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ON VOLTERRA TYPE INTEGRAL EQUATIONS IN HAUSDORFF SPACES

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Abstract: In this article we consider a generalization of the linear and non-linear Volterra integral equations of first and second kind in the case when the independent variable belongs to arbitrary first-countable Hausdorff space. Sufficient conditions for the uniqueness of the solutions of the Volterra type integral equations of first kind are obtained. As an application of this result the existence of solution of nonlinear Volterra type integral equations of second kind is proved.

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Key Words: Volterra type integral equations, First-countable Hausdorff space, Banach space

1. Introduction

Integral equations and their application have been of considerable significance of mathematics investigations and a lot of results are obtained in the last few decades (see [1], [2], [10] and the references there in). Consumer of this results are several applied fields, such as population dynamics, spread of epidemics, automatic control theory, network theory and the dynamics of nuclear reactors. The main idea of this paper is to develop the approach introduced for the linear

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case in the work [3] for the nonlinear one. The general idea introduced in [3] is to establish conditions under which the eventually unique solution of some integral equations is an upper bound of all solutions of the corresponding inequality. In view of the applications it is important to study equations and inequalities in the case when the domains of integration are arbitrary compact sets and not necessary Cartesian product of bounded and closed intervals. Following this concept it is obviously that the first step is the problem of the solvability of the integral equations. Note that this approach is used in [7], [11], [12], [13], [14] and some interesting ideas concerning equations with Volterra type operators and their applications are developed in [4] and [9].

The work is structured by following way:

In Section 2 we discuss the possibility to replace without loss of generality a compact domain of integration with another one which has the same measure, but better properties.

Section 3 is dedicated to prove the uniqueness of the solution of the integral equations of the first kind.

In Section 4, using an analog of the Fredholm alternative for nonlinear operators developed in [6], we apply the received in Theorem 3.3 of Section 3 result for the study of the second kind Volterra type integral equation introduced below.

2. Preliminaries

Let Ω be a first-countable Hausdorff space and B be a real Banach space with norm $||.||_B$. $B_\Omega \subset 2^\Omega$ denotes the σ-algebra of the Borel subsets of Ω and let $\mu : B_\Omega \to [0, \infty]$ be a nontrivial, nonatomic σ-finite Borel measure.

Let $G \in B_\Omega$ be an arbitrary set.

Definition 2.1. The point $x \in \Omega$ will be called essential for the set G, if for every open neighborhood $O(x)$ of x with $\mu(O(x)) > 0$ it follows that $\mu(G \cap O(x)) > 0$. Otherwise the point $x \in \Omega$ will be called nonessential.

We will denote by G^μ the set of all points $x \in G$, which are essential for G and by G^ν the set $G^\nu = G \setminus G^\mu$ including all nonessential points belonging to G.

Definition 2.2. [7] The sets $G, H \in B_\Omega$ will be called μ-equivalent $G \sim^\mu H$, if $\mu(G \Delta H) = 0$, where $G \Delta H = (G \setminus H) \cup (H \setminus G)$ is the symmetric difference.

Definition 2.3. [7] The set $G \in B_\Omega$ is called μ-dense, if each $x \in G$ is an essential point for the set G.

Definition 2.4. [7] We say that the set $G \subseteq \Omega$ is M-star if for every $x \in G$ the inclusion $M_x \subseteq G$ holds.

Remark 2.5. From definition 2.1 it follow that for arbitrary $G \in B_\Omega$ with $\mu(G) > 0$ every external or isolated point of G is nonessential and every internal point is essential for G. Than it follows if $\mu(G) = 0$ and $G \neq \emptyset$ then G does not include essential points. Moreover, if the point $x \in \Omega$ is nonessential for the set G, then there exists an open neighborhood $O(x)$ of x with $\mu(O(x)) > 0$ such that $\mu(G \cap O(x)) = 0$.

As in [3], we introduce the map $M : \Omega \to 2^\Omega$, which associates every point $x \in \Omega$ with a closed subset $M_x \subset \Omega$.

We will say that the conditions (A) hold if for the map $M : \Omega \to 2^\Omega$ the following conditions are fulfilled:

A1. For every point $x \in \Omega$ the set M_x is compact.

A2. For each $\epsilon > 0$ and every $x \in \Omega$, there exists an open neighborhood $O(x, \epsilon)$ of x, such that for each $y \in O(x, \epsilon)$ we have that $\mu(M_x \Delta M_y) < \epsilon$.

A3. For every $x \in \Omega$ the inclusion $M_y \subseteq M_x$ holds for each $y \in M_x$.

A4. There exists $x_0 \in \Omega$ such that $\mu(M_{x_0}) = 0$.

For every map $M : \Omega \to 2^\Omega$ for which the conditions (A) hold, we will denote $\mathbf{M} = \{M_x | x \in \Omega\}$ and $KerM = \{x \in \Omega | \mu(M_x) = 0\}$.

Remark 2.6. In general, the conditions (A) can be fulfilled for the map $M : \Omega \to 2^\Omega$, but $x \notin M_x$. A simply example is $\Omega = [0, \infty)$ and $M_x = [0, \alpha x]$ for some $\alpha \in (0, 1)$.

Remark 2.7. It is easy to see that condition $A3$ implies that for each $x \in \Omega$ the set M_x is M-star set. Moreover, the union and the intersection of an arbitrary family of M-star sets are M-star set. The sets $KerM$ and $M_\Omega = \bigcup\limits_{x \in \Omega} M_x$ are M-star sets too.

Let $\Omega_* \subseteq \Omega$ be an arbitrary set and denote by $C(\Omega_*, B)$ the linear space of all continuous mappings $\varphi : \Omega_* \to B$. Obviously when $\Omega_* \subseteq \Omega$ is a compact set, then $C(\Omega_*, B)$ is a Banach space with norm $||f||_{\Omega_*} = \sup\limits_{y \in \Omega_*} ||f(y)||_B$.

Consider the equations

$$f(x) = p(x) + \int_{M_x} Q(x, y, f(y)) \mathrm{d}\mu_y \tag{2.1}$$

$$f(x) = \lambda \int_{M_x} Q(x, y, f(y)) \mathrm{d}\mu_y \tag{2.2}$$

where the operator $Q : \Omega \times \Omega \times B \to B,\ \lambda \in R$ and $f, p \in C(M_\Omega, B)$.

Remark 2.8. If the operator Q is continuous in the set $\Omega \times M_\Omega \times B$ and for the map $M : \Omega \to 2^\Omega$ the condiion (A) hold, then the Bochner integral in (2.1) and (2.2) exists on each $M_x \subset \mathbf{M}$ (see [8], Chapter 3).

It is well known that even for finite dimensional metric spaces Ω, two compact sets M_x and M_y can be very close (even equal) in measure sense but the difference between the diameters of these sets can be arbitrary large. The aim of the next theorem is for every compact set $M_x \in \mathbf{M}$ to find a μ-dense compact set M_x^μ, such that the sets M_x and M_x^μ to be μ-equivalent.

Theorem 2.9. *[12] Let G be a compact subset of Ω and $\mu(G) > 0$.*

Then the set G^μ is a nonempty compact set and the sets G and G^μ are μ-equivalent.

Theorem 2.10. *[12] Let for the map $M : \Omega \to 2^\Omega$ the conditions (A) hold and lets Ω be a connected space with $0 < \mu(\Omega) \leq \infty$.*

Then for every $x \in \Omega$ we have that $\mu(M_x) < \infty$.

Besides, if all sets M_x are connected, then the set $M_\Omega = \bigcup\limits_{x \in \Omega} M_x$ is connected too.

Remark 2.11. The statement of Theorem 2.9 was proved in [14] in the case when Ω is arbitrary complete metric space.

Since the compact sets $M_x \in \mathbf{M}$ will be used as domains of integration then without loss of generalization using Theorem 2.9 we can assume that all sets M_x are μ-dense.

Everywhere below in our exposition we will assume that the sets M_x are μ-dense.

Lemma 2.12. *Let for the map $M : \Omega \to 2^\Omega$ the conditions (A) hold and Ω be a connected space with $0 < \mu(\Omega) \leq \infty$.*

Then for every $x \in \Omega$ and every $\epsilon \in (0, \mu(M_x))$ there exists an open neighborhood $O(x, \epsilon)$ of x, such that for each $y \in O(x, \epsilon)$ we have that

$$\mu(M_x \cap M_y) \geq \mu(M_x) - \epsilon.$$

Proof. Let $x \in \Omega$ be an arbitrary point.First we will prove that there exists an open neighborhood $O(x)$ of x, such that for each $y \in O(x)$ we have $M_x \cap M_y \neq \emptyset$. Assume the contrary,there exists a sequence $\{y_n\}_{n \in N} \subset \Omega$ such that $\lim\limits_{n \to N} y_n = x$ and number $n_0 \in N$, such that for each $n \geq n_0$ we have $M_x \cap M_y = \emptyset$. Therefore $\mu(M_x \Delta M_{y_n}) > \mu(M_x)$ for each $n \geq n_0$.

From other side condition A2 implies that there exits number $n_1 \geq n_0$, such that $\mu(M_x \Delta M_{y_n}) < 2^{-1}\mu(M_x)$ for every $n \geq n_1$ which is impossible.

Therefore $M_x \cap M_y \neq \emptyset$ for each $y \in O(x)$. Condition A2 implies that for every $\epsilon \in (0, \mu(M_x))$ there exists an open neighborhood $O(x, \epsilon) \subseteq O(x)$ of x, such that for each $y \in O(x, \epsilon)$ we have $\mu(M_x \Delta M_{y_n}) < \epsilon$, which implies that $\mu(M_x \cap M_y) \geq \mu(M_x) - \epsilon$. □

Theorem 2.13. *Let for the map $M : \Omega \to 2^\Omega$ the conditions (A) hold and Ω is a connected space with $0 < \mu(\Omega) \leq \infty$.*

Then for every $x \in \Omega$ the following statements hold $M_x \cap KerM \neq \emptyset$ for every $x \in \Omega$.

Proof. Assume that there exists some $y \in \Omega$, such that $M_y \cap KerM = \emptyset$. From condition A2 it follows that the function $\mu \circ M : \Omega \to [0, \infty)$ is continuous in Ω. Since M_y is compact, there exists $z \in M_y$, such that $\mu(M_z) = \min\limits_{x \in M_y} \mu(M_x)$ and $\mu(M_z) > 0$. Then from condition A3 it follows that for every $x \in \Omega$ for which $0 \leq \mu(M_x \cap M_z) < \mu(M_z)$ and $M_x \cap M_z \neq \emptyset$ there exists a point $s \in M_x \cap M_z$, such that $M_s \subset M_z$ and $\mu(M_s) \leq \mu(M_x \cap M_z) < \mu(M_z)$ which is impossible. Therefore for every point $x \in \Omega$ we have that either $0 \leq \mu(M_x \cap M_z) < \mu(M_z)$ and then $M_x \cap M_z = \emptyset$, or $\mu(M_x \cap M_z) = \mu(M_z)$. Since the sets M_x and M_z are μ-dense, then from condition A2 and the relation $\mu(M_x \cap M_z) = \mu(M_z)$ it follows that$M_z \subset M_x$.

Denote $G = \{x \in \Omega | M_z \subseteq M_x\}$ and $H = \Omega \setminus G$. We will proof that

G is closed. According our assumption $G \neq \emptyset$ and let $\{x_n\}_{n\in N}$ be an arbitrary convergent sequence, i.e. $\lim_{n\to\infty} x_n = s$, $s \in \Omega$.Condition A2 implies that there exits $n_0 \in N$ such that for every $n \geq n_0$, we have that the inequality $\mu(M_{x_n} \Delta M_s) < \frac{\mu(M_z)}{2}$ holds. If we assume that $M_s \cap M_z = \emptyset$ then we have that the estimation $\mu(M_{x_n} \Delta M_s) \geq \mu(M_{x_n} \backslash M_s) \geq \mu(M_z \backslash M_s) \geq \mu(M_z)$ holds, which is impossible. Therefore $M_z \subseteq M_s$, i.e. $s \in G$. Since $H \neq \emptyset$ and Ω is connected, then H cannot be a closed set. There exist a point $s \in \partial G$ with $M_s \supset M_z$ and sequence $\{y_n\}_{n\in N} \subseteq H$ such that $\lim_{n\to\infty} y_n = s$ and therefore there exits $n_0 \in N$, such that for $n \geq n_0$, we have $\mu(M_{y_n} \Delta M_s) < \frac{\mu(M_z)}{2}$. Since for $n \geq n_0$ we have $M_{y_n} \cap M_z = \emptyset$, then the inequalities $\mu(M_{y_n} \Delta M_s) \geq \mu(M_s \setminus M_{y_n}) \geq \mu(M_z)$ hold, which is impossible. $\square$

3. Main Results

The aim of our consideration in Section 3 is to prove thatuniqueness of the solution of the integral equation of first kind (2.2). In our discussion bellow we will assume that for the map $M : \Omega \to 2^\Omega$ the condition (A) hold and for the operator $Q : \Omega \times \Omega \times B \to B$ some of the following conditions are fulfilled:

S1. The operator $Q : \Omega \times \Omega \times B \to B$ is continuous in the set $\Omega \times M_\Omega \times B$.

S2. For every M- star continuum $\Omega_* \subset M_\Omega$ with $\mu(\Omega_*) < \infty$ there exists a constant $L(\Omega_*) > 0$ such that for every function $f \in C(\Omega_*, B)$ the inequality

$$\sup_{x\in\Omega_*,\, y\in M_x} ||Q(x, y, f(y))||_B \leq L(\Omega_*)||f||_{\Omega_*} \tag{3.1}$$

holds.

S3. The relations $Q(x, y, -u) = -Q(x, y, u)$ and $Q(x, y, tu) = tQ(x, y, u)$ hold for each $t > 0$ and every $(x, y, u) \in M_\Omega \times M_\Omega \times B$.

Consider the operator K defined by the equation

$$Kf(x) = \int_{M_x} Q(x, y, f(y))\mathrm{d}\mu(y) \tag{3.2}$$

where $f \in C(M_\Omega, B)$.

Lemma 3.1. *Let the following conditions are fulfilled:*

1. For the map $M : \Omega \to 2^{\Omega}$ the conditions (A) hold.

2. The condition S1 holds,the space Ω is connected and $0 < \mu(\Omega) \leq \infty$.

3. For each $x \in \Omega$ the sets M_x are connected and there exists a M-star continuum F_x with $\mu(F_x) < \infty$, such that $x \in intF_x$ and $M_x \subset intF_x$.

Then the operator K defined by (3.2) maps $C(M_\Omega, B)$ into $C(M_\Omega, B)$.

Proof. Let $x_0 \in M_\Omega$ be an arbitrary fixed point and let $\{x_n\}_{n\in N} \subset M_\Omega$ be an arbitrary sequence such that $\lim\limits_{n\to\infty} x_n = x_0$.If $f \in C(M_\Omega, B)$ is an arbitrary fixed element then

$$||Kf(x_n) - Kf(x_0)||_B \leq \int_{M_{x_n}\cap M_{x_0}} ||Q(x_n, y, f(y)) - Q(x_0, y, f(y))||_B \mathrm{d}\mu_y$$

$$+ \int_{M_{x_n}\setminus M_{x_0}} ||Q(x_n, y, f(y))||_B \mathrm{d}\mu_y + \int_{M_{x_0}\setminus M_{x_n}} ||Q(x_0, y, f(y))||_B \mathrm{d}\mu_y. \quad (3.3)$$

Since $x_0 \in M_\Omega$ there exists a set M_z with $x_0 \in M_z$ and then condition 3 implies that there exists a M-star continuum F_z, such that $M_z \subset intF_z$. Then there exists an open neighborhood $O(x_0)$ of x_0 with $O(x_0) \subset intF_z$. Since $\lim\limits_{n\to\infty} x_n = x_0$ then there exists $n_0 \in N$ such that for each $n \geq n_0$ we have that $x_n \in O(x_0)$ and therefore $M_{x_n} \subseteq F_z$. Then there exists a constant $Q_0(z, f) > 0$, such that $\sup\limits_{(s,y)\in F_z\times F_z} ||Q(s, y, f(y))||_B \leq Q_0(z, f)$. Let $\epsilon > 0$ is arbitrary. From condition S1 it follows that there exists $n_1 \geq n_0$, such that for each $n \geq n_1$ the inequalities $\sup\limits_{n\geq n_1, y\in F_z} ||Q(x_n, y, f(y)) - Q(x_0, y, f(y))||_B < \frac{\epsilon}{3\mu(M_{x_0})}$ and $\mu(M_{x_n}\Delta M_{x_0}) \leq \frac{\epsilon}{3Q_0(z,f)}$ hold. Therefore from (3.3) it follows, that for each $n \geq n_1$ we have $||(Kf)(x_n) - (Kf)(x_0)||_B < \epsilon$. □

The next example shows that if $\Omega = R^n$, $n \geq 1$ then obviously the condition 3 of lemma 3.1 is fulfilled.

Example. Let $\Omega = R^n$, M_x is arbitrary compact set and denote with $a = \min_{y\in M_x} d(x, y) < \infty$, where d is some metric in R^n. Since M_x is bounded, then there exists a n-dimentional cube F_x with edge lenght $b > a$,such that $x \in intF_x$ and $M_x \subset intF_x$.

Definition 3.2. We say that the equation (2.1) ((2.2)) has a local solution in some M-star set $\Omega_* \subset \Omega$ for some $p \in C(\Omega_*, B)$ if there exist a point $x_p \in \Omega_*$

with $\mu(M_{x_p}) > 0$ and a function $f \in C(M_{x_p}, B)$ which satisfies the equation (2.1) for each $s \in M_{x_p} \cup \{x_p\}$. If $p, f \in C(\Omega_*, B)$ and f satisfies the equation (2.1) ((2.2)) for each $x \in \Omega_*$ then we say that f is a solution of (2.1) ((2.2)) in $\Omega_* \subset \Omega$.

Theorem 3.3. *Let the following conditions are fulfilled:*

1. *The conditions of Lemma 3.1 hold.*

2. *The condition S2 holds.*

Then the equation (2.2) *has exactly one solution* $f \in C(M_\Omega, B)$ *for each* $\lambda \in R$.

Proof. From Lemma 3.1 it follows that every solution of equations (2.2) is continuous in M_Ω. Let $\lambda \neq 0, \lambda \in R$ be an arbitrary fixed real number and let assume that the equation (2.2) has two nontrivial solutions $f_1, f_2 \in C(M_\Omega, B)$. Consider the set $Kerf = \{x \in M_\Omega | f(y) = 0, y \in M_x\}$, where $f = f_1 - f_2$.

The set $Kerf \neq \emptyset$. Let $x \in M_\Omega$ be an arbitrary point. Therefore $M_x \subset M_\Omega$ and then from Theorem 2.13 it follows that there exists $x_0 \in M_x$, such that $\mu(M_{x_0}) = 0$. If $M_{x_0} \neq \emptyset$ then from condition A3 it follows that for each $y \in M_{x_0}$ we have $\mu(M_y) = 0$. Therefore we can conclude that $x_0 \in Kerf$.

The set $Kerf$ is M-star. Let $x \in Kerf$ and $z \in M_x$ be arbitrary. Then $M_z \subset M_x$ and hence $f(y) = 0$ for each $y \in M_z$. Thus $M_x \subset Kerf$.

The set $Kerf$ is closed. Let $\{x_n\}_{n \in N} \subset Kerf$ be an arbitrary convergent sequence, i.e. $\lim\limits_{n\to\infty} x_n = x_0$ and $x_0 \in M_\Omega$ If for arbitrary $z \in M_{x_0}$ we have $\mu(M_z) = 0$ then $f(z) = 0$, $z \in Kerf$ and therefore $x_0 \in Kerf$. Let for some $z \in M_{x_0}$ we have $\mu(M_z) > 0$. Then for each $\epsilon > 0$ there exists an open neighborhood $O(z, \epsilon)$ of z with $\mu(O(z, \epsilon)) > 0$, such that for every $y \in O(z, \epsilon)$ we have $||f(y) - f(z)||_B < \epsilon$. Since z is an essential point for M_{x_0} then we have that $\mu(M_{x_0} \cap O(z, \epsilon)) > 0$. Condition A2 and Lemma 2.12 imply that there exists $n_0 = n_0(z, \epsilon, x_0) \in N$, such that for each $n \geq n_0$ we have that $\mu(M_{x_0} \cap M_{x_n}) \geq \mu(M_{x_0}) - \frac{\mu(M_{x_0} \cap O(z,\epsilon))}{2} > 0$ and therefore $M_{x_0} \cap M_{x_n} \cap O(z, \epsilon) \neq \emptyset$. Then there exists at least one point $y_n \in M_{x_0} \cap M_{x_n} \cap O(z, \epsilon)$ for each $n \geq n_0$. Since $M_{x_n} \subset Kerf$ then $||f(z)||_B < \epsilon$ i.e. $||f(z)||_B = 0$. Thus we proved that $x_0 \in Kerf$.

The set $Kerf$ is open. Let $a \in Kerf$ be an arbitrary point. Then there exists a M-star continuum F_a, such that $a \in intF_a$ and $M_a \subset intF_a$.

Since $Kerf$ is a closed M-star set, then $F_a \cap Kerf$ is a M-star compact set. Let $\epsilon_a > 0$ be an arbitrary number. Condition A2 implies that for each $\epsilon \in$

$(0, \epsilon_a)$, there exists an open neighborhood $O(a, \epsilon)$ of the point a with $O(a, \epsilon) \subset intF_a$ such that for every $x \in O(a, \epsilon)$ we have that $\mu(M_a \Delta M_x) < \epsilon$.

Assume the contrary, that there exists a point $b \in O(a, \epsilon)$ with $b \notin Kerf$. Then there exists a point $x_0 \in M_b$ such that $||f(x_0)||_{M_b} > 0$. Since $O(a, \epsilon) \subset intF_a$, then for each $x \in O(a, \epsilon)$ we have that $\mu(M_x) < \mu(F_a)$. Denote by $L(F_a) > 0$ the constant in inequality (3.1) of condition S2 and by $g(x), g_1(x), g_2(x)$ the restrictions of $f(x), f_1(x), f_2(x)$ over M_b, i.e. $g(x) = f(x)|_{M_b}$, $g_1(x) = f_1(x)|_{M_b}$, $g_2(x) = f_2(x)|_{M_b}$. Let $\epsilon_0 \in (0, \epsilon_a]$ be an arbitrary number which satisfy the inequality $2\epsilon_0|\lambda|L(F_a)\frac{\max\{||f_1||_{F_a}, ||f_2||_{F_a}\}}{||f(x_0)||_B} < \frac{1}{2}$. Then from the conditions (A), condition S1 and S2 we obtain

$$\begin{aligned}
||f(x_0)||_B = ||g(x_0)||_B &= ||\lambda \int_{M_{x_0}} [Q(x_0, y, g_1(y)) - Q(x_0, y, g_2(y))] \, d\mu_y||_B \\
&\leq |\lambda| \sup_{x \in M_b} \int_{M_x} (||Q(x_0, y, g_1(y))||_B + ||Q(x_0, y, g_2(y))||_B) \, d\mu_y \\
&= |\lambda| \sup_{x \in M_b} \int_{M_x \setminus M_a} (||Q(x_0, y, g_1(y))||_B + ||Q(x_0, y, g_2(y))||_B) \, d\mu_y \\
&\leq |\lambda| \sup_{x \in M_b} \mu(M_x \setminus M_a)(||f_1||_{M_x} + ||f_2||_{M_x}) \\
&\leq 2\epsilon_0 L(F_a) \max\{||f_1||_{F_a}, ||f_2||_{F_a}\} < \frac{1}{2}||f(x_0)||_B
\end{aligned} \tag{3.4}$$

From (3.4) it follows that $f(x_0) = 0$ and hence $f(x) = 0$ for each $x \in M_b$. Thus we proved that $b \in Kerf$ and therefore $Kerf$ is open. Theorem 2.10 implies that the set M_Ω is connected and then we can conclude that $Kerf = M_\Omega$. Thus complete the proof. □

Corollary 3.4. *Let the conditions of Theorem 3.3 are fulfilled. Then equation* (2.2) *possess the trivial solution only for each* $\lambda \in R$.

Proof. Condition S2 implies that the function $f(x) \equiv 0, x \in M_\Omega$ is soluton of equation (2.2) for each $\lambda \in R$. Then the srarement of Corollary 3.4 follow from Theorem 3.3. □

4. Applications to the Second Kind Volterra Equations

In the present section we will use the results in the previous section to prove existece of global continouos solutions of equations (2.1). Our proof is essentially based of the following result, announced in [5]:

Theorem 4.1. *[5] Let X and Y are real Banach spaces and the following conditions are fulfilled:*

1. The operator $T : X \to Y$ is an odd homeomorphism and there exist constants $a, K, L > 0$ such that for each $x \in X$ we have that $K||x||_X^a \leq ||Tx||_Y \leq L||x||_X^a$.

2. The odd operators $S, W : X \to Y$ are compact and for each $x \in X$ and $t > 0$ we have that $S(tx) = t^a S(x)$.

3. $\lim\limits_{||x||_X \to \infty} \frac{||W(x)||_Y}{||x||_X^a} = 0$

Then if for every $\lambda \in R, \lambda \neq 0$ the equation $T(x) - \lambda S(x) = 0$ has the trivial solution only then the equation $T(x) - \lambda S(x) - W(x) = y$ has at least one solution for each $y \in Y$.

Theorem 4.2. *Let the following conditions are fulfilled:*

1. The conditions of Theorem 3.3 hold.

2. The condition S3 holds. xms

3. The operator K defined by (3.2) is compact in $C(\Omega_*, B)$ where $\Omega_* \subset M_\Omega$ is some M-star continuum with $\mu(\Omega_*) < \infty$.

Then the equation (2.1) has at least one solution for every $p \in C(\Omega_*, B)$.

Proof. Lemma 3.1 implies that the operator K maps $C(\Omega_*, B)$ into $C(\Omega_*, B)$. Obviously in our case we have that $a = K = L = 1$ and the operator W is equal to zero. Then the statement of the theorem immediately follows from Theorem 3.3 and Corollary 3.4. □

Remark 4.3. The condition 3 of Theorem 4.2 is unnecessarily for its validity, when the space B is finite dimentional and instead the condition S2 for the operator K defined with (3.2) the following condition is fulfilled:

*S*2*. For every M-star continuum $\Omega_* \subset M_\Omega$ with $\mu(\Omega_*) < \infty$ there exists a constant $L(\Omega_*) > 0$, such that for every two functions $f, g \in C(\Omega_*, B)$ the inequality

$$\sup_{x \in \Omega_*,\, y \in M_x} ||Q(x, y, f(y)) - Q(x, y, g(y))||_B \leq L(\Omega_*)||f - g||_{\Omega_*}$$

holds and $Q(x, y, 0) \equiv 0$, when $x \in \Omega_*$, $y \in M_x$.

Then it is not difficult to prove that the operato K is compact.

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