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ORBITAL HAUSDORFF DEPENDENCE ON IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we investigate a class of differential equations with variable impulsive moments. We introduce the concept of orbital Hausdorff dependence on the solutions of differential equations of this class. The dependence is concerning with the difference between consecutive impulsive moments. The sufficient conditions under which this specific continuous dependence is valid are found.

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1. Introduction

Impulsive differential equations (IDE) are divided into several classes dependent on the way of determining their impulsive moments. These classes could belong

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to one of the following groups:

— IDE with fixed impulsive moments;

— IDE with variable impulsive moments.

We define the impulsive moments as variable if they are specific to each solution. In the common case, the sets of impulsive moments corresponding to the non-coincident solutions are different. In particular, in IDE with variable impulsive moments, the basic solution and its respectively perturbed solution (both solutions are obtained for different initial conditions or parameters of the differential equation) have different impulsive moments. Apparently, the two solutions into the intervals between the respective impulsive moments of basic and perturbed solutions are subject to different numbers of impulsive effects. Let us denote the impulsive moments of the basic solution $x(t;t_0,x_0)$ by $t_1,t_2,\dots$ $(t_1<t_2<\dots)$, and the impulsive moments of perturbed solution $x^*(t;t_0^*,x_0^*)$ with $t_1^*,t_2^*,\dots$ $(t_1^*<t_2^*<\dots)$. Then, in the intervals between t_1 and t_1^*, between t_2 and t_2^* and etc., one of the two solutions is subjected to one more impulsive effect than the other solution. This means that we cannot expect "proximity" between them in the described intervals. In other words, the difference between the solutions in these intervals will be of the order of the "magnitude" of impulsive effects. Moreover, these comparatively large differences in the mentioned intervals are substantially independent on the closeness of the parameters of their respective initial problems. Therefore, we introduce the following measure $m(.,.)$ of deviation of the perturbed solution $x^*(t;t_0^*,x_0^*)$ from the initial solution $x(t;t_0,x_0)$:

$$
\begin{aligned}
m\Big(x^*(t;t_0^*,x_0^*),x(t;t_0,x_0)\Big)
&= \sup\Big\{\Big\|(x^*(t;t_0^*,x_0^*)-x(t;t_0,x_0)\Big\|\ ;\ t_0^{max}\le t\le t_0^{max}+T\,,\\
&\qquad |\, t-t_i \,|>\eta\,,\ i=1,2,\dots\Big\},
\end{aligned}
$$

where:

- (t_0^*,x_0^*) and (t_0,x_0) are the initial points of both solutions, respectively;

- $[t_0^{max},\infty)$ is the common interval of existence of these solutions, $t_0^{max}=\max\{t_0^*,t_0\}$;

- $\eta>0$ is the radius of the neighborhoods of impulsive moments $t_1,t_2,\dots$;

- T is a positive constant (it's possible $T=\infty$).

As it is seen from the above definition of the measure $m(.,.)$, in the neighborhoods of the impulsive moments of basic solution (with radii η), the proximity between $x^*(t;t_0^*,x_0^*)$ and $x(t;t_0,x_0)$ does not required. For more details, see monograph [13]. One of the basic preliminary problems in studying such class of equations is to determine (to be found sufficient conditions under which):

$$|t_i^* - t_i| < \eta\,, \quad i = 1,2,\ldots.$$

Let $t_1^*, t_2^*, \ldots$ be the impulsive moments of the perturbed solution. Given the definition of $m(.,.)$, we conclude that:

- this measure is similar to the classical uniform distance;

- its specific feature (and at the same time disadvantage) is the elimination of the time intervals between the impulsive effects by participating in the formation of the measure.

Unfortunately, $m(.,.)$ does not satisfy the conditions for distance. In the common case, the uniform distance between the solutions may be relatively large in the aforementioned neighborhoods (as we said, commensurate with the magnitudes of the impulsive effects).

The note above motivates the present research. Here the measure between the two solutions of IDE with variable impulsive moments is given in the whole domain, using the Hausdorff distance between their respective trajectories.

Scientific interest in the impulse equations is the results of their significant applications in modeling and studying the processes subjects to the short-time influences (see [2], [3], [6], [12], [16] and [17]). Various issues of fundamental and qualitative theory of differential equations (with and without impulses) are studied by the Hausdorff metrics (see [1], [5], [6], [7], [9], [10], [11], [15] and [17]).

2. Preliminary Results

The following notations are used. Let $a(a^1, a^2, \ldots, a^n)$ and $b(b^1, b^2, \ldots, b^n)$ be points of n-dimensional space $\mathbb{R}^n$. Their dot product is denoted by

$$\langle a, b\rangle = a^1b^1 + a^2b^2 + \cdots + a^nb^n.$$

The Euclidean distance between these points are introduced by

$$\rho(a,b) = \sqrt{(a^1-b^1)^2 + (a^2-b^2)^2 + ... + (a^n-b^n)^2}.$$

The Euclidean norm $\|a\|$ of the point a is equal to

$$\|a\| = \langle a, a\rangle^{\frac{1}{2}} = \sqrt{(a^1)^2 + (a^2)^2 + ... + (a^n)^2}.$$

The next equality is satisfied

$$\|a-b\| = \rho(a,b).$$

Let the nonempty sets $A, B \subset \mathbb{R}^n$. Then the Euclidean and Hausdorff distances between them are denoted by:

$$\rho_E(A,B) = \inf\Big\{ \inf\{\rho(a,b), a \in A\}, b \in B\Big\}$$

and

$$\rho_H(A,B) = \max\Big\{ \sup\big\{ \inf\{\rho(a,b), a \in A\}, b \in B\big\}, \\ \sup\big\{ \inf\{\rho(a,b), b \in B\}, a \in A\big\}\Big\},$$

respectively. If at least one of the sets A and B is empty, for convenience, we will assume that:

$$\rho_E(A,B) = 0 \quad \text{and} \quad \rho_H(A,B) = 0.$$

Further, we will denote by $\overline{A}$ the closure of the set A.

The following properties of the Hausdorff and Euclidean distance between the sets are valid in $\mathbb{R}^n$ (see [10] and [14]).

Remark 1. Let the sets $A, B, C \subset \mathbb{R}^n$ and a constant $\lambda \in \mathbb{R}^+$. Then:

1. $\rho_H(A,B) \geq \rho_E(A,B) \geq 0$;
2. $\rho_H(A,B) = 0 \iff A = B$;
3. $\rho_H(A,B) = \rho_H(B,A)$;
4. $\rho_H(\lambda A, \lambda B) = \lambda \rho_H(B,A)$;
5. $\rho_H(A \cup C, B \cup C) = \rho_H(A,B)$;
6. $\rho_H(A,B) \leq \rho_H(A,C) + \rho_H(C,B)$;
7. $\rho_H(\overline{A}, \overline{B}) = \rho_H(A,B)$;
8. If $\varnothing \neq A_1 \subset A_2 \subset \mathbb{R}^n$ and $\varnothing \neq B_1 \subset B_2 \subset \mathbb{R}^n$, then $\rho_E(A_1,B_1) \geq \rho_E(A_2,B_2)$.

Further, we will use the following statement.

Theorem 1. *[8] Assume that:*

1. *The sets* $A_1, A_2, \dots, A_k,\ B_1, B_2, \dots, B_k \subset \mathbb{R}^n$ *are bounded.*
2. $A = A_1 \cup A_2 \cup \cdots \cup A_k,\ B = B_1 \cup B_2 \cup \cdots \cup B_k$.

Then

$$\rho_H(A, B) \leq \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2), \dots, \rho_H(A_k, B_k)\}.$$

Let the functions $g, g^* : \mathbb{R}^+ \to \mathbb{R}^n$ and the constants $T_0, T_1, T_0^*, T_1^* \in \mathbb{R}^+$. We introduce the curves:

$$\gamma[T_0, T_1] = \begin{cases} \{g(t)\,; \quad T_0 \leq t \leq T_1\}, & \text{if } T_0 \leq T_1; \\ \emptyset\,, & \text{if } T_0 > T_1 \end{cases}$$

and

$$\gamma^*[T_0^*, T_1^*] = \begin{cases} \{g^*(t)\,; \quad T_0^* \leq t \leq T_1^*\}, & \text{if } T_0^* \leq T_1^*; \\ \emptyset\,, & \text{if } T_0^* > T_1^*. \end{cases}$$

Analogically, the next curves are introduced and defined in the half-open and open intervals, respectively:

$$\begin{array}{lll} \gamma(T_0, T_1]\,, & \gamma[T_0, T_1)\,, & \gamma(T_0, T_1)\,, \\ \gamma^*(T_0^*, T_1^*]\,, & \gamma^*[T_0^*, T_1^*)\,, & \gamma^*(T_0^*, T_1^*)\,. \end{array}$$

Remark 2. [8] Let $0 \leq T_0 \leq T_1$ and $0 \leq T_0^* \leq T_1^*$. The following definitional equalities are valid for the Euclidean, Hausdorff and uniform distances between the curves $\gamma^*[T_0^*, T_1^*]$ and $\gamma[T_0, T_1]$, respectively:

$$\begin{aligned} &\rho_E(\gamma^*[T_0^*, T_1^*], \gamma[T_0, T_1]) \\ &\quad = \inf\Big\{ \inf\big\{\rho(g^*(t^*), g(t))\,,\ T_0 \leq t \leq T_1\big\}\,,\ T_0^* \leq t^* \leq T_1^* \Big\}; \end{aligned}$$

$$\begin{aligned} &\rho_H(\gamma^*[T_0^*, T_1^*], \gamma[T_0, T_1]) \\ &= \max\Big\{ \sup\big\{ \inf\big\{\rho(g^*(t^*), g(t))\,,\ T_0 \leq t \leq T_1\big\}\,,\ T_0^* \leq t^* \leq T_1^*\big\}\,, \\ &\qquad \sup\big\{ \inf\{\rho(g^*(t^*), g(t))\,,\ T_0^* \leq t^* \leq T_1^*\}\,,\ T_0 \leq t \leq T_1\big\} \Big\}; \end{aligned}$$

$$\rho_R(\gamma^*[T_0, T_1], \gamma[T_0, T_1]) = \sup\{\rho(g^*(t), g(t))\,,\ T_0 \leq t \leq T_1\}.$$

Analogous differential equalities of the Euclidean, Hausdorff and uniform distances are valid for the curves defined in the half-open or open intervals.

The following symbols will be used in the next theorem:

$$T_0^{min} = \min\{T_0^*, T_0\}\,, \quad T_0^{max} = \max\{T_0^*, T_0\}\,,$$
$$T_1^{min} = \min\{T_1^*, T_1\}\,, \quad T_1^{max} = \max\{T_1^*, T_1\}.$$

Theorem 2. *[8] Assume that:*

1. *The function $g, g^* : \mathbb{R}^+ \to \mathbb{R}^n$ are continuous on the left-hand side in $\mathbb{R}^+$.*
2. *The inequality $T_0^{max} \leq T_1^{min}$ is true.*

Then the next estimate is valid:

$$\rho_H\Big(\gamma^*(T_0^*, T_1^*], \gamma(T_0, T_1]\Big)$$
$$\leq \max\Big\{\rho_R(\gamma^*(T_0^{max}, T_1^{min}], \gamma(T_0^{\max}, T_1^{\min}]),$$
$$\rho_H(g(T_0+0), \gamma^*(T_0^*, T_0])\,,\ \rho_H(g^*(T_0^*+0), \gamma(T_0, T_0^*]),$$
$$\rho_H(g(T_1), \gamma^*(T_1, T_1^*])\,,\ \rho_H(g^*(T_1^*), \gamma(T_1^*, T_1])\Big\}.$$

3. Class IDE with Variable Impulsive Moments

Subject of our study is this initial value problem for systems of IDE, which hereafter we will call basic:

$$\frac{dx}{dt} = f(t, x)\,, \quad t_{i-1} < t \leq t_i\,, \quad t_i = t_{i-1} + d\,; \tag{1}$$
$$x(t_i + 0) = x(t_i) + I_i\big(x(t_i)\big)\,, \quad i = 1, 2, \ldots\,; \tag{2}$$
$$x(t_0) = x_0, \tag{3}$$

where:

- function $f : \mathbb{R}^+ \times D \to \mathbb{R}^n$;
- D is a nonempty set in $\mathbb{R}^n$;
- impulsive functions $I_i : D \to \mathbb{R}^n$, $\big(Id + I_i\big) : D \to D$, $i = 1, 2, \ldots$;
- function Id is the identity for $\mathbb{R}^n$;
- constant $d > 0$ is the difference between any two sequential impulsive moments;
- initial point $(t_0, x_0) \in \mathbb{R}^+ \times D$.

The moments $t_1, t_2, \ldots$ are called impulsive. We determine successively the solution $x(t; t_0, x_0)$ of problem (1), (2), (3) as follows:

1.1 For $t_0 \le t \le t_1 = t_0 + d$, the solution of problem (1), (2), (3) coincides with the solution of problem without impulses

$$\frac{dx}{dt} = f(t,x)\,, \qquad x(t_0) = x_0\,;$$

1.2 At the moment t_1, an impulsive effect is performed:

$$x(t_1+0;t_0,x_0) = x(t_1;t_0,x_0) + I_1\big(x(t_1;t_0,x_0)\big) = x_1 + I_1(x_1) = x_1^+\,;$$

2.1 For $t_1 < t \le t_2 = t_1 + d$, the solution of this problem coincides with the solution of the problem without impulses

$$\frac{dx}{dt} = f(t,x)\,, \qquad x(t_1) = x_1^+\,;$$

2.2 At the moment t_2, an impulsive effect is performed:

$$x(t_2+0;t_0,x_0) = x(t_2;t_0,x_0) + I_2\big(x(t_2;t_0,x_0)\big) = x_2 + I_1(x_2) = x_2^+\,,$$

etc.

It is seen that the solution is a piecewise continuous function with the points of discontinuity $t_1, t_2, \dots$, in which the solution is continuous on the left-hand side.

The following notions are used:

$$x_i = x(t_i;t_0,x_0)\,,$$
$$x_i^+ = x(t_i;t_0,x_0) + I_i\big(x(t_i;t_0,x_0)\big) = \big(Id + I_i\big)(x_i)\,,\ i = 1,2,\dots\,.$$

Assume that the following conditions are satisfied:

H1. Function $f \in C[\mathbb{R}^+ \times D, \mathbb{R}^n]$.

H2. For any point $(t_0,x_0) \in \mathbb{R}^+ \times D$, the problem (1), (3) without impulses has a unique solution, defined in $t \in \mathbb{R}^+$.

H3. There exists a positive constant C_f such that

$$\big(\forall (t,x) \in \mathbb{R}^+ \times D\big) \Rightarrow \|f(t,x)\| \le C_f.$$

H4. The functions $I_i \in C\big[D, \mathbb{R}^n\big]$, $\big(Id + I_i\big) : D \to D$, $i = 1,2,\dots$.

The following theorem is a direct consequence of the conditions H1 and H2.

Theorem 3. *Let the conditions H1 and H2 are satisfied.*

Then for any point $(t_0, x_0) \in \mathbb{R}^+ \times D$, *the solution of the problem* (1), (2), (3) *exists and is unique for* $t \geq t_0$.

We study the basic and perturbed problem together:

$$\frac{dx}{dt} = f(t,x)\,, \quad t^*_{i-1} < t \leq t^*_i\,, \quad t^*_i = t^*_{i-1} + d^*; \tag{4}$$

$$x(t^*_i + 0) = x(t^*_i) + I_i\big(x(t^*_i)\big)\,, \quad i = 1, 2, \ldots; \tag{5}$$

$$x(t^*_0) = x^*_0, \tag{6}$$

where the initial point $(t^*_0, x^*_0) \in \mathbb{R}^+ \times D$ and the constants $d^*_i > 0$, $i = 1, 2, \ldots$. Denote the solution of perturbed problem by $x^*(t; t^*_0, x^*_0)$, and the impulsive moment by $t^*_1, t^*_2, \ldots$.

Let the constants $T_1, T_2, T^*_1, T^*_2 \in \mathbb{R}^+$. The trajectory $\chi\big(T_1, T_2\big]$ of the problem (1), (2), (3) is determined for $T_1 < t \leq T_2$. Similarly, let $\chi^*\big(T^*_1, T^*_2\big]$ be the trajectory of problem (4), (5), (6), which is located between the moments T^*_1 and T^*_2. The following equations are valid:

$$\chi\big(T_1, T_2] = \begin{cases} \big\{x(t; t_0, x_0)\,; \quad T_1 \leq t \leq T_2\big\}\,, & \text{if } \; T_1 < T_2; \\ \varnothing\,, & \text{if } \; T_1 \geq T_2 \end{cases}$$

and

$$\chi^*\big(T^*_1, T^*_2] = \begin{cases} \big\{x^*(t; t^*_0, x^*_0)\,; \quad T^*_1 \leq t \leq T^*_2\big\}\,, & \text{if } \; T^*_1 < T^*_2; \\ \varnothing\,, & \text{if } \; T^*_1 \geq T^*_2. \end{cases}$$

We denote $t^{min}_i = \min\{t^*_i, t_i\}$ and $t^{max}_i = \max\{t^*_i, t_i\}$, $i = 0, 1, 2, \ldots$.

Theorem 4. *Assume that:*

1. *The conditions H1 and H2 are valid.*
2. *The inequalities* $t^{max}_{i-1} < t^{min}_i$, $i = 1, 2, \ldots$, *are satisfied.*

Then for each $k \in \mathbb{N}$, *it is valid*

$$\begin{aligned} &\rho_H\Big(\chi^*[t^*_0, t^*_k], \chi[t_0, t_k]\Big) \\ &\leq \max\Big\{\rho_R\Big(\chi^*\big(t^{max}_{i-1}, t^{min}_i], \chi\big(t^{max}_{i-1}, t^{min}_i]\Big)\,, \\ &\qquad \rho_H\Big(x(t_{i-1}+0; t_0, x_0), \chi^*(t^*_{i-1}, t_i]\Big)\,, \\ &\qquad \rho_H\Big(x^*(t^*_{i-1}+0; t^*_0, x^*_0), \chi(t_{i-1}, t^*_{i-1}]\Big)\,, \end{aligned}$$

$$\rho_H\Big(x(t_i;t_0,x_0),\chi^*(t_i,t_i^*]\Big)\,,$$
$$\rho_H\Big(x^*(t_i^*;t_0^*,x_0^*),\chi(t_i^*,t_i]\Big)\,,i=1,2,...,k\Big\}.$$

Proof. The following equalities are fulfilled:

$$\chi(t_0,t_k]=\bigcup_{i=1,2,\dots,k}\chi(t_{i-1},t_i] \quad \text{and} \quad \chi^*(t_0^*,t_k^*]=\bigcup_{i=1,2,\dots,k}\chi^*(t_{i-1}^*,t_i^*].$$

By applying successively Theorem 1 and Theorem 2, we obtain:

$$\rho_H\Big(\chi^*(t_0^*,t_k^*],\chi(t_0,t_k]\Big) \tag{7}$$
$$=\rho_H\Big(\bigcup_{i=1,2,\dots,k}\chi^*(t_{i-1}^*,t_i^*],\bigcup_{i=1,2,\dots,k}\chi(t_{i-1},t_i]\Big)$$
$$\le\max\Big\{\rho_H\Big(\chi^*(t_{i-1}^*,t_i^*],\chi(t_{i-1},t_i]\Big)\,,\ i=1,2,...,k\Big\}$$
$$\le\max\Big\{\max\Big\{\rho_R\Big(\chi^*(t_{i-1}^{max},t_i^{min}],\chi(t_{i-1}^{max},t_i^{min}]\Big)\,,$$
$$\rho_H\Big(x(t_{i-1}+0;t_0,x_0),\chi^*(t_{i-1}^*,t_i]\Big)\,,$$
$$\rho_H\Big(x^*(t_{i-1}^*+0;t_0^*,x_0^*),\chi(t_{i-1},t_{i-1}^*]\Big)\,,$$
$$\rho_H\Big(x(t_i;t_0,x_0),\chi^*(t_{i-1},t_i^*]\Big)\,,$$
$$\rho_H\Big(x^*(t_i^*;t_0^*,x_0^*),\chi(t_i^*,t_i]\Big)\Big\}\,,\ i=1,2,...,k\Big\}$$
$$=\max\Big\{\rho_R\Big(\chi^*(t_{i-1}^{max},t_i^{min}],\chi(t_{i-1}^{max},t_i^{min}]\Big)\,,$$
$$\rho_H\Big(x(t_{i-1}+0;t_0,x_0),\chi^*(t_{i-1}^*,t_i]\Big)\,,$$
$$\rho_H\Big(x^*(t_{i-1}^*+0;t_0^*,x_0^*),\chi(t_{i-1},t_i^*]\Big)\,,$$
$$\rho_H\Big(x(t_i;t_0,x_0),\chi^*(t_i,t_i^*]\Big)\,,$$
$$\rho_H\Big(x^*(t_i^*;t_0^*,x_0^*),\chi(t_i^*,t_i]\Big)\,,\ i=1,2,...,k\Big\}.$$

Finally, taking into account that the solutions $x^*(t;t_0^*,x_0^*)$ and $x(t;t_0,x_0)$ are continuous on the initial moments t_0^* and t_0 respectively, we conclude that

$$\rho_H\big(\chi^*(t_0^*,t_k^*],\chi(t_0,t_k]\big)=\rho_H\big(\chi^*[t_0^*,t_k^*],\chi[t_0,t_k]\big). \tag{8}$$

From (7) and (8), we reach the correctness of the theorem. □

4. Orbital Hausdorff Continuous Dependence on the Difference between the Impulsive Moments

Definition 1. We will say that the solution of problem (1), (2), (3) is orbital Hausdorff continuous dependent on the difference between the impulsive moments (i.e. by the parameter d) if

$$(\forall \varepsilon > 0)(\forall T > t_0\,,\ T \neq t_i\,,\ i = 1, 2, \dots)$$
$$(\exists \delta = \delta(t_0, x_0, d, \varepsilon, T) > 0):$$
$$(\forall t_0^* \in \mathbb{R}^+\,,\ |t_0^* - t_0)| < \delta)(\forall x_0^* \in D\,,\ \|x_0^* - x_0)\| < \delta)$$
$$(\forall d_i^* > 0\,,\ |d_i^* - d)| < \delta\,,\ i = 1, 2, \dots)$$
$$\Rightarrow \rho_H\big(\chi^*[t_0^*, T], \chi[t_0, T]\big) < \varepsilon.$$

Theorem 5. *Assume that the conditions H1-H4 are satisfied.*
Then the solution of problem (1), (2), (3) *is orbital Hausdorff dependent on the difference between the impulsive moments.*

Proof. Let ε and T be positive constants, $T > t_0$ and $T \neq t_i$, $i = 1, 2, \dots$. Since

$$\lim_{i\to\infty} t_i = \lim_{i\to\infty} (t_0 + id) = \infty$$

there exists a number $k \in \mathbb{N}$ such that

$$t_0 + kd = t_k < T < t_{k+1} = t_0 + (k+1)d\,.$$

Therefore, we can choose the constant $\delta_T = \delta_T(d, T) > 0$ so small that

$$(\forall t_0^* \in \mathbb{R}^+\,,\ |t_0^* - t_0)| < \delta_T) \tag{9}$$
$$(\forall d_i^* > 0\,,\ |d_i^* - d)| < \delta_T,\ i = 1, 2, \dots, k+1)$$

$$\begin{aligned}
\Rightarrow\quad & t_0^* < t_1 \ \Leftrightarrow\ t_0^* < t_0 + d;\\
& t_1^* < t_2 \ \Leftrightarrow\ t_0^* + d_1^* < t_0 + 2d;\\
& \dots\dots\dots\dots\dots\dots;\\
& t_{k-1}^* < t_k \ \Leftrightarrow\ t_0^* + d_1^* + d_2^* + \dots + d_{k-1}^* < t_0 + kd;\\
& t_k^* < T \ \Leftrightarrow\ t_0^* + d_1^* + d_2^* + \dots + d_k^* < T;\\
& T < t_{k+1}^* \ \Leftrightarrow\ T < t_0^* + d_1^* + d_2^* + \dots + d_{k+1}^*.
\end{aligned}$$

For example, the inequalities (9) are satisfied if the corresponding inequalities are valid:

$$\delta_T < d\,,\ \delta_T < \frac{d}{2}\,, \ldots,\ \delta_T < \frac{d}{k}\,,\ \delta_T < \frac{T-t_k}{k+1}\,,\ \delta_T < \frac{t_{k+1}-T}{k+2}\,,$$

i.e.

$$0 < \delta_T < \min\left\{\frac{d}{k}, \frac{T-t_k}{k+1}, \frac{t_{k+1}-T}{k+2}\right\}.$$

By inequalities (9), we get

$$t_0^{max} < t_1^{min},\ t_1^{max} < t_2^{min}, \ldots,\ t_{k-1}^{max} < t_k^{min},\ t_k^{max} < T < t_{k+1}^{min}.$$

Let us consider the trajectories $\chi^*(t_0^*, t_1^*]$ and $\chi(t_0, t_1]$. From property 7 in Remark 1 and Theorem 2, it follows that

$$\rho_H\Big(\chi^*[t_0^*, t_1^*], \chi[t_0, t_1]\Big) \tag{10}$$

$$\begin{aligned}
&= \rho_H\Big(\chi^*(t_0^*, t_1^*], \chi(t_0, t_1]\Big) \\
&\le \max\Big\{\rho_R\big(\chi^*(t_0^{max}, t_1^{min}], \chi(t_0^{max}, t_1^{min}]\big)\,, \\
&\qquad \rho_H\big(x(t_0+0; t_0, x_0), \chi^*(t_0^*, t_0]\big)\,, \\
&\qquad \rho_H\big(x^*(t_0^*+0; t_0^*, x_0^*), \chi(t_0, t_0^*]\big)\,, \\
&\qquad \rho_H\big(x(t_1; t_0, x_0), \chi^*(t_1, t_1^*]\big)\,, \\
&\qquad \rho_H\big(x^*(t_1^*; t_0^*, x_0^*), \chi(t_1^*, t_1]\big)\Big\}.
\end{aligned}$$

Consider the case when $t_0 \le t_0^*$ and $t_1 \le t_1^*$. Similarly, the other cases are considered. In this case, we have

$$(t_0^*, t_0] = \varnothing \quad \text{and} \quad (t_1^*, t_1] = \varnothing\,.$$

Therefore,

$$\begin{aligned}
\rho_H\big(x(t_0+0; t_0, x_0), \chi^*(t_0^*, t_0]\big) &= 0, \\
\rho_H\big(x^*(t_1^*; t_0^*, x_0^*), \chi(t_1^*, t_1]\big) &= 0.
\end{aligned} \tag{11}$$

We will evaluate the remaining three addends in the right hand side of (10). For this purpose, we will use the Theorem of continuous dependence on initial point of the solutions of differential equations without impulses (see Theorem 7.1, § 7, Ch. I, [4]), which hereafter will be called Theorem of continuous dependence.

Let η_0, $0 < \eta_0 < \varepsilon$, be a constant which will be determined later. By the Theorem of continuous dependence, we have

$$\Big(\exists \delta_0 = \text{const}\,,\ 0 < \delta_0 < \min\Big\{\delta_T, \frac{\eta_0}{4C_f}\Big\}\Big):$$

$$\Big(\forall t_0^* \in \mathbb{R}^+\,,\ |t_0^* - t_0| < \delta_0\Big)$$

$$\Big(\forall x_0^* \in D\,,\ \|x_0^* - x_0\| < \delta_0\Big)$$

$$\Big(\forall d_1^* > 0,\ |d_1^* - d| < \delta_0\Big)$$

$$\Rightarrow\ \|x^*(t; t_0^*, x_0^*) - x(t; t_0, x_0)\| < \frac{1}{2}\eta_0\,, \quad t_0^{max} < t \le t_1^{min}$$

$$\Leftrightarrow\ \rho_R\big(\chi^*(t_0^{max}, t_1^{min}], \chi(t_0^{max}, t_1^{min}]\big) < \frac{1}{2}\eta_0 < \varepsilon. \tag{12}$$

For the difference $t_1^* - t_1$, we obtain

$$t_1^* - t_1 = |t_1^* - t_1| \le |t_0^* - t_0| + |d_1^* - d| < 2\delta_0 < \frac{\eta_0}{2C_f}. \tag{13}$$

Assuming that equality $t_1 = t_1^*$ is satisfied, it follows that $(t_1, t_1^*] = \varnothing$ and therefore,

$$\rho_H\big(x(t_1; t_0, x_0), \chi^*(t_1, t_1^*]\big) = 0.$$

Let $t_1 < t_1^*$. If $t_1 < t \le t_1^*$ and using (12) and (13), we assess

$$\|x(t_1; t_0, x_0) - x^*(t; t_0^*, x_0^*)\| \tag{14}$$

$$\begin{aligned}
&\le \|x(t_1; t_0, x_0) - x^*(t_1; t_0^*, x_0^*)\| + \|x^*(t; t_0^*, x_0^*) - x^*(t_1; t_0^*, x_0^*)\| \\
&< \frac{\eta_0}{2} + \int_{t_1}^{t} \|f(\tau, x^*(\tau; t_0^*, x_0^*)\|d\tau \\
&< \frac{\eta_0}{2} + C_f|t - t_1| \le \frac{\eta_0}{2} + C_f|t_1^* - t_1| \\
&< \frac{\eta_0}{2} + 2C_f\delta_0 < \eta_0 < \varepsilon.
\end{aligned}$$

The inequality above is equivalent to

$$\rho_H\big(x(t_1; t_0, x_0), \chi^*(t_1, t_1^*]\big) < \varepsilon. \tag{15}$$

As we mentioned above, if we assume that $t_0 = t_0^*$, then

$$\rho_H\big(x^*(t_0^*;t_0^*,x_0^*),\chi(t_0,t_0^*]\big)=0.$$

Let $t_0 < t_0^*$. For any t, $t_0 < t \leq t_0^*$, we have

$$\begin{aligned}
&\|x(t;t_0,x_0)-x^*(t_0^*+0;t_0^*,x_0^*)\|\\
&\quad=\|x(t;t_0,x_0)-x^*(t_0^*;t_0^*,x_0^*)\|\\
&\quad\leq\|x(t_0^*;t_0,x_0)-x(t;t_0,x_0)\|+\|x^*(t_0^*;t_0^*,x_0^*)-x(t_0^*;t_0,x_0)\|\\
&\quad<\int_t^{t_0^*}\|f(\tau,x(\tau;t_0,x_0)\|d\tau+\frac{\eta_0}{2}\\
&\quad\leq\frac{\eta_0}{2}+C_f|t_0^*-t|\leq\frac{\eta_0}{2}+C_f|t_0^*-t_0|\\
&\quad<\frac{\eta_0}{2}+C_f\delta_0<\eta_0<\varepsilon\,,
\end{aligned}$$

i.e.

$$\rho_H\big(x^*(t_0^*+0;t_0^*,x_0^*),\chi(t_0,t_0^*]\big)=\rho_H\big(x^*(t_0^*;t_0^*,x_0^*),\chi(t_0,t_0^*]\big)<\varepsilon. \tag{16}$$

Considering (10), (11), (12), (15) and (16), we get

$$\begin{gathered}
\big(\forall\varepsilon>0\big)\big(\exists\delta_0=\delta_0(\varepsilon)>0\big):\\
\big(\forall t_0^*\in\mathbb{R}^+,\ |t_0^*-t_0|<\delta_0\big)\\
\big(\forall x_0^*\in D,\ \|x_0^*-x_0|<\delta_0\big)\\
\big(\forall d_1^*\in\mathbb{R}^+,\ |d_1^*-d|<\delta_0\big)\\
\Rightarrow\ \rho_H\big(\chi^*[t_0^*,t_1^*],\chi[t_0,t_1]\big)=\rho_H\big(\chi^*(t_0^*,t_1^*],\chi(t_0,t_1]\big)<\varepsilon.
\end{gathered}\tag{17}$$

We will define the constant η_0. Let δ_1 be an arbitrary positive constant, which will be defined later. Initially, we assume that $\eta_0 < 2C_f\delta_1$. Then from (13), it follows that

$$|t_1^*-t_1|<\delta_1 \tag{18}$$

is satisfied.

From (14), for $t = t_1^*$, it follows that

$$\|x(t_1;t_0,x_0)-x^*(t_1^*;t_0^*,x_0^*)\|=\|x_1-x_1^*\|<\eta_0\,.$$

Using written above and the continuity of function I_1 (see condition H4), we obtain that

$$\big(\forall\delta_1 > 0\big)\big(\exists\eta_0 = \eta_0(\delta_1) > 0\big) : \big(\forall x_1^*, x_1 \in D\,,\ \|x_1^* - x_1\| < \eta_0\big)$$
$$\Rightarrow\ \|x_1^{*+} - x_1^+\| = \|x_1^* + I_1(x_1^*) - x_1 - I_1(x_1)\| < \delta_1\,. \tag{19}$$

Further, we assume that $\eta_0 = \eta_0(\delta_1) > 0$ is a sufficiently small constant, such that the inequality (19) is valid.

From (17), (18) and (19), we receive

$$\big(\forall\delta_1 > 0\big)\big(\exists\delta_0 = \text{const} > 0\big) : \tag{20}$$
$$\big(\forall t_0^* \in \mathbb{R}^+,\ |t_0^* - t_0| < \delta_0\big)$$
$$\big(\forall x_0^{*+} = x_0^* \in D\,,\ \|x_0^{*+} - x_0^+\| = \|x_0^* - x_0\| < \delta_0\big)$$
$$\big(\forall d_1^* > 0,\ |d_1^* - d| < \delta_0\big)$$

$$\Rightarrow\quad \begin{aligned}&\rho_H\big(\chi^*[t_0^*, t_1^*], \chi[t_0, t_1]\big) < \varepsilon\\ &|t_1^* - t_1| < \delta_1\,;\\ &\|x_1^{*+} - x_1^+\| < \delta_1\,.\end{aligned}$$

Let us consider the trajectories $\chi^*(t_{i-1}^*, t_i^*]$ and $\chi(t_{i-1}, t_i],\ i = 2, 3, \ldots, k$. Similarly to (20), we get:

$$\big(\forall\delta_i > 0\big)\big(\exists\delta_{i-1} = \text{const} > 0\big) : \tag{21}$$
$$\big(\forall t_{i-1}^* \in \mathbb{R}^+,\ |t_{i-1}^* - t_{i-1}| < \delta_{i-1}\big)$$
$$\big(\forall x_{i-1}^{*+} \in D\,,\ \|x_{i-1}^{*+} - x_{i-1}^+\| < \delta_{i-1}\big)$$
$$\big(\forall d_i^* > 0,\ |d_i^* - d| < \delta_{i-1}\big)$$

$$\Rightarrow\quad \begin{aligned}&\rho_H\big(\chi^*(t_{i-1}^*, t_i^*], \chi(t_{i-1}, t_i]\big) < \varepsilon;\\ &|t_i^* - t_i| < \delta_i\,;\\ &\|x_i^{*+} - x_i^+\| < \delta_i\,.\end{aligned}$$

Finally, we consider the trajectories $\chi^*(t_k^*, T]$ and $\chi(t_k, T]$. As a consequence of Theorem 2, we get the following estimate:

$$\rho_H\Big(\chi^*(t_k^*, T], \chi[t_k, T]\Big) \tag{22}$$

$$\leq \max\Big\{\rho_R\big(\chi^*(t_k^{max},T],\chi(t_k^{max},T]\big)\,,\\ \rho_H\big(x(t_k+0;t_0,x_0),\chi^*(t_k^*,t_k]\big)\,,\\ \rho_H\big(x^*(t_k^*+0;t_0^*,x_0^*),\chi(t_k,t_k^*]\big)\Big\}.$$

We will evaluate each of the addends on the right hand side of (22).

According to the Theorem of continuous dependence, we get

$$\big(\forall\varepsilon>0\big)\Big(\exists\delta_k,\ 0<\delta_k<\frac{\varepsilon}{2C_f}\Big):$$

$$\big(\forall t_k^*\in\mathbb{R}^+,\ |t_k^*-t_k|<\delta_k\big)$$
$$\big(\forall x_k^{*+}\in D\,,\ \|x_k^{*+}-x_k^+\|<\delta_k\big)$$
$$\big(\forall d_{k+1}^*>0,\ |d_{k+1}^*-d|<\delta_k\big)$$

$$\Rightarrow\ \|x^*(t;t_k^*,x_k^{*+})-x(t;t_k,x_k^+)\|\\ =\|x^*(t;t_0^*,x_0^*)-x(t;t_0,x_0)\|\\ <\varepsilon\,,\quad t_k^{max}<t\leq T$$

$$\Leftrightarrow\ \rho_R\big(\chi^*(t_k^{max},T],\chi(t_k^{max},T]\big)<\frac{\varepsilon}{2}. \tag{23}$$

Suppose that $t_k^*\leq t_k$ (similarly seen in the other cases). Then $\chi(t_k,t_k^*]=\varnothing$ and therefore

$$\rho_H\big(x^*(t_k^*+0,t_0^*,x_0^*),\chi(t_k,t_k^*]\big)=0. \tag{24}$$

Assume that $t_k^*=t_k$, we obtain

$$(t_k^*,t_k]=\varnothing\ \Rightarrow\ \chi^*(t_k^*,t_k]=\varnothing\ \Rightarrow\rho_H\big(x(t_k+0,t_0,x_0),\chi^*(t_k^*,t_k]\big)=0.$$

Let $t_k^*<t_k$. For $t_k^*<t\leq t_k$, we have

$$\|x(t_k+0;t_0,x_0)-x^*(t;t_0^*,x_0^*)\|\\ \leq\|x(t_k+0;t_0,x_0)-x^*(t_k+0;t_0^*,x_0^*)\|\\ +\|x^*(t_k+0;t_0^*,x_0^*)-x^*(t;t_0^*,x_0^*)\|\\ \leq\frac{\varepsilon}{2}+\int_t^{t_k}\|f\big(\tau,x^*(\tau;t_0^*,x_0^*)\big)\|d\tau\\ \leq\frac{\varepsilon}{2}+C_f|t_k-t|\leq\frac{\varepsilon}{2}+C_f|t_k-t_k^*|<\varepsilon.$$

This means that

$$\rho_H\big(x(t_k+0,t_0,x_0),\chi^*(t_k^*,t_k]\big)<\varepsilon. \tag{25}$$

From (22), (23), (24) and (25), we obtain:

$$\begin{gathered}\big(\forall\varepsilon>0\big)\big(\exists\delta_k>0\big): \\ \big(\forall t_k^*\in\mathbb{R}^+,\ |t_k^*-t_k|<\delta_k\big) \\ \big(\forall x_k^{*+}\in D,\ \|x_k^{*+}-x_k^+|<\delta_k\big) \\ \big(\forall d_{k+1}^*>0,\ |d_{k+1}^*-d|<\delta_k\big) \\ \Rightarrow\ \rho_H\Big(\chi^*(t_k^*,T],\chi(t_k,T]\Big)<\varepsilon.\end{gathered} \tag{26}$$

We define the constants $\delta_0,\delta_1,...,\delta_k$ in the reverse order. First we find

$$\delta_k=\delta_k(\varepsilon).$$

Thus we determine

$$\delta_{k-1}=\delta_{k-1}(\varepsilon,\delta_k),\ \delta_{k-2}=\delta_{k-2}(\varepsilon,\delta_{k-1}),\ldots,\ \delta_1=\delta_1(\varepsilon,\delta_2).$$

Finally, we fix

$$\delta_0=\delta_0(\varepsilon,\delta_1,\delta_T).$$

From (20), (21) and (26), it is valid:

$$\begin{gathered}\big(\forall\varepsilon>0\big)\big(\exists\delta_0=\text{const}>0\big) \\ \big(\forall t_0^*\in\mathbb{R}^+,\ |t_0^*-t_0|<\delta_0\big) \\ \big(\forall x_0^*\in D,\ \|x_0^*-x_0|<\delta_0\big) \\ \big(\forall d_i^*>0,\ |d_i^*-d|<\delta_0,\ i=1,2,...,k+1\big)\end{gathered}$$

$$\begin{aligned}\Rightarrow\ &\rho_H\Big(\chi^*[t_0^*,t_1^*],\chi[t_0,t_1]\Big)<\varepsilon; \\ &\rho_H\Big(\chi^*(t_{i-1}^*,t_i^*],\chi(t_{i-1},t_i]\Big)<\varepsilon\,,\ i=1,2,\ldots,k\,; \\ &\rho_H\Big(\chi^*(t_k^*,T],\chi(t_k,T]\Big)<\varepsilon.\end{aligned} \tag{27}$$

The next inequalities are satisfied:

$$\chi[t_0,T]=\chi[t_0,t_1]\cup\Big(\bigcup_{i=2,\ldots,k}\chi(t_{i-1},t_i]\Big)\cup\chi(t_k,T]$$

and

$$\chi^*[t_0^*, T] = \chi^*[t_0^*, t_1^*] \cup \Big(\bigcup_{i=2,\dots,k} \chi^*(t_{i-1}^*, t_i^*] \Big) \cup \chi^*(t_k^*, T].$$

Using Theorem 1 and the estimates (27), we obtain

$$\begin{aligned}
&\rho_H\Big(\chi^*[t_0^*, T], \chi[t_0, T]\Big) \\
&\quad = \rho_H\Big(\chi^*[t_0^*, t_1^*] \cup \Big(\bigcup_{i=2,\dots,k} \chi^*(t_{i-1}^*, t_i^*] \Big) \cup \chi^*(t_k^*, T], \\
&\qquad\quad \chi[t_0, t_1] \cup \Big(\bigcup_{i=2,\dots,k} \chi(t_{i-1}, t_i] \Big) \cup \chi(t_k, T]\Big) \\
&\quad \le \max\Big\{\rho_H\Big(\chi^*[t_0^*, t_1^*], \chi[t_0, t_1]\Big), \\
&\qquad\quad \rho_H\Big(\chi^*(t_{i-1}^*, t_i^*], \chi(t_{i-1}, t_i]\Big),\ i = 2, ..., k, \\
&\qquad\quad \rho_H\Big(\chi^*(t_k^*, T], \chi(t_k, T]\Big)\Big\} \\
&\quad < \varepsilon.
\end{aligned}$$

The theorem is proved. □

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