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A CLASS OF KOLMOGOROV SYSTEM WITH EXACT ALGEBRAIC LIMIT CYCLE

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Abstract: A multi-parameter quartic Kolmogorov system is introduced. It is shown under suitable assumptions that at least one limit cycle can occur. This limit cycle is algebraic, analytically given and its hyperbolicity and stability fully controlled.

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1. Introduction

In the framework of population dynamics, differential equations modelling the interaction of two species constitute the well known Kolmogorov system

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$$\dot{x} = \frac{dx}{dt} = xf(x,y), \qquad (1)$$
$$\dot{y} = \frac{dx}{dt} = yg(x,y),$$

where $x(t)$ and $y(t)$ denote the population density of these species at time t.

When f and g are polynomials of degrees ≥ 2, limit cycles can occur and there is an extensive literature dealing with their existence, number and stability (see for instance Xun C. Huang and Lemin Zhu [8], N. G. Lloyd and J. M. Pearson [10], N. G. Lloyd et al. [11] and references therein), but to our knowledge, the exact analytic expressions of the limit cycles for a given kolmogorov system are still unknown even in simplest cases.

This paper is a contribution in that direction, motivated by the recent publication of some research papers exhibiting planar polynomial systems with one or more algebraic limit cycles analytically given (see for instance [[3], [4], [9]] and references therein). It is also surprising that similar results have been achieved by M. Abdelkadder [1] and A. Bendjeddou and R. Cheurfa [2] for the Liénard equation.

Let us first recall some useful notions (for more details, the reader can consult with profit the book of F. Dumortier et al. [5]. For $U \in \mathbb{R}[x,y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1), if for some polynomial $K \in \mathbb{R}[x,y]$ called the cofactor of the algebraic curve, we have

$$xf(x,y)U_x + yg(x,y)U_y = KU,$$

where a notation such as $U_x = \dfrac{\partial U(x,y)}{\partial x}$ is assumed in this paper. Simple analysis of eq.(5) shows that when $\max(\deg f, \deg g) = n$, the degree of the cofactor is at most $n-1$ and that the curve $U = 0$ is formed by trajectories of the system (1). Also, if the curve $U = 0$ is non-singular, it will be interesting that the equilibrium points of the system are contained either in its unbounded components or are located on the curve $K = 0$.

A limit cycle $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$, is a T-periodic solution isolated with respect to all other possible periodic solutions of the system.

To be successful, we restrict ourselves to algebraic limit cycles searched as non-singular compact components (or ovals) of real invariant algebraic curves.

We construct here a multi-parameter quartic Kolmogorov system admitting the circle

$$\Gamma : (x-\alpha)^2 + (y-\beta)^2 - r^2 = 0, \qquad (2)$$

with $\alpha > 0$, $\beta > 0$,and $0 < r < \min\{\alpha, \beta\}$, as limit cycle. For that, we consider in the first quadrant $(x > 0, y > 0)$, the system

$$\begin{aligned} \dot{x} &= x\left(Ax + By - 2\alpha a_1 x^2 + Cxy + Dy^2 + a_1 x^3 + b_1 x^2 y + a_1 xy^2 + b_1 y^3\right), \\ \dot{y} &= y\left(Ex + Fy + Gx^2 + Hxy - 2\beta b_2 y^2 + a_2 x^3 + b_2 x^2 y + a_2 xy^2 + b_2 y^3\right), \end{aligned} \tag{3}$$

where a_i, b_i, $1 \leq i \leq 2$, A, B, C, D, E, F and G are real constants such as

$$\begin{aligned} A &= \left(\alpha^2 + \beta^2 - r^2\right) a_1, \\ B &= \left(\alpha^2 + \beta^2 - r^2\right) b_1 - \beta, \\ C &= -2\left(\beta a_1 + \alpha b_1\right), \\ D &= -2\beta b_1 + 1, \\ E &= \left(\alpha^2 + \beta^2 - r^2\right) a_2 + \alpha, \\ F &= \left(\alpha^2 + \beta^2 - r^2\right) b_2, \\ G &= -2\alpha a_2 - 1, \\ H &= -2\left(\beta a_2 + \alpha b_2\right). \end{aligned}$$

As a main result, we have

Theorem 1. *The system* (3) *admits the circle* Γ *as an hyperbolic limit cycle if*

$$a_1\left(\beta - \sqrt{\beta^2 - r^2}\right) + b_2\left(\alpha - \sqrt{\alpha^2 - r^2}\right) \neq 0. \tag{4}$$

To prove the above theorem, we begin by sume additional results.

Lemma 2. *The circle* Γ *defines a periodic orbit of system* (3).

Proof. We remark, that this system can be put on the form

$$\begin{aligned} \dot{x} &= x\left(\left(a_1 x + b_1 y\right)\left((x-\alpha)^2 + (y-\beta)^2 - r^2\right) + y\left(y-\beta\right)\right), \\ \dot{y} &= y\left(\left(a_2 x + b_2 y\right)\left((x-\alpha)^2 + (y-\beta)^2 - r^2\right) - x\left(x-\alpha\right)\right), \end{aligned}$$

So, it is obvious that the circle Γ is an invariant curve of system (3) with associated cofactor

$$\begin{aligned} &K(x,y) \\ &= 2\left(a_1 x^3 + b_1 x^2 y + a_2 xy^2 + b_2 y^3 - \alpha a_1 x^2 - \left(\alpha b_1 + \beta a_2\right) xy - \beta b_2 y^2\right). \end{aligned} \tag{5}$$

It is clear that Γ is a trajectory of (3). Suppose that Γ contains an equilibrium point (x_0, y_0) of the system, that is

$$\begin{aligned} x_0\left(\left(a_1 x_0 + b_1 y_0\right)\left((x_0-\alpha)^2 + (y_0-\beta)^2 - r^2\right) + y_0\left(y_0-\beta\right)\right) &= 0, \\ y_0\left(\left(a_2 x_0 + b_2 y_0\right)\left((x_0-\alpha)^2 + (y_0-\beta)^2 - r^2\right) - x_0\left(x_0-\alpha\right)\right) &= 0. \end{aligned}$$

In the first quadrant it reduces to $y_0 = \beta$ and $x_0 = \alpha$, which is absurd, hence is a periodic orbit. □

Proof of Theorem 1. Let T be the period of the periodic solution

$$\Gamma : \{(x(t), (y(t))), t \in [0, T]\}.$$

To see that Γ is in fact a limit cycle, we recall a classic result characterizing limit cycles among other periodic orbits (see for instance Perko [12] for more details), which means that $\Gamma(t)$ is a limit cycle when $\int_0^T \operatorname{div}(\Gamma)\, dt \neq 0$, stable if $\int_0^T \operatorname{div}(\Gamma)\, dt < 0$,and instable if $\int_0^T \operatorname{div}(\Gamma)\, dt > 0$. We shall compute this integral explicitly, using also a practical result of J. Giné et al. [7], which asserts that

$$\int_0^T \operatorname{div}(\Gamma)\, dt = \int_0^T K(x, y)\, dt. \tag{6}$$

This allow us to deal with a more simple integral as follow. Let $\lambda \in \mathbb{R}^*$ an arbitrary constant, we have

$$\begin{aligned}\int_0^T \operatorname{div}(\Gamma)\, dt &= \int_0^T \operatorname{div}(\Gamma)\, dt + \lambda\left(\int_0^T \operatorname{div}(\Gamma)\, dt - \int_0^T K(x, y)\right) dt \\ &= \int_0^T ((1+\lambda)\operatorname{div}(\Gamma) - \lambda K(x, y))\, dt.\end{aligned}$$

Here

$$\begin{aligned}\int_0^T \operatorname{div}(\Gamma)\, dt &= \int_0^T y(a_2x + b_2y + \lambda)(y - \beta)\, dt \\ &\quad + \int_0^T x(a_1x + b_1y - \lambda)(x - \alpha)\, dt.\end{aligned}$$

Via contour integral over Γ oriented in the positive sense,

$$\int_0^T \operatorname{div}(\Gamma)\,dt = \oint_\Gamma \frac{y\left(a_2x+b_2y+2\lambda\right)\left(y-\beta\right)}{x\left(\left(a_1x+b_1y\right)\left((x-\alpha)^2+\left(y-\beta\right)^2-r^2\right)+y\left(y-\beta\right)\right)}dx$$
$$+\oint_\Gamma \frac{x\left(a_1x+b_1y-2\lambda\right)\left(x-\alpha\right)}{y\left(\left(a_2x+b_2y\right)\left((x-\alpha)^2+\left(y-\beta\right)^2-r^2\right)-x\left(x-\alpha\right)\right)}dy$$
$$=\oint_\Gamma \frac{a_2x+b_2y+2\lambda}{x}dx-\oint_\Gamma \frac{a_1x+b_1y-2\lambda}{y}dy.$$

Now, we apply the Green formula to get

$$\int_0^T \operatorname{div}(\Gamma)\,dt = -b_2\iint_{\operatorname{int}(\Gamma)} \frac{1}{x}dxdy - a_1\iint_{\operatorname{int}(\Gamma)} \frac{1}{y}dxdy, \tag{7}$$

where $\operatorname{int}(\Gamma)$ denotes the interior of Γ.

Integrating $I_1 = \iint_{\operatorname{int}(\Gamma)} \left(\frac{1}{x}\right)dxdy$, with respect to y, we obtain

$$I_1 = 2\int_{\alpha-r}^{\alpha+r}\left(\frac{\sqrt{r^2-\left(x-\alpha\right)^2}}{x}\right)dx.$$

The change of variable $x=\alpha+r\sin\theta$ gives

$$I_1 = \pi\frac{\alpha-\sqrt{\alpha^2-r^2}}{r^2}. \tag{8}$$

Now, we consider the second integral, that is $I_2 = \iint_{\operatorname{int}(\Gamma')} \left(\frac{1}{y}\right)dxdy$.

By symmetry, we have

$$I_2 = \iint_{\operatorname{int}(\Gamma')} \left(\frac{1}{x}\right)dydx,$$

where Γ' is the circle given by the equation $\left(y-\alpha\right)^2+\left(x-\beta\right)^2-r^2=0$. So we have just to replace α by β in the expression of I_1, to obtain

$$I_2 = \pi\frac{\beta-\sqrt{\beta^2-r^2}}{r^2}. \tag{9}$$

Taking into account (8) and (9), the formula (7) becomes,

$$\int_0^T \operatorname{div}(\Gamma(t))\,dt = -\tfrac{\pi}{r^2}\left(a_1\left(\beta-\sqrt{\beta^2-r^2}\right)+b_2\left(\alpha-\sqrt{\alpha^2-r^2}\right)\right).$$

Form the hypothesis of the theorem, we conclude. □

2. Applications

As applications of the our theorem, we present two examples of Kolmogorov systems admitting circles as exact limit cycles.

Example 1. For $\alpha = 3, \beta = 2, r = \sqrt{2}, a_1 = -1, b_1 = 2,\ a_2 = -1, b_2 = 1$, the system

$$\begin{aligned}\dot{x} &= x\left(-11x+20y+6x^2-8xy-7y^2-x^3+2x^2y-xy^2+2y^3\right),\\ \dot{y} &= y\left(-8x+11y+5x^2-2xy-4y^2-x^3+x^2y-xy^2+y^3\right),\end{aligned} \tag{10}$$

admits the circle $\Gamma_1 : (x-3)^2+(y-2)^2-2=0$ as an unstable limit cycle, since $\int_0^T \operatorname{div}(\Gamma(t))\,dt = 0.363\,70$.

This limit cycle encloses a saddle point, and two unstable strong focus.

Example 2. For $\alpha = 3, \beta = 4, r = 1, a_1 = 1, b_1 = 0, a_2 = -\frac{4}{3}$ and $b_2 = -\frac{3}{4}$, the system

$$\begin{aligned}\dot{x} &= 3x\left(96x-88y-24x^2-14xy+28y^2+4x^3+4xy^2-3yx^2-3y^3\right),\\ \dot{y} &= 4y\left(-87x+72y+21x^2+14xy-24y^2-4x^3-4xy^2+3yx^2+3y^3\right),\end{aligned} \tag{11}$$

admits the circle $\Gamma_2 : (x-3)^2+(y-4)^2-1=0$ as an stable limit cycle, since $\int_0^T \operatorname{div}(\Gamma(t))\,dt = -0.29859\pi$.

This limit cycle encloses a stable strong focus at the point $(3,4)$.

Conclusion 3. *The simple choice of the circle as a limit cycle for our system, has allow us to carry out all the needed calculations. Moreover, the elementary method used in this paper seems to be fruitful to investigate more general Kolmogorov systems in order to obtain explicitly some or all their limit cycles at least when it is question of the algebraic ones. We will devote a further publication to a more general class of Kolmogorov systems.*

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