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ERROR ANALYSIS OF EXTENDED EXPONENTIAL GENERAL LINEAR METHODS OF TYPE 524

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Abstract: The inspiration for this paper is drawn from the work of Butcher [1], Calvo and Palencia [3] and Osisiogu and Bazuaye[5]. EEGLM of type **524** is derived and error estimate analysed. The numerical experiments indicate that extended exponential general linear methods of type **524** perform better than the existing methods. Stability analysis reveals that the method is zero stable as the parasitic root lie in a unit disc.

AMS Subject Classification: 76WXX, 76DXX
Key Words: error estimate, stability analysis, exponential methods

1. Introduction

Our interest in this paper is to construct and develop error estimate for **EEGLM** of **TYPE 524**. Although the theory of numerical methods for time integration is well established for a general class of problems, recently due to improvements in the efficient computation of the exponential function, exponential integrators

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for problems

$$u'(t) = Mu(t)) + N(u(t)), \quad 0 \leq t \leq T, \quad \text{given } u(0),$$

have emerged as a viable alternative. In the early fifties, the phenomenon of stiffness was first discovered by Curtis and Hirschfelder [4]. The stiffiness effectively yields explicit integrators useless, as stability rather than accuracy governs how the integrator performs. It could be said that more integrators have been developed to overcome the phenomenon of stiffiness, than any other property that a differential equation may have. Butcher and Wright [2], provided a novel approach to solving stiff problems. Integrators were constructed, which solve exactly the linear part of the problem and then used a change of variables to cast the problem in a form, which a traditional explicit method can be used to solve the transformed equation, the approximate solution is then back transformed. These methods are commonly known as Integrating Factor (IF) methods. Calvo and Palencia [3] constructed a class of methods known as the RKMK methods. Enright and Muir [5] introduced the Generalized Integrating Factor (GIF methods). Recently Osisiogu and Bazuaye [5] extended the general linear methods as developed by the above researchers to higher degree of the exponential and its related matrix functions.The order conditions derived which open possibilities for the construction of methods of higher order

2. Mathematical Formulation

In the past few years the exponential general linear methods for the problem

$$u'(t) = Mu(t)) + N(u(t)), \quad 0 \leq t \leq T, \text{ given } \ u(0) \tag{1}$$

have attracted a lot of interest. Hence, this paper dwells on the construction of extended exponential general linear methods (EEGLM) of type **524** for the autonomous problem (1). For given starting values $u_0, u_1, \cdots u_{q-1}$, the theoretical approximation u_{n+1} at time t_{n+1}, $n \leq q-1$, is given by the recurrence relation or formula

$$u_{n+1} = e^{hl}u_n + h\sum_{i=1}^{s} B_i(hL)N(U_{ni}) + h\sum_{k=1}^{q-1} V_k(hL)N(u_{n-k}) \ . \tag{2}$$

The internal stages U_{ni}, $1 \leq i \leq s$, are defined through

$$U_{ni} = e^{c_i hl} u_n + h \sum_{j=1}^{i-1} A_{ij}^{(1)}(hl) N(u_{nj}) + h \sum_{k=1}^{q-1} W_{ik}^{(1)}(hl) N(u_{n-k}) + h^2 \sum_{j=1}^{i-1} A_{ij}^{(2)}(hl) N'(u_{nj}) + h^2 \sum_{k=1}^{q-1} W_{ik}^{(2)}(hl) N'(u_{n-k}). \quad (3)$$

Throughout this paper we assume that these conditions are satisfied. The coefficients can be represented in the following table:

$A_{21}^{(1)}$	$W_{21}^{(1)} \cdots W_{2,q-1}^{(1)}$	$A_{21}^{(2)}$	$W_{21}^{(2)} \cdots W_{2,q-1}^{(2)}$
$\vdots \ddots$	$\vdots \cdots \vdots$	$\vdots \ddots$	$\vdots \cdots \vdots$
$A_{s1}^{(1)} \cdots A_{s,s-1}^{(1)}$	$W_{s1}^{(1)} \cdots W_{s,q-1}^{(1)}$	$A_{s,s-1}^{(2)} \cdots A_{s,s-1}^{(2)}$	$W_{s1}^{(2)} \cdots W_{s,q-1}^{(2)}$
$B_1 \cdots B_{s-1} B_s$	$V_1 \cdots V_{q-1}$		

Table 1: Coefficient table

Osisiogu and Bazuaye obtained the order conditions for extended exponential general linear methods (**EEGLM**) which opens up possibilities of deriving methods of higher order.so, they obtained the order conditions as:

$$c_i^{\ell} \psi_{\ell}(c_i hL) = \sum_{j=1}^{i-1} \frac{c_j^{\ell-1}}{(\ell-1)!} A_{ij}^{(1)}(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!} U_{ik}^{(1)}(hL) + \sum_{j=1}^{i-1} \frac{c_j^{\ell-2}}{(\ell-2)!} A_{ij}^{(2)}(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-2}}{(\ell-2)!} U_{ik}^{(2)}(hL), \quad (4)$$

$$\psi_{\ell}(hL) = \sum_{i=1}^{s} \frac{c_j^{\ell-1}}{(\ell-1)!} B_i(hL) + \sum_{k=1}^{q-1} \frac{(-k)^{\ell-1}}{(\ell-1)!} V_k(hL), \quad (5)$$

and so by definition $c_i = 0$ for all $1 \leq i \leq s$.

3. Construction of Extended Exponential General Linear Methods Order Five Step Two Stage Order Four

The extended exponential general linear methods order five step two stage order four (known as methods 524) are given as

$$u_{n+1} = e^{hL}u_n + hB_1(hL)N(u_n) + hB_2(hL)N(W_{n2}) + hV_1N(u_{n-1}) + hV_2N(u_{n-2}) + hV_3N(u_{n-3}), \quad (6)$$

$$U_{n2} = e^{c_2hL}y_n + hA_{21}N(u_n) + hW_{21}^{(1)}N(u_{n-1}) + hW_{22}^{(1)}N(u_{n-2}) + hW_{23}^{(1)}N(u_{n-3}) + h^2A_{21}^{(2)}N'(u_n) + h^2W_{21}^{(2)}N'(u_{n-1}) + h^2W_{22}^{(2)}N'(u_{n-2}) + h^2W_{23}^{(2)}N'(u_{n-3}). \quad (7)$$

Using the order conditions (4) and (5) to determine the coefficient matrix, we obtain:

$$\begin{aligned}
c_1^1A_{21}^{(1)} + (-1)^1W_{21}^{(1)} + (-2)^1W_{22}^{(1)} + (-3)^1W_{23}^{(1)} + c_1^0A_{21}^{(2)} \\
+(-1)^0W_{21}^{(2)}(-2)^0W_{22}^{(2)} + (-3)^0W_{23}^{(2)} &= \psi_2, \\
-W_{21}^{(1)} - 2W_{22}^1 + A_{21}^{(2)} + W_{21}^{(2)} + W_{22}^{(2)} + W_{23}^{(2)} &= \psi_2, \quad (8) \\
c_1^2\frac{1}{2!}A_{21}^{(1)} + \frac{(-1)^2}{2!}W_{21}^{(1)} + \frac{(-2)^2}{2!}W_{22}^{(1)} + \frac{(-3)^2}{2!}W_{23}^{(1)} \\
+(-1)^1W_{21}^{(2)} + (-2)^{(1)}W_{22}^{(2)} + (-3)^1W_{23}^{(2)} &= \psi_3, \\
\frac{1}{2}W_{21}^{(1)} + 2W_{22}^{(1)} + \frac{9}{2}W_{23}^{(1)} - W_{21}^{(1)} - 2W_{22}^{(2)} - 3W_{23}^{(2)} &= \psi_3, \quad (9) \\
\frac{1}{3!}c_1^3A_{21}^{(1)} + \frac{(-1)^3}{3!}W_{21}^{(1)} + \frac{(-2)^3}{3!}W_{22}^{(1)} + \frac{(-3)^3}{3!}U_{23}^{(1)} \\
+\frac{(-1)^2}{2!}W_{21}^{(2)} + \frac{(-2)^2}{3!}W_{22}^{(2)} + \frac{(-3)^2}{3!}W_{23}^{(2)} &= \psi_4, \\
\frac{-1}{3!}W_{21}^{(1)} + \frac{-8}{3!}W_{22}^{(1)} + \frac{-27}{3!}W_{23}^{(1)} + \frac{1}{2!}W_{21}^{(2)} + 2W_{22}^{(2)} + \frac{9}{2}W_{23}^{(2)} &= \psi_4, \quad (10) \\
\frac{1}{4!}c_1^4A_{21}^{(1)}\frac{(-1)^4}{4!}W_{21}^{(1)} + \frac{(-2)^4}{4!}W_{22}^{(1)} + \frac{(-3)^4}{4!}W_{23}^{(1)}, \\
\frac{1}{3!}c_1^3A_{21}^{(2)}\frac{(-1)^3}{3!}W_{21}^{(2)} + \frac{(-2)^3}{3!}W_{22}^{(2)} + \frac{(-3)^3}{3!}W_{23}^{(2)} &= \psi_5, \\
\frac{1}{4!}W_{21}^{(1)} + \frac{-16}{4!}W_{22}^{(1)} + \frac{-81}{4!}W_{23}^{(1)} + \frac{-1}{3!}W_{21}^{(2)} + \frac{-8}{3!}W_{22}^{(2)} + \frac{-27}{3!} &= \psi_5, \quad (11)
\end{aligned}$$

$$\frac{1}{5!}c_1^5A_{21}^{(1)}+\frac{(-1)^5}{5!}W_{21}^{(1)}+\frac{(-2)^5}{5!}W_{22}^{(1)}+\frac{(-3)^5}{5!}W_{23}^{(1)}$$
$$+\frac{(-1)^4}{4!}W_{21}^{(2)}+\frac{(-2)^4}{4!}W_{22}^{(2)}+\frac{(-3)^4}{4!}W_{23}^{(2)}=\psi_6,$$
$$\frac{(-1)}{5!}W_{21}^{(1)}-\frac{32}{5!}W_{22}^{(1)}-\frac{243}{5!}W_{23}^{(1)}+\frac{1}{4!}W_{21}^{(2)}+\frac{16}{4!}W_{22}^{(2)}+\frac{81}{4!}W_{23}^{(2)}=\psi_6, \quad (12)$$
$$\frac{1}{6!}c_1^6A_{21}^{(1)}+\frac{(-1)^6}{6!}W_{21}^{(1)}+\frac{(-2)^6}{6!}W_{22}^{(1)}+\frac{(-3)^6}{6!}W_{23}^{(1)}$$
$$+\frac{(-1)^5}{5!}W_{21}^{(2)}+\frac{(-2)^5}{5!}W_{22}^{(2)}+\frac{(-3)^5}{5!}W_{23}^{(2)}=\psi_7,$$
$$\frac{1}{6!}W_{21}^{(1)}+\frac{64}{6!}W_{22}^{(1)}+\frac{729}{6!}W_{23}^{(1)}+\frac{1}{5!}W_{21}^{(2)}-\frac{32}{5!}W_{22}^{(2)}-\frac{243}{5!}W_{23}^{(2)}=\psi_7. \quad (13)$$

Similarly:

$$c_1^0B_1+c_2^0B_2+(-1)^0V_1+(-2)^0V_2+(-3)^0V_3=\psi_1,$$

therefore

$$B_1+B_2+V_1+V_2+V_3=\psi_1, \quad (14)$$

$$c_1^1B_1+c_2^1B_2+(-1)^1V_1+(-2)^1V_2+(-3)^1V_3=\psi_2,$$

therefore

$$B_2-V_1-2V_2-3V_3=\psi_2; \quad (15)$$
$$\frac{1}{2!}c_1^2B_1+\frac{1}{2!}c_2^2B_2+\frac{1}{2!}(-1)^2V_1+\frac{1}{2!}(-2)^2V_2+\frac{1}{2!}(-3)^2V_3=\psi_3,$$

therefore

$$\frac{1}{2}B_2+\frac{1}{2}V_1+2V_2+\frac{9}{2}V_3=\psi_3; \quad (16)$$
$$\frac{1}{3!}c_1^3B_1+\frac{1}{3!}c_2^3B_2+\frac{1}{3!}(-1)^3V_1+\frac{1}{3!}(-2)^3V_2+\frac{1}{3!}(-3)^3V_3=\psi_4,$$

therefore

$$\frac{1}{6}B_2-\frac{1}{6}V_1-\frac{8}{6}V_2-\frac{27}{6}V_3=\psi_4, \quad (17)$$
$$\frac{1}{4!}c_1^4B_1+\frac{1}{4!}c_2^4B_2+\frac{1}{4!}(-1)^4V_1+\frac{1}{4!}(-2)^4V_2+\frac{1}{4!}(-3)^4V_3=\psi_5,$$
$$\frac{1}{24}B_2+\frac{1}{24}V_1+\frac{16}{24}V_2+\frac{81}{24}V_3=\psi_5. \quad (18)$$

Analogously to the equations above, we have:

$$B_1=\frac{1}{6}(6\psi_1+5\psi_2-10\psi_4-24\psi_5),$$

$$
\begin{aligned}
B_2 =& \frac{1}{12}(3\psi_2 + 11\psi_3 + 18\psi_4 + 12\psi_5),\\
V_1 =& \frac{1}{2}(-3\psi_2 + \psi_3 + 12\psi_4 + 12\psi_5),\\
V_2 =& \frac{1}{6}(3\psi_2 + 2\psi_3 - 18\psi_4 - 25\psi_5),\\
V_3 =& \frac{1}{12}(-\psi_2 - \psi_3 + 6\psi_4 + 12\psi_5),
\end{aligned}
$$

and

$$
\begin{aligned}
W_{21}^{(1)} =& \frac{3}{23}(6\psi_2 + 44\psi_3 + 157\psi_4 + 336\psi_5 + 420\psi_6 + 240\psi_7),\\
W_{22}^{(1)} =& \frac{3}{46}(66\psi_2 + 277\psi_3 + 692\psi_4 + 936\psi_5 + 480\psi_6 - 120\psi_7),\\
W_{23}^{(1)} =& \frac{1}{207}(78\psi_2 + 1124\psi_3 + 2961\psi_4 + 1608\psi_5 - 5580\psi_6 - 7920\psi_7),\\
W_{21}^{(2)} =& \frac{6}{23}(18\psi_2 + 63\psi_3 + 126\psi_4 + 157\psi_5 + 110\psi_6 + 30\psi_7),\\
W_{22}^{(2)} =& \frac{3}{23}(9\psi_2 - 3\psi_3 - 6\psi_4 + 136\psi_5 + 400\psi_6 + 360\psi_7),\\
W_{23}^{(2)} =& \frac{3}{69}(6\psi_2 + 67\psi_3 + 180\psi_4 + 129\psi_5 - 270\psi_6 - 450\psi_7).
\end{aligned}
$$

4. Derivation of the Error Estimate for EEGLM

For simplicity, we introduce the following notations

$$\bar{u} = u_n, \bar{U_{ni}} = y(t_n + c_i h), \quad 1 \le i \le s,\ n \ge 0. \tag{19}$$

Substituting the above into (1) and (2), we have

$$\bar{u}_{n+1} = e^{hL}\bar{u_n} + h\sum_{i=1}^{s} B_i(hL)N(\bar{U_{ni}}) + h\sum_{k=1}^{q-1} u_k(hL)N\bar{u}_{(n-k)} + T_{n+1}. \tag{20}$$

Also the internal stages U_{ni}, $1 \le i \le s$ will now be defined as

$$\bar{U}_{ni} = e^{c_i hL}\bar{u}_n + h\sum_{j=1}^{i-1} A_{ij}^{(1)}(hL)N(\bar{U}_{nj})$$

$$+ h\sum_{k=1}^{q-1} W_{ik}(hL)^{(1)} N(\bar{u}_{(n-k)}) + D_{ni}. \quad (21)$$

Subtracting (1) from (5) we receive

$$\bar{u}_{n+1} - y_{n+1} = e^{hL}[\bar{u_n} - u_n] + h\sum_{i=1}^{s} B_i(hL)[N((\bar{U_{ni}})) - N(U_{ni})] + h\sum_{k=1}^{q-1} V_k(hL)[N(\bar{u}_{(n-k)}) - N(u_{n-k})] + T_{n+1}. \quad (22)$$

Also, subtracting (2) from (6), we obtain

$$\begin{aligned}\bar{U}_{ni} - U_{ni} =& e^{c_i hL}[\bar{u_n} - u_n] + h\sum_{j=1}^{i-1} A_{ij}^{(1)}(hL)[N(\bar{U_{nj}}) - N(U_{nj})] \\ &+ h\sum_{k=1}^{q-1} W_{ik}(hL)^{(1)}[N(\bar{u}_{n-k}) - N(u_{n-k})] \\ &+ h^2\sum_{j=1}^{i-1} A_{ij}^{(2)}(hL)[N'(\bar{U_{nj}}) - N'(U_{nj})] \\ &+ h^2\sum_{k=1}^{q-1} W_{ik}(hL)^{(2)}[N'(\bar{u}_{n-k}) - N'(u_{n-k})] + D_{ni}. \end{aligned} \quad (23)$$

But

$$\begin{aligned}&[\bar{u}_{n+1} - u_{n+1}] = e_{n+1}, \\ &[\bar{u_n} - y_n] = e_n, \\ &[\bar{U}_{ni} - U_{ni}] = E_{ni}, \\ &[N(\bar{U_{nj}}) - N(U_{nj})] = NE_{nj}, \\ &[N'(\bar{u}_{n-k}) - N'(u_{n-k})] = N'E_{nk}, \\ &[N(\bar{u}_{n-k}) - N(u_{n-k})] = NE_{nk}. \end{aligned} \quad (24)$$

Substituting (9) into (7) and (8), we obtain

$$e_{n+1} = e^{hL}e_n + h\sum_{i=1}^{s} B_i(hL)E_{ni} + h\sum_{k=1}^{q-1} V_k(hL)NE_{nk} + T_{n+1}, \quad (25)$$

also (from a simplification of (8)):

$$E_{ni} = e^{c_i hL} e_n + h \sum_{j=1}^{i-1} A_{ij}^{(1)}(hL) N E_{nj} + h \sum_{k=1}^{q-1} W_{ik}(hL)^{(1)} N E_{nk} + h^2 \sum_{j=1}^{i-1} A_{ij}^{(2)}(hL) N' E_{nj} + h^2 \sum_{k=1}^{q-1} W_{ik}^{(2)}(hL) N' E_{nk} + D_{ni}. \quad (26)$$

4.1. Stabilities of Order Five Step Two and Stage Order Four (524) Scheme

We recall order five step two and stage order four (524) Scheme given by

$$u_{n+1} = e^{hL} u_n + hB_1(hL) N(u_n) + hB_2(hL) N(U_{n2}) + hV_1 N(u_{n-1}) + hV_2 N(u_{n-2}) + hV_3 N(u_{n-3}), \quad (27)$$

$$U_{n2} = e^{c_2 hL} u_n + hA_{21} N(u_n) + hW_{21}^{(1)} N(u_{n-1}) + hW_{22}^{(1)} N(u_{n-2}) + hW_{23}^{(1)} N(u_{n-3}) + h^2 A_{21}^{(2)} N'(u_n) + h^2 W_{21}^{(2)} N'(u_{n-1}) + h^2 W_{22}^{(2)} N'(u_{n-2}) + h^2 W_{23}^{(2)} N'(u_{n-3}), \quad (28)$$

The first characteristic polynomial is given by

$$u_{n+1} = e^z u_n = 0$$

because $hL = z$. Dividing through by u_n

$$\frac{u_{n+1}}{u_n} - \frac{e^z u_n}{u_n} = 0,$$

$$r - e^z = 0. \quad (29)$$

The stability graph of order five step two and stage order four (524) scheme is shown in Figure 1.

The graph shows by *MATHEMATICA* that order five step two and stage order four (524) scheme is zero stable as the first characteristics polynomial lies in a unit circle

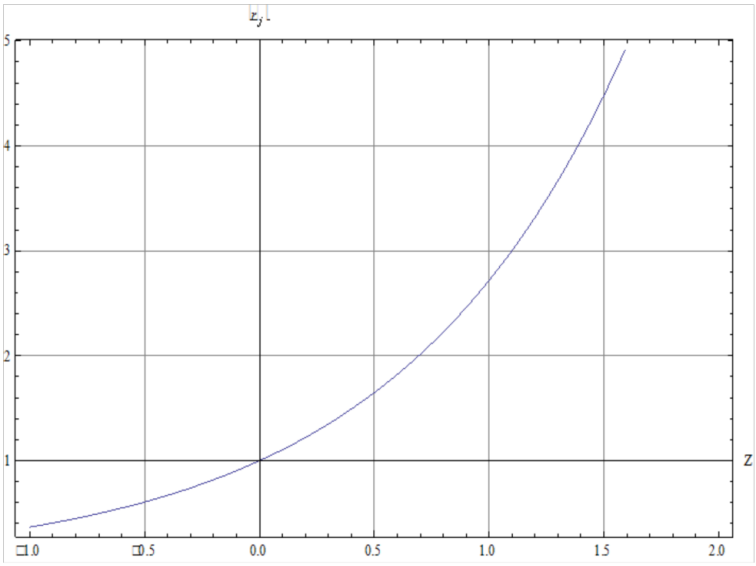


Figure 1

5. Numerical Experiments

In this section, we illustrate the accuracies of different extended exponential general linear methods.

Problem 1. Let

$$y' = \frac{y}{4}(1 - \frac{y}{20}), \quad y(0) = 1,$$

with explicit solution given as

$$y = \frac{20}{1 + 19\exp(-t^4)}.$$

The graph of extended exponential general linear method 322, 523 and 524 is shown in Figure 2.

Problem 2. Given

$$y' = -10(y-1)^2, \ y(0) = 2 \ \text{ with } \ t \in [0, 1],$$

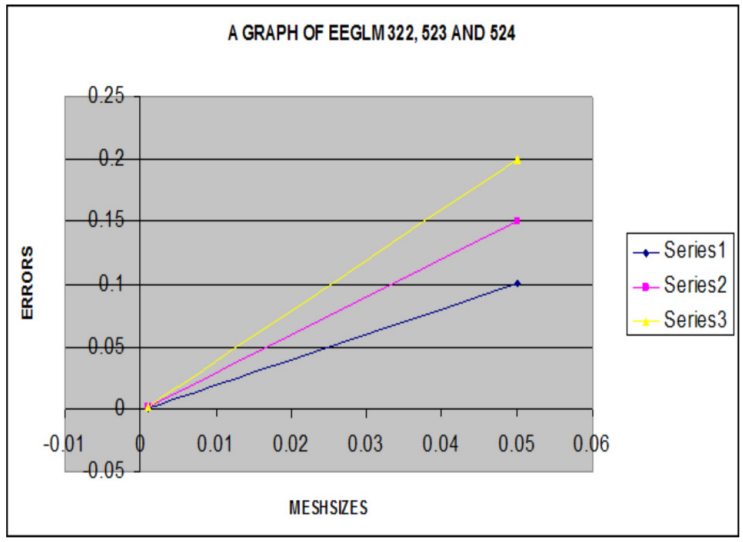


Figure 2

with exact solution

$$y = \frac{2 + 10t}{1 + 10t}.$$

The graph of EEGKM 524, CEERK and CRKM is shown in Figure 3. Our scheme 524 compares favorably with the convergent Explicit Runge-Kutta methods and Classical Runge-Kutta methods.

6. Conclusions

In this paper we have derived and analyse the error of Extended Exponential General linear methods(**EEGLM**) using the order conditions of [5]. The stability analysis reveals that the method is stable as the characteristic root lie in the unit disk. The experimental experience indicates that **524** methods perform better than the already existing ones compared with.

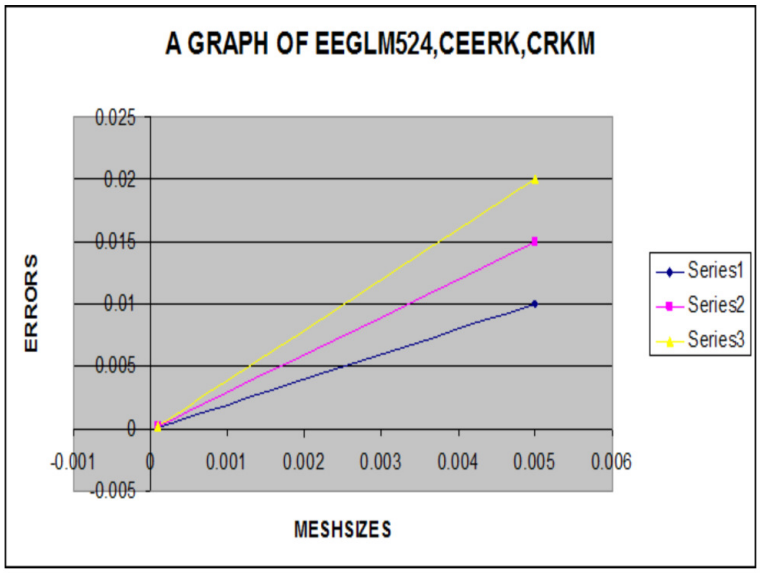

A GRAPH OF EEGLM524,CEERK,CRKM
ERRORS
0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.001
0
0.001
0.002
0.003
0.004
0.005
0.006
MESHSIZES
Series1
Series2
Series3

Figure 3

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