Eq. (18) into the Eq. (8) along with using Eq. (19) and Eq. (12) and then equating the coefficients of $\cos\varphi_2$ and $\cos 3\varphi_2$, the following equations are obtained

$$-(1-u)a_0\omega_2^2 + \frac{4}{\pi} = 0, \tag{20}$$

$$-9a_0u\omega_2^2 - \frac{4}{3\pi} = 0. \tag{21}$$

After simplification, Eq. (20) takes the form

$$\omega_2^2 = \left(\frac{4}{\pi}\right)/a_0(1-u) \tag{22}$$

By elimination of ω_2^2 from Eq. (21) with the help of Eq. (22), the equation of u is obtained as

$$u = -\lambda_0, where \lambda_0 = \frac{4}{104}. \tag{23}$$

Substituting the value of u from Eq. (23) into the Eq. (22) we have got the second-order approximate angular frequency and period are

$$\omega_2 = \sqrt{\left(\frac{4}{\pi}\right) / a_0(1-u)} = \frac{1.10729}{\sqrt{a_0}} \tag{24}$$

$$T_2 = \frac{2\pi\sqrt{a_0}}{1.10729} = 5.67438\sqrt{a_0} \tag{25}$$

In a similar way, the method can be used to determine higher order approximations. In this article, a third approximate solution is of the form,

$$x(t) = a_0 \cos\varphi_3 + a_0 u(\cos 3\varphi_3 - \cos\varphi_3) + a_0 v(\cos 5\varphi_3 - \cos\varphi_3), \tag{26}$$

where $\varphi_3 = \omega_3 t$. Substituting Eq. (26) into the Eq. (8) and also using Eq. (12)-(13) and then setting the coefficients of $\cos\varphi_3$, $\cos 3\varphi_3$, and $\cos 5\varphi_3$ the related functions are obtained from the following equations

$$-(1-u-v)a_0\omega_3^2 + \frac{4}{\pi} = 0, \tag{27}$$

$$-9ua_0\omega_3^2 - \frac{4}{3\pi} = 0, \tag{28}$$

$$-25va_0\omega_3^2 + \frac{4}{5\pi} = 0. \tag{29}$$

From the Eq. (27) we can easily written as

$$\omega_3^2 = \left(\frac{4}{\pi}\right) / a_0(1-u-v) \tag{30}$$

Now using Eq. (30) into the Eq. (28)-(29) we get the equation of u and v are

$$u = \lambda_0\left(-1+v\right), \tag{31}$$

$$v = \mu_0(1-u), \tag{32}$$

where λ_0 is define as Eq. (23) and $\mu_0 = \frac{4}{504}$.
The algebraic relation between λ_0 and μ_0 is

$$\mu_0 = 104\lambda_0/504. \tag{33}$$

Substituting this value of μ_0 from the Eq. (33) into the Eq. (32), and then solved Eq. (31) and Eq. (32) in powers of λ_0 is

$$u = \frac{-63\lambda_0 + 13\lambda_0^2}{63 + 13\lambda_0^2} \tag{34}$$

$$v = \frac{13\lambda_0 + 13\lambda_0^2}{63 + 13\lambda_0^2} \tag{35}$$

Substituting the values of u and v from Eq. (34)-(35) into Eq. (30), we get the third-order approximate angular frequency and period is

$$\omega_3 = \sqrt{\left(\frac{4}{\pi}\right) / a_0(1 - u - v)} = \frac{1.111876}{\sqrt{a_0}} \tag{36}$$

$$T_3 = \frac{2\pi\sqrt{a_0}}{1.111876} = 5.65098\sqrt{a_0} \tag{37}$$

4. Results and Discussions

We illustrate the accuracy of a simple analytical method by comparing the approximate angular frequencies and periods previously obtained with the exact frequency ω_e and period T_e. For this nonlinear problem, the exact angular frequency and period are

$$\omega_e(a_0) = \frac{1.110721}{\sqrt{a_0}}$$

$$T_e(a_0) = 5.656854\sqrt{a_0}$$

as stated by Wu *et al.*[20].

The periodic values and their relatives errors (RE) obtained in this paper by applying a simple analytical technique of mentioned nonlinear oscillator with discontinuities are the following

$$T_1(a_0) = 5.568328\sqrt{a_0} RE = 1.6\%$$

$$T_2(a_0) = 5.67438\sqrt{a_0} RE = 0.30\%$$

$$T_3(a_0) = 5.65098\sqrt{a_0} RE = 0.10\%$$

Where the percentage errors (RE) were calculated using the following equation

$$RE = 100 \times \left| \frac{T_i(a_0) - T_e(a_0)}{T_e(a_0)} \right| i = 1, 2, 3.$$

Belendez *et al.*[25] approximately solved Eq. (8) using He's homotopy perturbation method (HPM). They achieved the following results for the first and second and third approximation orders

$$T_1(a_0) = 5.568328\sqrt{a_0} RE = 1.6\%$$

$$T_2(a_0) = 5.693731\sqrt{a_0} RE = 0.65\%$$

$$T_3(a_0) = 5.670590\sqrt{a_0} RE = 0.24\%$$

Belendez *et al.*[26] approximately solved Eq. (8) using He's homotopy perturbation method (HPM). They achieved the following results for the first and second approximation orders

$$T_{B1}(a_0) = \pi\sqrt{\pi a_0} \approx 5.568328\sqrt{a_0} RE = 1.6\%$$

$$T_{B2}(a_0) = \pi\sqrt{\frac{2\pi a_0}{1 + \sqrt{4-\pi}}} \approx 5.673551\sqrt{a_0} RE = 0.30\%$$

Wu *et al.*[20] approximately solved Eq. (8) using an improved harmonic balance method that incorporates the salient features of both CityplaceNewton's method and the harmonic balance method. They achieved the following results for the first and second approximation orders

$$T_{WSL1}(a_0) = \pi\sqrt{\pi a_0} \approx 5.568328\sqrt{a_0} RE = 1.6\%$$

$$T_{WSL2}(a_0) = \pi\sqrt{\frac{27\pi a_0}{26}} \approx 5.674401\sqrt{a_0} RE = 0.31\%$$

Observing all the approximate angular frequencies and periods, the correctness of the result obtained in this paper is better than those obtained previously by Belendez *et al.*[25] and only second-order approximate angular frequency and period is all most similar those obtained previously by Wu *et al.*[20] and Belendez *et al.*[26]. But it can be observed that these equations provide excellent approximations to the exact period regardless of the oscillation amplitude a_0. It has been mentioned that the procedure of Wu *et al.*[20], Belendez *et al.*[25] and Belendez *et al.*[26] is laborious especially for obtaining the higher approximations. The advantages of this method include its simplicity and computational efficiency, and the ability to objectively better agreement in higher-order approximate solution.

5. Conclusion

Based on a harmonic balance method (HBM), a simple analytical technique has been presented to determine higher-order approximate frequencies and periods for a conservative anti-symmetric, constant force nonlinear oscillator for which the elastic force term is proportional to $sgn(x)$. In compared with the previously published methods, determination of frequencies and periods is straightforward and simple. Excellent agreement between approximate periods and the exact one has been demonstrate and discussed, and the discrepancy of the third order approximate period with respect to the exact one is as low as 0.10% but Belendez et. al. [25] those obtained by 0.24%. Finally, we can say that the method presented in this article for solving strongly nonlinear differential equations can be considered as an efficient alternative of the previously proposed methods.

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