THE PRODUCT OF LAPLACE OPERATOR AND ULTRAHYPERBOLIC OPERATOR RELATED TO THE BIHARMONIC EQUATION AND WAVE EQUATION

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Abstract: In this paper, we study the nonlinear equation

\[ \Delta_{c_1}^{k_1} \Delta_{c_2}^{k_2} u(x) = f(x, \Delta_{c_1}^{k_1-1} \Delta_{c_2}^{k_2} u(x)) \]

and

\[ \Delta_{c_1}^{k_1} \Box_{c_2}^{k_2} u(x) = f(x, \Delta_{c_1}^{k_1-1} \Box_{c_2}^{k_2} u(x)), \]

where the operator \( \Delta \) and \( \Box \) are the Laplace operator and the Ultrahyperbolic operator, respectively. \( n \) is the dimension of the Euclidean space \( \mathbb{R}^n \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( k_1 \) and \( k_2 \) are nonnegative integer, \( u(x) \) is an unknown and \( f \) is a given function.

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1. Introduction

Gelfand and Shilov [3] have shown that the iterated Laplace equation \( \Delta^k u(x) = f(x) \) will be solved when we have obtained an elementary solution \( E(x) \). Kananthai [5], [6] has shown that \( u(x) = (-1)^k R_{2k}^c(x) \) be the elementary solution of the equation \( \Delta^k u(x) = \delta(x) \), where \( R_{2k}^c(x) \) defined by (5) and \( u(x) = \delta(x) \).
$((-1)^{k-1} R_{2(k-1)}^e (x))^{(l)}$ be a solution of $\Delta^k u(x) = 0$.

R. Courant and D. Hilbert [1] have studied the nonlinear equation of the form $\Delta u(x) = f(x, u(x))$ with $f$ defined and continuous function for all $x \in \Omega \cup \partial \Omega$ where $\Omega$ is an open set in $\mathbb{R}^n$, $\partial \Omega$ denotes the boundary of $\Omega$ and $\Delta$ is the Laplace operator, defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1}$$

They found that the solution $u(x)$ of such equation is unique under the condition $|f(x, u(x))| \leq N$ where $N$ is a constant for all $x \in \Omega$ and the boundary condition $u(x) = 0$ for all $x \in \partial \Omega$.

In this paper, we study the solution of nonlinear equation

$$\Delta_{c_1}^{k_1} \Delta_{c_2}^{k_2} u(x) = f(x, \Delta_{c_1}^{k_1-1} \Delta_{c_2}^{k_2} u(x))$$

and

$$\Delta_{c_1}^{k_1} \Box_{c_2}^{k_2} u(x) = f(x, \Delta_{c_1}^{k_1-1} \Box_{c_2}^{k_2} u(x)),$$

where the operator $\Delta$ and $\Box$ are the Laplace operator and the Ultrahyperbolic operator, respectively. $n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $k_1$ and $k_2$ are nonnegative integer, $u(x)$ is an unknown and $f$ is a given function. Moreover the solution $u(x)$ related to the nonhomogeneous biharmonic equation and nonhomogeneous wave equation depend on the conditions of $k_1$ and $k_2$.

2. Preliminaries

**Definition 1.** Let $c_1, c_2$ be positive numbers, $p + q = n$ and $k_1, k_2$ is a nonnegative integer. The Laplace operator iterated $k_1, k_2$ times are defined by

$$\Delta_{c_1}^{k_1} = \left[ \frac{1}{c_1^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^{k_1} \tag{2}$$

$$\Delta_{c_2}^{k_2} = \left[ \frac{1}{c_2^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^{k_2} \tag{3}$$

The Ultrahyperbolic operator iterated $k_2$ times is defined by

$$\Box_{c_2}^{k_2} = \left[ \frac{1}{c_2^2} \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^{k_2} \tag{4}$$
**Definition 2.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and the function \( R_\beta^e(X) \) and \( R_\gamma^e(Y) \) are defined by

\[
R_\beta^e(X) = 2^{-\beta} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n - \beta}{2} \right) \frac{X^{\frac{\beta - n}{2}}}{\Gamma \left( \frac{\beta}{2} \right)}
\]  

and

\[
R_\gamma^e(Y) = 2^{-\gamma} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n - \gamma}{2} \right) \frac{Y^{\frac{\gamma - n}{2}}}{\Gamma \left( \frac{\gamma}{2} \right)},
\]

where

\[
X = c_1^2 (x_1^2 + x_2^2 + \ldots + x_p^2) + (x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2),
\]

\[
Y = c_2^2 (x_1^2 + x_2^2 + \ldots + x_p^2) + (x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2), p + q = n.
\]

The function \( R_\beta^e(X) \) and \( R_\gamma^e(Y) \) are called the elliptic kernel of Marcel Riesz and is ordinary function for \( Re(\beta) \geq n \) and is a distribution of \( \beta \) for \( Re(\beta) < n \).

**Definition 3.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and

\[
Z = c_2^2 (x_1^2 + x_2^2 + \ldots + x_p^2) - (x_{p+1}^2 + x_{p+2}^2 + \ldots + x_{p+q}^2), p + q = n
\]

the nondegenerated quadratic form. Denote the interior of the forward cone by

\[
\Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } Z > 0 \}
\]

and the closure of \( \Gamma_+ \) by \( \overline{\Gamma}_+ \). For any complex number \( \alpha \), define

\[
R_\alpha^H(Z) = \begin{cases} 
\frac{Z^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\
0 & \text{for } x \notin \Gamma_+,
\end{cases}
\]

where

\[
K_n(\alpha) = \frac{\pi^{-\frac{n-1}{2}} \Gamma \left( \frac{\alpha + 2 - n}{2} \right) \Gamma \left( \frac{1 - \alpha}{2} \right) \Gamma(\alpha)}{\Gamma \left( \frac{\alpha + 2 - p}{2} \right) \Gamma \left( \frac{\alpha - p}{2} \right)}.
\]

The function \( R_\alpha^H(Z) \) was introduced by Nozaki [5, p.72]. It is well known that \( R_\alpha^H(Z) \) is ordinary function for \( Re(\alpha) \geq n \) and it is a distribution of \( \alpha \) for \( Re(\beta) < n \).

**Lemma 4.** Given the equation

\[
\triangle^k_{c_1} u(x) = \delta(x)
\]

and

\[
\triangle^k_{c_2} v(x) = \delta(x),
\]

where \( \triangle^k \) is the \( k \)-th order Poisson equation. For \( c_1 \) and \( c_2 \) are positive constants, and \( \delta(x) \) is the Dirac delta function.
where $\triangle_{c_1}^{k_1}$ and $\triangle_{c_2}^{k_2}$ are defined by (2) and (3) respectively. Then we obtain $u(x) = (-1)^{k_1} R_{2k_1}^e (X)$ and $v(x) = (-1)^{k_2} R_{2k_2}^e (Y)$ is an elementary solution of (9) and (10) respectively. $R_{2k_1}^e (X)$ and $R_{2k_2}^e (Y)$ are defined by (5) and (6) respectively, with $\beta = 2k_1, \gamma = 2k_2$.

Proof. See [1].

Lemma 5. Given the equation
\[ \square_{c_2}^{k_2} w(x) = \delta(x), \]
where $\square_{c_2}^{k_2}$ is defined by (4). Then we obtain $w(x) = R_{2k_2}^H (Z)$ is an elementary solution of (10). $R_{2k_2}^H (Z)$ is defined by (8) with $\alpha = 2k_2$.

Proof. See [7].

Lemma 6. Given the equation
\[ \triangle_{c_1}^{k_1} u(x) = 0, \]
where $\triangle_{c_1}^{k_1}$ is defined by (2). We obtain $u(x) = ((-1)^{k_1-1} R_{2(k_1-1)}^e (X))^{(l)}$ as a solutions of (12) where $l = (n-4)/2, n \geq 4$ is nonnegative integer and $n$ is even and $R_{2(k_1-1)}^e (X)$ defined by equation (5) with $l$ derivatives and $\beta = 2(k_1-1)$.

Proof. See [6].

Lemma 7. Given the equation
\[ \square_{c_2}^{k_2} u(x) = 0, \]
where $\square_{c_2}^{k_2}$ is the Ultrahyperbolic operator iterated $k$-times defined by equation (4). Then we obtain $u(x) = (R_{2(k-1)}^H (Z))^{(m)}$ as a solutions of (13) with $m = (n-4)/2, n \geq 4$ and $n$ is even. The function $(R_{2(k-1)}^H (Z))^{(m)}$ is defined by equation (8) with $m$ derivatives and $\alpha = 2(k-1)$.

Proof. See [6].

Lemma 8. Given the equation
\[ \triangle_{c_1} u(x) = f(x, u(x)), \]
where $f$ is defined and has continuous first derivatives for all $x \in \Omega \cup \partial \Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial \Omega$ denotes the boundary of $\Omega$. Assume $f$ is a bounded, that is $|f(x,u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial \Omega$. Then we obtain $u(x)$ as a unique solution of (14).

Proof. We can prove this lemma by the method of iterations and the Schauder’s estimates, see [1].
3. Main Results

**Theorem 9.** Given the nonlinear equation

\[ \triangle_{c_1}^{k_1} \triangle_{c_2}^{k_2} u(x) = f(x, \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x)) \]  

(15)

where \( \triangle_{c_1}^{k_1}, \triangle_{c_2}^{k_2} \) are defined by (2) and (3) respectively. Let \( f \) be defined and having continuous first derivative for all \( x \in \Omega \cup \partial \Omega \), \( \Omega \) is an open subset of \( \mathbb{R}^n \) and \( \partial \Omega \) denotes the boundary of \( \Omega \) and \( n \) is even with \( n \geq 4 \). Suppose \( f \) be a bounded function, that is

\[ |f(x, \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x))| \leq N \]  

(16)

and the boundary condition

\[ \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x) = 0 \]  

(17)

for all \( x \in \partial \Omega \). Then we obtain

\[ u(x) = (-1)^{k_1-1} R^{e}_{2(k_1-1)}(X) \ast (-1)^{k_2} R^{e}_{2k_2}(Y) \ast W(x) \]  

(18)

as a solution of (15) with the boundary condition

\[ u(x) = ((-1)^{k_1-2} R^{e}_{2(k_1-2)}(X))^{(l)} \ast (-1)^{k_2} R^{e}_{2k_2}(Y) \]  

for all \( x \in \partial \Omega \). And we have

\[ u(x) = (-1) R^{e}_{2}(Y) \ast W(x) \]  

(21)

as a solution of (19).
Proof. From equation (15), we have
\[ \triangle_{c_1}^{k_1} \triangle_{c_2}^{k_2} u(x) = \triangle (\triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x)) = f(x, \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x)). \] (22)
Since \( u(x) \) has continuous derivatives up to order \( k_1 + k_2 \) for \( k_i = 1, 2, 3, \ldots ; i = 1, 2 \) we can assume
\[ \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x) = W(x) \] (23)
for all \( x \in \partial \Omega \). Thus, (22) can be written in the form
\[ \triangle_{c_1}^{k_1} \triangle_{c_2}^{k_2} = \triangle_{c_1} W(x) = f(x, W(x)). \] (24)
by (16)
\[ |f(x, W(x))| \leq N. \] (25)
and by (17), \( W(x) = 0 \) or
\[ \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x) = 0 \] (26)
for all \( x \in \partial \Omega \). Thus by Lemma 8 there exist a unique solution \( W(x) \) of (24) which satisfies (25). The function \((-1)^{k_1-1} R_{2(k_1-1)}^{e}(X) \) and \((-1)^{k_2} R_{2k_2}^{e}(Y) \) are the elementary solution of the operators \( \triangle_{c_1}^{k_1-1} \) and \( \triangle_{c_2}^{k_2} \) respectively. Thus we have \( \triangle_{c_1}^{k_1-1}(-1)^{k_1-1} R_{2(k_1-1)}^{e}(X) = \delta, \triangle_{c_2}^{k_2}(-1)^{k_2} R_{2k_2}^{e}(Y) = \delta, \) where \( \delta \) is the Dirac-delta distribution. The functions \( R_{2(k_1-1)}^{e}(X) \) and \( R_{2k_2}^{e}(Y) \) are defined by (5) and (6) respectively, with with \( \beta = 2k_1, \gamma = 2k_2 \). Thus, convolving both sides of (23) by \((-1)^{k_1-1} R_{2(k_1-1)}^{e}(X) \) \((-1)^{k_2} R_{2k_2}^{e}(Y) \) and by the properties of convolution, we obtain
\[ u(x) = (-1)^{k_1-1} R_{2(k_1-1)}^{e}(X) \ast (-1)^{k_2} R_{2k_2}^{e}(Y) \ast W(x) \] (27)
as required. Consider the condition \( \triangle_{c_1}^{k_1-1} \triangle_{c_2}^{k_2} u(x) = 0 \) for all \( x \in \partial \Omega \). By Lemma 6, we have
\[ \triangle_{c_2}^{k_2} u(x) = ((-1)^{k_1-2} R_{2(k_1-2)}^{e}(X))^{(l)} \]
\[ u(x) = ((-1)^{k_1-2} R_{2(k_1-2)}^{e}(X))^{(l)} \ast (-1)^{k_2} R_{2k_2}^{e}(Y) \]
where \( l = (n-4)/2, n \geq 4 \) is nonnegative integer and \( n \) is even and \( R_{2(k_1-2)}^{e}(X) \) defined by equation (5) with \( l \) derivatives and \( \beta = 2(k_1 - 2). \) for all \( x \in \partial \Omega \) and \( k = 2, 3, 4, \ldots. \)
Moreover, for \( k_1 = 1, k_2 = 1 \) then (15) becomes
\[ \triangle_{c_1} \triangle_{c_2} u(x) = f(x, \triangle_{c_2} u(x)) \] (28)
with boundary condition
$$\Delta_{c_2} u(x) = 0$$
for all $x \in \partial \Omega$. We have
$$u(x) = (-1) R^e_{2(k_1-1)}(X) * W(x)$$
as a solution of (28), which is called the nonhomogeneous biharmonic equation.

**Theorem 10.** Given the nonlinear equation
$$\Delta_{c_1}^{k_1} \Box_{c_2}^{k_2} u(x) = f(x, \Delta_{c_1}^{k_1-1} \Box_{c_2}^{k_2} u(x))$$
where $\Delta_{c_1}^{k_1}, \Box_{c_2}^{k_2}$ are defined by (2) and (4) respectively. Let $f$ be defined and having continuous first derivative for all $x \in \Omega \cup \partial \Omega$, $\Omega$ is an open subset of $\mathbb{R}^n$ and $\partial \Omega$ denotes the boundary of $\Omega$ and $n$ is even with $n \geq 4$. Suppose $f$ be a bounded function, that is
$$|f(x, \Delta_{c_1}^{k_1-1} \Box_{c_2}^{k_2} u(x))| \leq N$$
and the boundary condition
$$\Delta_{c_1}^{k_1-1} \Box_{c_2}^{k_2} u(x) = 0$$
for all $x \in \partial \Omega$. Then we obtain
$$u(x) = (-1)^{k_1-1} R^e_{2(k_1-1)}(X) * R^H_{2k_2}(Z) * W(x)$$
as a solution of (30) with the boundary condition
$$u(x) = ((-1)^{k_1-2} R^e_{2(k_1-2)}(X))^{(l)} * R^H_{2k_2}(Z)$$
$W(x)$ is a continuous function for $x \in \Omega \cup \partial \Omega$, where $l = (n - 4)/2, n \geq 4$ is nonnegative integer and $n$ is even and $R^e_{2(k_1-2)}(X)$ defined by equation (5) with $l$ derivatives and $\beta = 2(k_1 - 2)$ and $R^H_{2k_2}(Z)$ are defined by (8) with $\alpha = 2k_2$. Moreover, for $k_1 = 1, k_2 = 1$ then (30) becomes
$$\Delta_{c_1} \Box_{c_2} u(x) = f(x, \Box_{c_2} u(x))$$
with boundary condition
$$\Box_{c_2} u(x) = 0$$
for all $x \in \partial \Omega$. Then we have

$$u(x) = R_2^H(Z) \ast W(x)$$

(36)

as a solution of (34) with the boundary condition $u(x) = \delta^{(m)}(Z)$ and $m = (n-4)/2, n \geq 4$ and $n$ is even. Also, if we put $k_2 = 1, p = 1$ and $q = n - 1$ in (4) the operator $\square_{c_2}^{k_2}$ reduce to the wave operator

$$\square_{c_2} = \frac{1}{c_2^2} \frac{\partial^2}{\partial x_1^2} - \left( \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right).$$

Thus, the solution $u(x)$ of (36) reduce to the solution of the wave equation $\square_{c_2} u(x) = W(x)$ with boundary condition $\square_{c_2} u(x) = 0$.

Proof. The proof of this Theorem is similar to the proof of Theorem 3.1, by including Lemma (5) and (7).

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References


