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STABILITY FOR A PERTURBED POLLUTION PROCESS

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Abstract: In this paper we consider an evolution model of the process of polluted flow going throughout a channel with some reaction between active substances. Criteria for asymptotic stability in classical sense for a reaction-diffusion process as well existence and uniqueness are discussed.

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Key Words: stability, parabolic PDE, initial and boundary value problem, uniqueness

1. Introduction

The evolution model of the process of polluted flow going throughout a channel is studied by many authors, [8]-[10],[15], [18], [24]. In this paper we investigate a mathematical model of the pollution in a channel, which occurs very often in different technological processes, in the nature, especially when the flow is an aggressive liquid (see, e.g., [1]-[6], [16], [17]). Here we consider a simple convection-diffusion model of a pollutant on the surface of a narrow channel, along with a water stream with a speed, which can be constant or not. The transport of the pollutant throughout the channel is directed towards to the

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positive direction of the x axis. Assume that the pollutant floats throughout the very narrow channel. Many authors investigate a mathematical model by which one may describe the evolution of the concentration of the pollutant (see, e.g. [8]-[10], [24]), $C = C(t,x)$ $\big([C] = [mass]/[length]\big)$. Thus in the real situation the evolution of the concentration appears as a solution of the partial differential equation (PDE) of parabolic type,

$$C_t = DC_{xx} - vC_x, \tag{1}$$

under certain initial and boundary conditions. Here $D > 0$ is the diffusion constant, and v denotes the stream speed. Such mathematical models of diffusive and convective processes of aggressive fluids moving in nonmetal channels as well as their applications in certain technologies are investigated in [1]-[6]. If we take into account that the pollutant undergoes of a decay due to some biological decomposition, then the resulting evolution can be expressed by

$$C_t = DC_{xx} - vC_x - \gamma C, \tag{2}$$

where γ is a rate of decay $[\gamma] = [1/time]$. However, in lots of cases the speed v may depend on the time, space or/and may be nonlinear w.r.t. the concentration C. Thus the model (2) should be

$$C_t = DC_{xx} - v(t,x,C)C_x - \gamma C. \tag{3}$$

This differential equation is obviously quasilinear due to $v = v(t,x,C)$. There are methods of solving which are very complicated and require different approaches taking into account the domain and its smoothness, as well the smoothness of $v(t,x,C)$.

In this paper we are interested in the case when the equation (3) is perturbed by any nonlinearity $\varphi(t,x,C)$. Similar mathematical models are investigated in [12]-[18], where existence and uniqueness results are discussed under special initial and boundary conditions. In the case when $v = v(t,x)$ and $\varphi = \varphi(t,x)$ depending only on t,x then we have a linear problem,

$$C_t = DC_{xx} - v(t,x)C_x - \gamma C + \varphi(t,x) \ \text{ in } \ (0,T)\times\Omega \quad (T>0) \tag{4}$$

under the boundary condition

$$BC = h(t,x) \ \text{ on } \ \partial\Omega \ \ (t>0), \tag{5}$$

and the initial condition

$$C(0,x) = u_0(x) \ \ (t=0). \tag{6}$$

Particularly, in one dimensional case the domain Ω is a subset of $\mathbb{R}$ with a boundary $\partial\Omega$ containing two different points $\partial\Omega = \{0; T\}$. The solving of (4)-(6) is standard: the Fourier method, Laplace transform, or other approaches for which the reader should be referred to [8]-[10], [21]-[25], and Galerkin Finite Element Methods can be found in [18] and [25].

There exist a lot of examples in which the perturbation is represented by a functional form $\tilde{\varphi}(t, x, l(C(t, x)))$ ($\tilde{\varphi} : \ C(t, x) \to \tilde{\varphi}(t, x, l(C(t, x)))$), where l is a linear functional $l : \ C \to \mathbb{R}$. The functional perturbation can be defined by different ways. So it can be taken in: the integral form ([23]), delay ([7], [23]), "maxima" ([7], [11], [19], [26]), and so on, where existence, uniqueness and stability results are considered. The criteria for existence, uniqueness and stability for parabolic equations with "maxima" are investigated also in [11], [19], [20].

Therefore, the equation (4) will be modified in a new PDE of functional type:

$$C_t = DC_{xx} - v(t, x)C_x - \gamma C + \tilde{\varphi}(t, x, l(C(t, x))) \ \text{ in } \ (0, T) \times \Omega \ \ (T > 0), \ (7)$$

where the boundary and initial conditions may be similar to (5), (6), respectively. Here the problems connected with the existence, uniqueness, stability, continuity and blow-up are of special interest. Also of special interest are the cases when one looks for solutions in Sobolev classes (see, e.g., [12], [15], [18], [21], [24], [25]).

2. Preliminaries

Physical Interpretation

Introduce the function $Q = Q(x, t)$, that is, the mass flux entering the interval $[x, x + \triangle x]$, through the point x at time t. So we determine a constitutive relation for Q, $[Q] = [mass] \times [time]^{-1}$ (see, e.g., [8]-[10], [23], [24]), and consider the physical meaning of the convection and diffusion.

Convection. Consider the flux determined by the water or another liquid. There is a simple model of this phenomenon, which can be represented as "a bulk of pollutant that is driven by the stream, without deformation or expansion" (see e.g. [24]), and may conclude that $Q(x, t) = vC(x, t)$.

Diffusion. The pollutant expands from higher concentration regions to lower. Here there exists the Ficks law $Q(x, t) = DC_x(x, t)$, where the diffusion constant D depends on the polluting and has the usual dimension $[D] =$

$[length]^2 \times [time]^{-1}$. In our consideration the convection and diffusion are taken into account, and their superposition, can be expressed by the equality $Q(x,t) = vC(x,t)DC_x(x,t)$, and thus one may deduce $C_t = DC_{xx} - vC_x$.

Define the diffusion medium Ω which is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. Construct a cylinder $\Sigma_T \equiv (0,T)\times\Omega$ $(T > 0)$, and define the surface $\Lambda_T \equiv (0,T) \times \partial\Omega$.

Remark 1. The points (t,x) of Σ_T do not lay on the plane $t = T$.

The initial and boundary value problem (IBVP) to a parabolic PDE has the form:

$$\begin{aligned} &u_t = Lu + f(x,u) \ \text{ in } \ \Sigma_T \\ &Bu = h(x) \ \text{ on } \ \Lambda_T \\ &u(0,x) = u_0(x) \ \text{ in } \ \Omega, \end{aligned} \tag{8}$$

where

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(t,x)\frac{\partial u}{\partial x_j},$$

and L is uniformly elliptical differential operator in the sense that the entries of the matrix $\{a_{ij}(t,x)\}$ satisfy the inequality:

$$K_0|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le K_1|\xi|^2, \quad (t,x) \in \overline{\Omega}, \quad |\xi|^2 = \sum_{i=1}^{n}\xi_i^2,$$

B is the boundary operator, $Bu \equiv p_0(x)\partial u/\partial\nu + q_0(x)$, $(t,x) \in \Lambda_T$, $p_0, q_0 \ge 0$, and $p_0 + q_0 > 0$.

Further, we consider a particular case when the IBVP has the form:

$$\begin{aligned} &u_t = D\nabla^2 u + f(x,u) \ \text{ in } \ \Sigma_T \\ &Bu = h(x) \ \text{ on } \ \Lambda_T \\ &u(0,x) = u_0(x) \ \text{ in } \ \Omega, \end{aligned} \tag{9}$$

where $\nabla^2 u \equiv \triangle u$ is the classical Laplacian. Here $D > 0$ is the above stated diffusion coefficient. Assume that $\partial\Omega$ has the outside strong sphere property (see, e.g. [12]) at $p_0 = 0$, and is of the Hölder class $C^{1+\alpha}$ for $p_0 > 0$. The functions f, h and u_0 are assumed also Hölder continuous w.r.t. their arguments.

It is known that every solution of (9) is a lower solution, and at the same time upper one. To ensure the existence of a solution to the problem (9) we need to impose certain conditions on the reaction function f.

Assume that there is a sector $\langle \hat{u}, \tilde{u} \rangle \equiv \{u : \ \hat{u} \le u \le \tilde{u}\}$, such that one-sided Lipschitz condition hold, [23]:
H1.

$$f(x, z_1) - f(x, z_2) \ge -\underline{c}(z_1 - z_2), \quad \hat{u} \le z_2 \le z_1 \le \tilde{u}, \tag{10}$$

and

H2.

$$f(x, z_1) - f(x, z_2) \le \overline{c}(z_1 - z_2), \quad \hat{u} \le z_2 \le z_1 \le \tilde{u}. \tag{11}$$

Remark 2. In the case when $f \in C^1$, the constants $\underline{c}$ and $\overline{c}$ from H1 (10) and H2 (11) can be found by

$$\underline{c} = \max\{-f_u(x, u) : \ x \in \overline{\Omega}, \ \hat{u} \le u \le \tilde{u}\},$$

and

$$\overline{c} = \max\{f_u(x, u) : \ x \in \overline{\Omega}, \ \hat{u} \le u \le \tilde{u}\},$$

respectively.

Following [21] and [23] it is clear that if f satisfies H1, then there exists a pair of sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ with monotone property

$$\hat{u} \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le \tilde{u} \ \text{ in } \ \overline{\Sigma}_T \ \ (k = 1, 2, \ldots). \tag{12}$$

Also if the pair of functions $\hat{u}$, $\tilde{u}$ are ordered lower and upper solutions of (8), and f satisfies both H1 and H2, then there exists a unique solution $U(t, x)$, such that the pair $\hat{u}, \tilde{u}$, appears as a unique limit of both sequences $\{\overline{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$ (see, e.g., [21], [23]).

We note that in the case when $f \in C^1$ in the sector $\langle \hat{u}, \tilde{u} \rangle$, then the problem (9) has a unique solution in the above sector.

Remark 3. When f satisfies only the Lipschitz condition in the argument u, then the Lipschitz constant $L = \underline{c} = \overline{c}$, and the above stated existence and uniqueness follow as well.

Elliptic Problem

Consider the elliptic BVP:

$$\begin{aligned} &-Lu + c_1(x)u = \tilde{q}(x) \ \text{ in } \ \Omega, \\ &Bu = h(x) \ \text{ on } \ \partial\Omega, \end{aligned} \tag{13}$$

where $Bu \equiv p(x)\partial u/\partial \nu + q(x)$, and L, B are required to be the same operators as in (8) and (9). The operator L is uniformly elliptic in $\overline{\Omega}$ in the sense that

the matrix (a_{ij}) is symmetric positive definite in $\overline{\Omega}$. The coefficients of L and $c_1(x)$ are in $C^{\alpha}(\overline{\Omega})$.

Here note that a classical solution exists when Ω is of the class $C^{2+\alpha}$, and $p, q \in C^{1+\alpha}(\partial\Omega)$ with either $p = 0$, $q > 0$ or $p > 0$, $q \geq 0$ on $\partial\Omega$. The internal and boundary data $\tilde{q}$, h are assumed in $C^{\alpha}(\overline{\Omega})$ and $C^{1+\alpha}(\partial\Omega)$, respectively, and when $p = 0$, $q > 0$ the function h/q is assumed in $C^{2+\alpha}(\partial\Omega)$. These assumptions are needed to ensure that the problem (13) has a unique classical solution in $C^{2+\alpha}(\overline{\Omega})$.

Consider the elliptic eigenvalue problem:

$$\begin{aligned} &(L - \tilde{c}_0 + \lambda r)\phi = 0 \quad \text{in} \quad \Omega, \\ &B\phi = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{14}$$

There is the following theorem in [23]:

Theorem 1. ([23]) *Let $\tilde{c}_0, r \in C(\Omega)$, $\tilde{c}_0 \geq 0$, $r > 0$. Then the principal eigenvalue $\lambda^*(r)$ of (14) is real and nonnegative, and $\phi^* > 0$ in Ω. Moreover, $\lambda^*(r) > 0$ when $\tilde{c}_0$ and q are not both identically zero, and $\phi^* > 0$ in $\overline{\Omega}$ when $p > 0$; ϕ^* and $\lambda^*(r)$ are the principal eigenfunction and eigenvalue, respectively.*

The following example answers why we must investigate a priori the parabolic IBVP (8). The following example is the dynamical model of a flow going through a channel along with a pollutant moving from the initial point $x = 0$ ($t_0 = 0$) to the point $x = 1$ ($t = T = 5$). The pollutant is distributed linearly as at the initial moment $t = 0$ the pollutant is zero, and at $t = 1$ it is $C(0, 1) = 1$. The rate is linear $\gamma = 0.1t + 2x$, and the perturbation $\varphi = 0.01t - 0.02x$ is linear as well. On this example one may see the complicated behavior of the solution in a simple case.

Example A. Consider the IBVP

$$\begin{aligned} &u_t = 0.01u_{xx} - 0.5u_x - (0.1t + 2x)u + 0.01t - 0.02x, \\ &\quad (t, x) \in (0; 5] \times (0; 1), \\ &u(t, 0) = 0; \ u(t, 1) = 1, \quad 0 \leq t \leq 5 \quad \text{(boundary condition)}, \\ &u(0, x) = x, \quad x \in (0; 1) \quad \text{(initial condition)}. \end{aligned}$$

The graph of the approximate solution may be seen on the following figure.

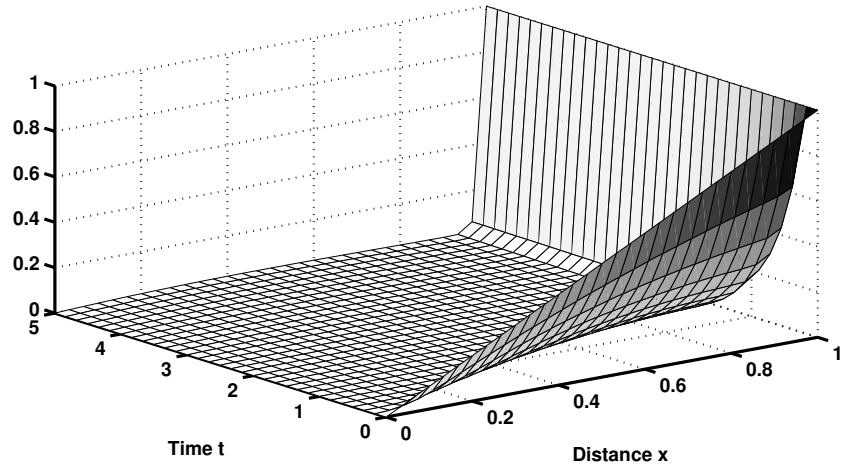

Figure 1: Graph of the solution for Example A

3. Main Result

Here we discuss some stability criteria for the problem under consideration. Introduce for our purposes the following notations:

$$\Sigma_T \equiv (0,T] \times [0;1],\ \Lambda_T \equiv (0,T] \times \partial\{[0;1]\},\quad E_T \equiv [0,T] \times \{\overline{[0;1]}\}.$$

Here the boundary $\partial\{[0;1]\} \equiv \{x_0 = 0; x = 1\}$.

Consider the mathematical model of the pollutant flow going throughout the channel with some perturbation φ:

$$\begin{aligned} &C_t = DC_{xx} - v(t,x)C_x - \gamma C + \varphi(t,x,C) \ \text{ in } \ \Sigma_T, \\ &BC = h(t,x) \ \text{ on } \ \Lambda_T \ \ (t>0), \\ &C(0,x) = u_0(x) \ \text{ in } \ \Omega \ \ (t=0), \end{aligned} \tag{15}$$

where C is the concentration of the fluid going throughout a pipe as it was described in the Introduction. There is an important statement in [23]:

Lemma 1. ([23]) *Let $u(t,x)$ and $u_s(x)$ be the respective solutions of (15) and (13), respectively, and let $\lambda^*(1)$ ($r=1$) and ϕ^* be the principal eigenvalue and eigenfunction to the problem (14), respectively. If $\lambda^*(1) > 0$ and $\tilde{q}(t,x)$ satisfies*

$$|\tilde{q}(t,x) - \tilde{q}_0(x)| \le Ke^{-\gamma t} \ \ (t>0,\ \ x \in \Omega,\ \ K,\gamma > 0), \tag{16}$$

then for any initial function $u_0(x)$ there exist positive constants ρ and α such that

$$|u(t,x) - u_s(x)| \leq \rho e^{-\alpha t} \quad (t > 0, \quad x \in \overline{\Omega}. \tag{17}$$

On the other hand, if $\lambda^*(1) < 0$ and $\tilde{q}(t,x) \geq 0$, and $h(x) \geq 0$, then for any constant $\delta > 0$

$$u(t,x) \geq \delta e^{|\lambda^*(1)|t}\phi^*(x) \quad (t > 0, \quad x \in \overline{\Omega}, \tag{18}$$

whenever it holds at $t = 0$.

Let $f(t,x,C) \equiv -\gamma C + \varphi(t,x,C)$ in (8). Next we will prove a statement to the problem (15) which is a modified case in [23].

Theorem 2. (modified of that in [23]) *Let f be C^1 function in the variable $C \in \mathbb{R}^+$ and let $\varphi(t,x,0) = 0$. If there exists $\alpha \in (0; \lambda_0 + \gamma)$, such that*

$$\varphi(t,x,\eta) \leq (\gamma\eta + \lambda_0 - \alpha)\eta, \tag{19}$$

then a unique nonnegative solution U of (15) exists, whenever it holds at $t = 0$.

Proof. We use the same method as it is in the proof in [23]. From $\varphi(t,x,0) = 0$ one has that for any $u_0 \geq 0$, the trivial function $\hat{u} = 0$ is a lower solution. Define $\tilde{u} = p(t)\phi(x)$ for some $p(t) > 0$. Then it follows that

$$\begin{aligned} &\tilde{u}_t - D\tilde{u}_{xx} + v(t,x)\tilde{u}_x = p'\phi - pL\phi = (p' + \lambda_0 p)\phi, \\ &B\tilde{u} = pB\phi = 0, \end{aligned} \tag{20}$$

where we have set $-L\tilde{u} \equiv -D\tilde{u}_{xx} + v(t,x)\tilde{u}_x$. Thus it is shown that $\tilde{u}$ is an upper solution if $p(0)\phi \geq u_0$ and then

$$(p' + \lambda_0 p)\phi \geq -\gamma p(t)\phi(x) + \varphi(t,x,p(t)\phi(x)),$$

i.e.

$$p' + (\lambda_0 + \gamma)p(t) \geq \varphi(t,x,p(t)\phi(x)).$$

Let $p(0) \geq \rho$ and then one has to prove that

$$p' + (\lambda_0 + \gamma)p(t) \geq (\lambda_0 + \gamma - \alpha)p,$$

thus $p' \geq -\alpha p$, and the statement is proved. □

The following statement is also an modification of Theorem 5.2.4 in [23].

Theorem 3. *Let $\omega(s)$ be a positive nondecreasing function in $\mathbb{R}^+$ such that $\omega(0) = 0$, $h = 0$, and $\varphi(t, x, \eta)$ is C^1 function, provided that $\varphi(t, x, 0) = 0$. Also assume that for each $\rho > 0$ with $\omega(\rho) < \lambda_0$,*

$$\int_0^\rho \frac{\omega(\tau)}{\tau} d\tau < \infty. \tag{21}$$

If the following inequality hold

$$\varphi_\eta(t, x, \eta) \le \omega(\eta) + \gamma \quad \text{for} \quad \eta \ge 0, \tag{22}$$

then for any nonnegative initial function $u_0 \le \delta\phi$ with $\omega(\delta) < \lambda_0$ there exists a constant $K \ge \delta$ such that a unique solution $C = C(t, x)$ to IBVP (15) exists estimated by:

$$0 \le C(t, x) \le Ke^{-\lambda_0 t} \quad (t > 0,\ x \in \overline{\Omega}) \tag{23}$$

Proof. To prove statement we use the same method developed in [23]. Define an upper solution $\tilde{u} = p(t)\phi(x)$ for (15), and our goal is to find $p = p(t)$ such that $p(0) \ge \delta$, then in order $\tilde{u}$ to be an upper solution we require $(p' + \lambda_0 p)\phi(x) \ge -\gamma p\phi(x) + \varphi(t, x, p\phi)$, i.e. one has that

$$p' + \lambda_0 p \ge -\gamma p + \varphi(t, x, p\phi). \tag{24}$$

Using the mean value theorem and by the requirement (22) obtain the inequality $-\gamma + \varphi(t, x, \hat{\eta}) \le \omega(\eta)$, where $0 \le \hat{\eta} \le \eta$. Therefore, we look for a differentiable function $p(t)$ such that (24) should be satisfied, hence need to solve the following Cauchý problem:

$$\begin{aligned} p' + \lambda_0 p &= p\omega(p) \\ p(0) &= \delta. \end{aligned} \tag{25}$$

Here we note that $\omega(p(0)) = \omega(\delta) < \lambda_0$. Next consider the equalities

$$p' + \lambda_0 p = p\omega(p) \equiv \frac{p'\omega(p)}{-\lambda_0 + \omega(p)},$$

and after integrating obtain

$$\int_\delta^{p(t)} \frac{dq}{q} = -\lambda_0 t - \int_\delta^{p(t)} \frac{\omega(z)}{z[\lambda_0 - \omega(z)]} dz,$$

hence

$$\ln(p/\delta) = -\lambda_0 t + \int\limits_{p(t)}^{\delta} \frac{\omega(z)}{z[\lambda_0 - \omega(z)]} dz \leq -\lambda_0 t + \int\limits_{p(t)}^{\delta} \frac{\omega(z)}{z[\lambda_0 - \omega(\delta)]} dz,$$

and

$$\ln(p/\delta) = -\lambda_0 t + [\lambda_0 - \omega(\delta)]^{-1} \int\limits_{p(t)}^{0} \frac{\omega(z)}{z} dz.$$

Thus obtain the estimate

$$p(t) = \delta e^{-\lambda_0 t + [\lambda_0 - \omega(\delta)]^{-1} [K_1 + \int\limits_{p(t)}^{0} \frac{\omega(z)}{z} dz]} \leq K e^{-\lambda_0 t},$$

whence the statement follows.

□

Following the ideas given in [23] if assume that $\omega(z)$ is a polynomial for instance, $\omega(z) = \sum\limits_{j=0}^{\mu} b_j z^j$ $(\mu \geq 1)$ satisfying the requirements in Theorem 3, and $T = +\infty$, then obtain that: If the perturbing function is such that $\varphi_\eta \leq \gamma + \rho\eta^\kappa$ $(\rho, \eta > 0,\ \kappa \geq 1)$, then for any nonnegative $u_0 \leq \delta\phi$ and $\delta < (\lambda_0/\rho)^{1/\kappa}$ the solution of (4) satisfies (15).

4. Conclusion

As a conclusion we may accent on the perturbation effect, which may be a different function or functional. In the case of functional we may solve the diffusion equation in Sobolev classes. For this purposes we should impose specific conditions on the elliptic operator L as well on the initial and boundary data as it is in Theorem 3.

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