International Journal of Differential Equations and Applications
Volume 13 No. 3 2014, 177-183
ISSN: 1311-2872
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijdea.v13i3.1753

acadpubl.eu

ABEL'S FORMULA AND WRONSKIAN FOR CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we discuss and present the form of the Wronskian for conformable fractional linear differential equations with variable coefficients. Further, we prove that there is an Able's formula for fractional differential equations with variable coefficients.

AMS Subject Classification: 26A33
Key Words: Wronskian, Abel formula, fractional derevative

1. Introduction

Consider the differential equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{1}$$

It is known that such equation has two linearly independent solutions, say y_1 and y_2. The Wronskian of the solutions is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_2' & y_2' \end{vmatrix}.$$

It is not easy to solve equation (1) in general. But if one knows one solution say y_1, then Abel's formula gives the second solution y_2 in terms of the first

Received: April 28, 2014

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solution y_1:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx.$$

In this paper we discuss equation (1) when the derivative is replaced by conformable fractional derivative. We study the fractional Wronskian, and the fractional Abel's formula. It turns out that conformable derivative gives results similar to the classical case of ordinary differential equations.

For $0 < \alpha \leq 1$, let $D^\alpha y$ denote the α-conformable fractional derivative of y. We will discuss the equation

$$D^\alpha D^\alpha y + P(x) D^\alpha y + Q(x) y = 0. \tag{2}$$

We prove a similar formula for the fractional Wronskian of (2) compatible with the classical case. Further, we prove a fractional Abel's formula for (2). Some applications of the construction is given.

2. Preliminaries

The subject of fractional derivative is as old as calculus. In 1695, L'Hopital asked if the expression $\frac{d^{0.5}}{dx^{0.5}} f$ has any meaning. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives. These days, many definitions for the fractional derivative are available. Most of these definitions use an integral form. The most popular definitions are:

(i) Riemann-Liouville Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α^{th} derivative of f is given by

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo Definition: For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(i) The Riemann-Liouville derivative **does not** satisfy $D_a^\alpha(1) = 0$ ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives **do not** satisfy the known product rule:

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f).$$

(iii) All fractional derivatives **do not** satisfy the known quotient rule:

$$D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives **do not** satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)\, g^{(\alpha)}(t).$$

(v) All fractional derivatives **do not** satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general

(vi) Caputo definition **assumes that the function f is differentiable**. Further one can see that $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

In [3], a new definition called conformable fractional derivative was introduced. The new definition satisfies:

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
2. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0, 1]$ and and f, g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$. Then

3. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
4. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$

We list here the fractional derivatives of certain functions,

5. $T_\alpha(t^p) = p\, t^{p-\alpha}$.
6. $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
7. $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.
8. $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives. One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(t) = 1$. Hence $T_{\frac{1}{2}}(f)(0) = 1$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

We refer to [1,2] for the very recent results on conformable derivative. For classical results on fractional calculus we refer to [4], [5], [6], and [7].

3. The Wronskian

In this section we discuss the Fractional wronskian of two functions. As in [4], we let $I_\alpha(f) = \int \frac{f(x)}{x^{1-\alpha}} dx$. It is known (see [5]), that $D^\alpha I_\alpha(f) = f$.

Definition 2.1. For two functions y_1 and y_2 satisfying (2) and $0 < \alpha \leq 1$, we set

$$W_\alpha[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ D^\alpha y_1 & D^\alpha y_2 \end{vmatrix}.$$

Theorem 2.2. $W_\alpha[y_1, y_2] = e^{-I_\alpha(P)}$.

Proof. We apply the operator D^α on $W_\alpha[y_1, y_2]$ to get

$$\begin{aligned} D^\alpha(W_\alpha[y_1, y_2]) =& D^\alpha(y_1 D^\alpha y_2 - y_2 D^\alpha y_1) \\ =& D^\alpha y_1 D^\alpha y_2 + y_1 D^\alpha D^\alpha y_2 - D^\alpha y_2 D^\alpha y_1 - y_2 D^\alpha D^\alpha y_1 \end{aligned}$$

But, y_1 and y_2 satisfies (2). Hence

$$D^\alpha D^\alpha y_1 = -P(x) D^\alpha y_1 - Q(x) y_1$$

and

$$D^\alpha D^\alpha y_2 = -P(x) D^\alpha y_2 - Q(x) y_2.$$

Therefore:

$$\begin{aligned} D^\alpha(W_\alpha[y_1, y_2]) =& - P(x)(y_1 D^\alpha y_2 - y_2 D^\alpha y_1) \\ =& - P(x) W_\alpha[y_1, y_2]. \end{aligned}$$

Thus

$$\frac{D^\alpha(W_\alpha[y_1, y_2])}{W_\alpha[y_1, y_2]} = -P(x).$$

Consequently,

$$W_\alpha[y_1, y_2] = I_\alpha(-P(x)). \tag{3}$$

This completes the proof. $\square$

4. Abel's Formula

First we need to discuss linear fractional differential equations:

$$D^\alpha y + a(x)y = b(x), \tag{4}$$

where, $0 < \alpha \leq 1$.

Multiply (4) by $e^{I(a)}$ to get

$$e^{I(a)} D^\alpha y + e^{I(a)} a(x)y = e^{I(a)} b(x).$$

So using results on conformable fractional derivatives from [4], we get

$$D^\alpha(e^{I(a)} y) = e^{I(a)} b(x).$$

Hence

$$y = e^{-I(a)} I_\alpha(e^{I(a)} b(x)) \tag{5}$$

is a solution of (4).

Now, let y_1 be a solution of (2). Our goal is to find a second solution y_2 for equation (2).

To do this, we have from (3), $W_\alpha[y_1, y_2] = I_\alpha(-P(x))$, from which we get

$$y_1 D^\alpha y_2 - y_2 D^\alpha y_1 = I_\alpha(-P(x)),$$

and so

$$D^\alpha y_2 - y_2 \frac{D^\alpha y_1}{y_1} = \frac{I_\alpha(-P(x))}{y_1}. \tag{6}$$

Equation (6) is a fractional linear equation, with $a(x) = \frac{D^\alpha y_1}{y_1}$, and $b(x) = \frac{I_\alpha(-P(x))}{y_1}$. Hence, using the fact

$$I_\alpha\left(\frac{D^\alpha y_1}{y_1}\right) = \ln y_1,$$

and formula (5) to receive

$$y_2 = y_1 I_\alpha\left(\frac{e^{-I_\alpha(P)}}{y_1^2}\right). \tag{7}$$

5. Some Applications

(1) Consider the differential equation

$$D^{\frac{1}{2}}D^{\frac{1}{2}}y+\sqrt{x}D^{\frac{1}{2}}y=0.$$

Clearly, $y_1 = 1$ is a solution of such equation. Using formula (7) and the definition of $I_{\frac{1}{2}}(f)$ to get

$$y_2=y_1 I_{\frac{1}{2}}\left(\frac{e^{-I_{\frac{1}{2}}(P)}}{y_1^2}\right)=I_{\frac{1}{2}}\left(e^{-x}\right).$$

Clearly, y_2 satisfies the above equation.

(2) Consider the differential equation

$$D^{\frac{1}{2}}D^{\frac{1}{2}}y+\frac{1}{2}\tan\sqrt{x}D^{\frac{1}{2}}y=0.$$

Clearly, $y_1 = 1$ is a solution of such equation, noting that $D^{\alpha}1 = 0$,, see [2]. Hence using formula (7), to get

$$y_2=y_1 I_{\frac{1}{2}}\left(\frac{e^{-I_{\frac{1}{2}}(P)}}{y_1^2}\right)=I_{\frac{1}{2}}\left(e^{-I_{\frac{1}{2}}(\frac{1}{2}\tan\sqrt{x})}\right).$$

But using formulas (5), (6), and (7) in the preliminaries section we get $y_2 = I_{\frac{1}{2}}(\cos\sqrt{x})$. Using using formulas (5), (6), and (7) in the preliminaries section again we see that such y_2 is a solution of the equation.

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