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NUMERICAL SIMULATION OF THE SOME ILL-POSED PROBLEMS FOR THE HEAT TRANSFER EQUATION

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Abstract: In this paper the regularization methods for solving some one-dimensional ill-posed problems for heat transfer equation are considered. The solution is obtained by solving the retrospective problem for linear and nonlinear initial-boundary value problem of singularly perturbed heat transfer equations.

For regularization of some inverse problems or unstable retrospective problems the well-known Lattes-Lions technique – introduction in the linear equations the differential operators of higher order with small coefficients – is used. In this case finite difference approximations in the space are obtained using method of lines with two small parameters ϵ in three ways:

1) the finite difference scheme (FDS method) with the second and the fourth orders of approximation in the uniform grid,

2) the difference scheme with exact spectrum (FDSES method) in the uniform grid,

3) the global approximation (GAN method) in nonuniform grid with grid points on the roots of the Chebyshev polynomials.

The nonlinear heat transfer equations with nonmonotonous heat conductivity without Lattes-Lions technique numerical solution is regularized with two methods: By introducing the differential operator of higher order with mixed derivatives in the special form $\epsilon\frac{\partial^5 u}{\partial^4 x \partial t}$ and constructing monotonous continuous functions for approximation the heat conductivity.

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To investigate stability of bounded solutions for the continuous and discrete problems, solvability in the Sobolev space and to determine the parameters ϵ some theoretical estimations are obtained using the method of logarithmic convexity and integral identity.

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1. Introduction

Many applied problems formulated as inverse problems of mathematical physics belong to the class of problems that are ill-posed in the classical sense. An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics. If we know a physical device completely, we have a classical mathematical description of this device including uniqueness, stability and existence of the solution of the corresponding mathematical problem. But if one of the parameters describing this device is to be found from experimental data, then we arrive at an inverse problem. For their approximate solution regularization methods (approximation by well-posed problems) are widely used. The theory of stable numerical solutions of ill-posed problems using regularization was developed in the 1950-1960s by John and Tikhonov (see the books [5], [11], [17]). The inverse ill-posed problems for heat transfer equation are considered in the books [13], [3], [12], [9].

For inverse problems for time-dependent equations the generalized inverse method or quasi-reversibility (R.Lattes and J-L. Lions [15]) is also used. In the pioneering paper [8] the numerical solution of the backward parabolic equation is consider. In [16] some typical results are presented on the construction of difference schemes for unstable inverse problems of mathematical physics, based on the regularization principle. In book [15] for solving physically unstable retrospective problem (ill-posed time reverse problem for backward parabolic equation) of the linear heat transfer equation the regularization term $\epsilon\frac{\partial^4 u}{\partial x^4}$ is added, where $u = u(x,t)$ is the solution of the heat transfer equation with homogenous boundary condition (BCs) of the first kind and ϵ is a small coefficient.

In paper [6] the backward parabolic equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, t \in (T,0)$ is regularized by pseudo-parabolic equation $\frac{\partial}{\partial t}(u_\epsilon + \epsilon\frac{\partial^2 u_\epsilon}{\partial x^2}) = \frac{\partial^2 u_\epsilon}{\partial x^2}$.

When approximately solving ill-posed problems the choice of regularization parameter ϵ must correspond to the amount of error in the input data. Here we merely construct stable computational algorithms for ill-posed time-dependent problems and the influence of the regularization parameter only on the stability of the corresponding difference scheme.

Inverse problems for partial differential equations and their methods of regularization are considered in the book by V.Isakov [7].

The nonlinear heat transfer ill-posed problem with heat conductivity in the form of trigonometrical function was derived in [1], [2]. Its numerical solution was regularized with two methods: by introducing the differential operator of higher order in the following form $\epsilon \frac{\partial^5 u}{\partial x^4 \partial t}$ and constructing monotonous continuous functions. Here similar simple nonlinear ill-posed problem with heat conductivity in the form of power function is consider.

2. Mathematical Model

Similarly to [1], [2] for numerical simulation the magnetic droplet dynamics in a rotating field we consider simplest nonlinear problem of heat transfer partial differential equation (PDE) in the following form:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 F(u(x,t))}{\partial x^2} - \epsilon \dfrac{\partial^5 u(x,t)}{\partial x^4 \partial t} + g_0, \quad x \in (0,L),\ t \in (0,T), \\ u(0,t) = u(L,t) = 0, \dfrac{\partial^2 u(0,t)}{\partial x^2} = \dfrac{\partial^2 u(L,t)}{\partial x^2}, \quad t \in [0,T), \\ u(x,0) = u_0(x), \quad x \in [0,L] \end{cases} \tag{1}$$

where $F(u) = 0.4u^3 - 1.8u^2 + 2.4u$ is the nonlinear function –the polynomial of the third order, ϵ is a small coefficient, $g_0 = const \geq 0$ is the constant heat source term, T, L are the final time for stationary solution and the length, $u_0(x)$ is continuously differentiable function with $u_0(0) = u_0(L) = 0$.

In [1], [2] the nonlinear function is in the form $F(u) = \frac{1}{Bm}u + \sin(2u)$, where Bm is the magnetic Bond number, $g_0 = \omega\tau$, ω – angular frequency, τ – time scale, ϵ is the parameter for the regularization, about equal to 10^{-4}, obtained from physical considerations.

The function $F(u)$ is not monotonic with $F'(1) = F'(2) = 0$, $F(1) = 1$ (local maximum), $F(2) = 0.8$ (local minimum) (see Figure 1).

The last term in the equations (1) is used for the regularization of the numerical calculations.

By setting $\epsilon = 0$ we obtain the ill-posed problem. In the numerical experiments by changing (increasing or decreasing) the right side constant g_0 we

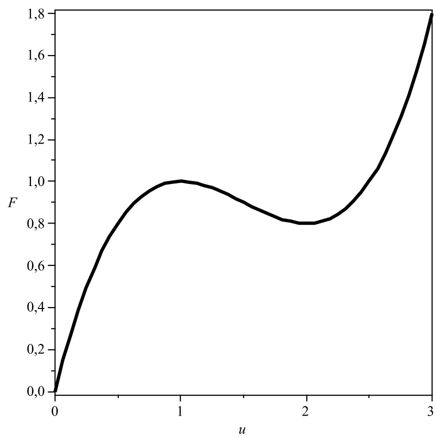

Figure 1: Function $F(u)$

can take $u_0(x)$ equal to the stationary solution obtained at the previous assigned value of g_0. This leads to hysteresis phenomena in the droplet shape transformations at change of rotating field [2].

If $F(u) = gu$, $g = const > 0$, we have the linear problem of heat transfer equation with the constant source term g_0. If $g_0 = 0$, we have the following simple linear problem for homogenous heat transfer equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = g\dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,L),\ t \in (0,T), \\ u(0,t) = u(L,t) = 0, \quad t \in [0,T), \\ u(x,0) = u_0(x), \quad x \in [0,L]. \end{cases} \tag{2}$$

Here $u_0(x)$ is continuously differentiable function.

In book [15] the method of quasi-reversibility has been suggested for solving physically unstable retrospective problem of the linear homogenous heat transfer equation (ill-posed time reverse problem). The problem is replaced by regularized "higher"-order equation with the added regularization term $\epsilon\frac{\partial^4 u}{\partial x^4}$.

The retrospective or reverse in the time problem for linear homogenous backward heat transfer equation is in the following form:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = g\dfrac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,L),\ t \in (T,0), \\ u(0,t) = u(L,t) = 0, \quad t \in [T,0], \\ u(x,T) = u_T(x), \quad x \in [0,L], \end{cases} \tag{3}$$

where with the initial function $u_T(x)$, $u_T(0) = u_T(L) = 0$ by $t = T$ the final data is given. This function also can be obtained by solving the direct problem (2) for $t \in [0, T]$.

The obtained solution $u(x, 0)$ for the backward parabolic problem (3) for $t = 0$ needs to be compared with the initial function $u_0(x)$. The inverse (retrospective) problem (3) is ill-posed as it is unstable with respect to relatively small perturbations of the initial data.

Using the Fourier's series

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) w_k(x), \quad w_k(x) = \sqrt{2/L} \sin \frac{k\pi x}{L}$$

we obtain that the Fourier coefficients of the solution are

$$a_k(t) = \exp(g\lambda_k(T-t)) a_{Tk},$$

where

$$a_{Tk} = \int_0^L u_T(x) w_k(x) dx, \quad \lambda_k = (\frac{k\pi}{L})^2.$$

The coefficients $a_k(0)$ are uniquely determined from the relations $a_k(0) = \exp(g\lambda_k T) a_{Tk}$. These relations show that for any u_T a solution $u(x,t)$ does not exist or is exponentially unstable: for u that is the k-th term of the sum $u(x,t) = \sum_{k=1}^{\infty} a_k(0) \exp(-g\lambda_k t) a_{Tk} w_k(x)$ with $a_k(0) = \epsilon \exp(g\lambda_k T)$ we have for the norm for this term in $L_2[0, L]$- $||u(0)|| = \epsilon \exp(g\lambda_k T)$, while $||u_T|| = \epsilon$. The linear ill-posed problem (3) we can regularize in the following form:

$$\begin{cases} \dfrac{\partial u_\epsilon(x,t)}{\partial t} = g\dfrac{\partial^2 u_\epsilon(x,t)}{\partial x^2} + \epsilon_1 \dfrac{\partial^4 u_\epsilon(x,t)}{\partial x^4} - \epsilon_2 \dfrac{\partial^6 u_\epsilon(x,t)}{\partial x^6}, \\ x \in (0,L), \ t \in (T,0), \\ u_\epsilon(0,t) = u_\epsilon(L,t) = 0, \\ \dfrac{\partial^2 u_\epsilon(0,t)}{\partial x^2} = \dfrac{\partial^2 u_\epsilon(L,t)}{\partial x^2} = 0, \dfrac{\partial^4 u_\epsilon(0,t)}{\partial x^4} = \dfrac{\partial^4 u_\epsilon(L,t)}{\partial x^4} = 0, \\ u_\epsilon(x,T) = u_T(x), \quad x \in [0,L], \end{cases} \tag{4}$$

where $\epsilon_1 \geq 0$, $\epsilon_2 \geq 0$ are small coefficients.

In [15] the solution is analyzed with $\epsilon_2 = 0$. Then $a_k(t) = \exp(g\lambda_k - \epsilon_1 \lambda_k^2)(T-t)) a_{Tk}$ and the series for $u(x,t)$ is convergent in $L_2(0,L)$ for any u_T in this space and any $t < T$, if $\epsilon_1 \lambda_1 \geq g$.

Moreover, when ϵ_1 goes to 0, the regularized solutions are convergent to the solutions u of the initial problem (3) [7].

If $\epsilon_2 \neq 0$ then $a_k(t) = \exp(g\lambda_k - \epsilon_1\lambda_k^2 - \epsilon_2\lambda_k^3)(T-t))a_{Tk}$ and we have exponentially stable solution when $\epsilon_1\lambda_1 + \epsilon_2\lambda_1^2 \geq g$.

For the comparison of the maximal error $\delta = max|u_\epsilon(x,0) - u_0(x)|$ we consider following regularization problem with the mixed derivative:

$$\begin{cases} \dfrac{\partial u_\epsilon(x,t)}{\partial t} = g\dfrac{\partial^2 u_\epsilon(x,t)}{\partial x^2} + \epsilon_1 \dfrac{\partial^4 u_\epsilon(x,t)}{\partial^4 x} - \epsilon_2 \dfrac{\partial^5 u_\epsilon(x,t)}{\partial x^4 \partial t}, \\ x \in (0,L), t \in (T,0), \\ u_\epsilon(0,t) = u_\epsilon(L,t) = 0, \dfrac{\partial^2 u_\epsilon(0,t)}{\partial x^2} = \dfrac{\partial^2 u_\epsilon(L,t)}{\partial x^2} = 0, \quad t \in [T,0], \\ u_\epsilon(x,T) = u_T(x), \quad x \in [0,L]. \end{cases} \tag{5}$$

Then $a_k(t) = \exp((g\lambda_k - \epsilon_1\lambda_k^2)/(1+\epsilon_2\lambda_k^2))(T-t))a_{Tk}$ and we have exponentially stable solution when $\epsilon_1\lambda_1 \geq g$.

Using the simple regularization from H. Gajevski, K.Zacharias [6] for (3) we have the following problem ($g > 0$):

$$\begin{cases} \dfrac{\partial u_\epsilon(x,t)}{\partial t} = g\dfrac{\partial^2 u_\epsilon(x,t)}{\partial x^2} - \epsilon \dfrac{\partial^3 u_\epsilon(x,t)}{\partial x^2 \partial t}, \quad x \in (0,L),\ t \in (T,0), \\ u_\epsilon(0,t) = u_\epsilon(L,t) = 0, u_\epsilon(x,T) = u_T(x), \quad x \in [0,L], \end{cases} \tag{6}$$

$a_k(t) = \exp(\frac{g\lambda_k}{1-\epsilon\lambda_k}(T-t))a_{Tk}$ and we have exponentially stable solution when $\epsilon_1\lambda_1 \geq 1$.

We can also consider parabolic equation directly in the inverse time $\bar{t} = T-t$. Then we have ill-posed problem which differs from the standard parabolic equation only in the sign of the derivatives in space $\frac{\partial}{\partial \bar{t}} = -\frac{\partial}{\partial t}$, [15], [10]. In this case we have ill-posed direct problem with the negative coefficient g of the heat conductivity. Then from direct problem (6) we obtain $a_k(t) = \exp(\frac{-g\lambda_k}{1-\epsilon\lambda_k}t)a_k(0)$ and we have exponentially stable solution when $\epsilon_1\lambda_1 \geq 1$.

3. Some Theoretical Estimations

We can use the method of **logarithmic convexity** [7] for investigation of the stability in the backward evolutionary equations (4). For the squared norm $f(t) = ||u(t)||^2$ we can prove that the logarithm $K(t) = \ln(f(t))$ is a convex function or $K''(t) \geq 0$, where $K'' = (ff'' - (f')^2)/f^2$.

∇ By using the definition of the L_2-norm and the differential equation (4) we obtain

$$f' = 2\int_0^L u\frac{\partial u}{\partial t}dx = 2\int_0^L u(g\frac{\partial^2 u}{\partial x^2} + \epsilon_1\frac{\partial^4 u}{\partial x^4} - \epsilon_2\frac{\partial^6 u}{\partial x^6})dx$$
$$= \int_0^L (-2g(\frac{\partial u}{\partial x})^2 + 2\epsilon_1(\frac{\partial^2 u}{\partial x^2})^2 + 2\epsilon_2(\frac{\partial^3 u}{\partial x^3})^2)dx,$$

where we have used integration by parts and the BCs (4).

Further

$$f'' = 4\int_0^L (-g\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x, \partial t} + \epsilon_1\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^2 \partial t} + \epsilon_2\frac{\partial^3 u}{\partial x^3}\frac{\partial^4 u}{\partial x^3 \partial t})dx$$
$$= 4\int_0^L (g(\frac{\partial u^2}{\partial x^2})^2\frac{\partial u}{\partial t} + \epsilon_1(\frac{\partial^4 u}{\partial x^4})^2\frac{\partial u}{\partial t} + \epsilon_2(\frac{\partial^6 u}{\partial x^6})^2\frac{\partial u}{\partial t})dx = 4\int_0^L (\frac{\partial u}{\partial t})^2 dx,$$

where we integrated by parts again, used the boundary conditions and expressed $\frac{\partial u}{\partial t}$ from the differential equations.

Now, we have $f''f - (f')^2 = 4||u||^2||\frac{\partial u}{\partial t}||^2 - (2\int_0^L u\frac{\partial u}{\partial t}dx)^2 \geq 0$ according to the Schwarz inequality.

Therefore $K = K(t)$ is a convex function and $K(t) \leq (1 - t/T)K(0) + t/TF(T)$, or $f(t) \leq f(0)^{1-t/T}f(T)^{t/T}$.

$\triangle$ Using the definition of f, we get

$$||u(t)|| \leq ||u(0)||^{1-t/T}||u(T)||^{t/T}. \tag{7}$$

Stability for the class of bounded solutions follows from this estimate. Corresponding estimation can not be obtained from the problem (5. This estimate remained for the direct ill-posed problem obtained from (4) with the transformation $\bar{t} = T - t$.

Using the **regularization** (4) with $\epsilon_2 = 0$ for the direct nonlinear problem (1) we have the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(g(u)\frac{\partial u}{\partial x}\Big) - \epsilon_1\frac{\partial^4 u}{\partial x^4} + g_0 \tag{8}$$

and the following integral identity

$$\frac{\partial}{2\partial t}\int_0^L u^2 dx + \int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2 dx + \epsilon_1\int_0^L \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 dx = g_0\int_0^L u dx, \tag{9}$$

where $g(u) = F'(u) = 1.2u^2 - 3.6u + 2.4, -0.3 \leq g(u), g'(u) = 2.4u - 3.6.$

For **investigating the solvability** of the corresponding initial-value problem in the Sobolev space $\overset{0}{W}{}_2^2$ for the weak solution and for obtaining the apriori estimations for fixed t we need to determine the parameter ϵ_1 from the following inequality:

$$\int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2 dx + \epsilon_1 \int_0^L \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 dx \geq k_0 \int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx,$$

where $k_0 = const > 0$. Similarly to [2] for fixed t follows the nonlinear differential equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{\kappa}\Big(0.5 g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + (k_0 - g(u))\frac{\partial^2 u}{\partial x^2}\Big) = 0, \tag{10}$$

where

$$\epsilon_1 \geq \kappa = \max I(u), \quad I(u) = \int_0^L ((k_0 - g(u))\Big(\frac{\partial u}{\partial x}\Big)^2 dx / \int_0^L \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 dx.$$

The **numerical solution** by $L = 2$ using Matlab solver "bvp4c" with 5 boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = 0, \frac{\partial u(0,t)}{\partial x} = b_c > 0$$

is $k_0 = 1.29$, $b_c = 0.55$ (for these values k_0 and b_c the minimal value of κ is obtained).

If $g(u) = -|g| = const < 0$ then we have $\frac{k_0+|g|}{\kappa} = \lambda_1$, where $\lambda_1 = \frac{\pi^2}{L^2}$ is the first eigenvalues of the differential operator $-\frac{\partial^2}{\partial x^2}$ for homogenous BCs. Therefore, $\kappa = \frac{(k_0+|g|)L^2}{\pi^2}$. Using Matlab solver "bvp4c" with $g = -1, k_0 = 0, L = 2$ from (10) we get $\kappa = 0.4053 \approx \frac{4}{\pi^2}$.

For the **Fourier's series** $u(x,t) = \sum_{k=1}^{\infty} a_k(t) w_k(x)$, for constant function g we obtain

$$\frac{d a_k(t)}{dt} = -g\lambda_k a_k(t) - \epsilon_1 (\lambda_k)^2 a_k(t) + b_k,$$

$$b_k = g_0 \int_0^L w_k(x) dx = g_0 \frac{L}{k\pi}(1 - (-1)^k),$$

or

$$a_k(t) = \exp(\rho_k t) a_k(0) + \frac{b_k}{\rho_k}(\exp(\rho_k t) - 1),$$

where

$$\rho_k = -g\lambda_k - \epsilon_1(\lambda_k)^2, a_k(0) = \int_0^L u(x,0)w_k(x)dx.$$

We have a bounded solution for $g = -|g| < 0$, when $\rho_k \leq 0$ or $\epsilon_1 \geq \max \frac{|g|}{\lambda_k} = \frac{|g|}{\lambda_1} = \frac{L^2|g|}{\pi^2}$.

From (9) using Hölder's inequality $|\int_0^L u(x,t)dx| \leq \sqrt{L}||u(t)||$ follows the inequality

$$\frac{d}{2dt}||u(t)||^2 + k_0||u_x(t)||^2 \leq \sqrt{L}g_0||u(t)||. \tag{11}$$

Using Friedrichs inequality [14] $||u(t)||^2 \leq \frac{L^2}{\pi^2}||u_x(t)||^2$ we obtain

$$\frac{d||u(t)||}{dt} + \frac{k_0\pi^2}{L^2}||u(t)|| \leq \sqrt{L}g_0$$

or

$$||u(t)|| \leq ||u(0)|| \exp(-\frac{k_0\pi^2}{L^2}t) + \sqrt{L}g_0\frac{L^2}{k_0\pi^2}(1 - \exp(-\frac{k_0\pi^2}{L^2}t)).$$

Here $||u(t)|| = (\int_0^L u(x,t)^2dx)^{1/2}$, $||u_x(t)|| = (\int_0^L (\frac{\partial u(x,t)}{\partial x})^2dx)^{1/2}$.

For the stationary solutions $u_s(x)$ follows the estimation $||u_s|| \leq C_s$, where $C_s = \sqrt{L}g_0\frac{L^2}{k_0\pi^2}$.

Using Matlab by $N = 100$, $L = 2$ ($\epsilon_1 = 0.001$) for $g_0 = 5$ we obtain $||u_s|| = 2.21$ ($C_s = 2.23$, $\kappa = 0.0051$, $k_0 = 1.29$).

Using the **regularization** (4) with $\epsilon_1 = 0$ for the direct nonlinear problem (1) we have the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(g(u)\frac{\partial u}{\partial x}\Big) + \epsilon_2\frac{\partial^6 u}{\partial x^6} + g_0 \tag{12}$$

and the following integral identity

$$\frac{\partial}{2\partial t}\int_0^L u^2dx + \int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2dx + \epsilon_2\int_0^L \Big(\frac{\partial^3 u}{\partial x^3}\Big)^2dx = g_0\int_0^L udx. \tag{13}$$

To **obtain the apriori estimations** we need to determine the parameter ϵ_2 from the following inequality:

$$\epsilon_2 \geq \kappa = \max I(u),$$

where $I(u) = \int_0^L \Big((k_0 - g(u))(\frac{\partial u}{\partial x})^2\Big)dx / \int_0^L \Big(\frac{\partial^3 u}{\partial x^3}\Big)^2 dx, u \in \overset{0}{W}{}_2^2$.

From $\frac{dI(u+\epsilon\phi)}{d\epsilon} \to 0, \epsilon \to 0$ follows the nonlinear differential equation in the form:

$$\frac{\partial^6 u}{\partial x^6} - \frac{1}{\kappa}\Big(0.5g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + (k_0 - g(u))\frac{\partial^2 u}{\partial x^2}\Big) = 0. \qquad (14)$$

The **numerical solution** by $L = 2$ using Matlab solver "bvp4c"with 7 BCs:

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = \frac{\partial^4 u(0,t)}{\partial x^4} = \frac{\partial^4 u(L,t)}{\partial x^4} = 0,$$

$$\frac{\partial u(0,t)}{\partial x} = b_c$$

is

$$k_0 = 1.0, \quad \kappa = 0.0041, \quad b_c = 0.7.$$

If $g(u) = -|g| = const < 0$ then we have $\frac{k_0+|g|}{\kappa} = \lambda_1^2$. Therefore, $\kappa = \frac{(k_0+|g|)L^4}{\pi^4}$.

Using Matlab solver "bvp4c" with $g = -1, k_0 = 0, L = 2$ from (14) we get $\kappa = 0.1643 \approx \frac{16}{\pi^4}$.

From (13) using Hölder's inequality and Friedrichs inequality we obtain the inequality (11) and previous estimates for $||u(t)||$ and $||u_s||$.

Using the **Fourier's series** for constant function g we obtain

$$\frac{da_k(t)}{dt} = -g\lambda_k a_k(t) - \epsilon_2(\lambda_k)^3 a_k(t) + b_k,$$

$$b_k = g_0\frac{L}{k\pi}(1-(-1)^k),$$

or

$$a_k(t) = \exp(\rho_k t)a_k(0) + \frac{b_k}{\rho_k}(\exp(\rho_k t) - 1),$$

where

$$\rho_k = -g\lambda_k - \epsilon_2(\lambda_k)^3, a_k(0) = \int_0^L u(x,0)w_k(x)dx.$$

We have bounded solution for $g = -|g| < 0$, when $\rho_k \le 0$ or $\epsilon_2 \ge \max\frac{|g|}{\lambda_k^2} = \frac{|g|}{\lambda_1^2} = \frac{L^4|g|}{\pi^4}$.

Using the **regularization** (4) with two ϵ, $\epsilon_1 \ne 0$, $\epsilon_2 \ne 0$ for the direct nonlinear ill-posed problem (1) we have the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(g(u)\frac{\partial u}{\partial x}\Big) - \epsilon_1\frac{\partial^4 u}{\partial x^4} + \epsilon_2\frac{\partial^6 u}{\partial x^6} + g_0 \qquad (15)$$

and the following integral identity

$$\frac{\partial}{2\partial t}\int_0^L u^2dx+\int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2dx+\epsilon_1\int_0^L\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2dx \\ +\epsilon_2\int_0^L\Big(\frac{\partial^3 u}{\partial x^3}\Big)^2dx=g_0\int_0^L udx. \tag{16}$$

For **obtaining the apriori estimations** we need to determine the parameter ϵ_2 from the previous inequality:

$$\epsilon_2\geq\kappa,$$

where the parameter κ can be obtained from the equation (14). We have the inequality (11), where k_0 is replaced with $k_0+\epsilon_1$.

4. Approximations and Solution of the Problems

For analyzing the nonstationary solution of the problem (1), the stationary solution is considered.

4.1. The Stationary Solution of the Problem (1)

The stationary solution $u_s(x)$ of the problem (1) can be obtained from the following transcendental equation $F(u_s(x))=0.5g_0x(L-x)$ by fixed values of g_0 and $x\in(0,L)$. The maximal value $u_m=u_s(L/2)$ is the solution of the transcendental equation $F(u_m)=\frac{L^2}{4}0.5g_0$..

The solution $(u(x,t)\geq 0)$ is symmetric with respect to $x=L/2$:

$$u(L/2-x_1,t)=u(L/2+x_1,t)$$

, $x_1\in(0,L/2)$ or $\frac{\partial u(L/2,t)}{\partial x}=0$.

The nonstationary $u(x,t)$ as a function of the variable x is discontinuous for $g_0=\frac{4}{L^2}2F(u_*)$, where $u_*=1$, $u_*=2$ are the roots of equation $F'(u_*)=0$ (the local maximum or minimum of the function $F(u)$). The ill-posed problem (1) with $\epsilon=0$ can be also regularized similarly [1] by modifying the function $F(u)$ in a such way, that in the intervals where the function's derivative is negative $(F'(u)<0)$ the function is replaced with a constant value. There are two possible variations depending on the behaviour of the value g_0 (direct F_d or reverse f functions for increasing or decreasing g_0, see Figs. 2,3 by L=2).

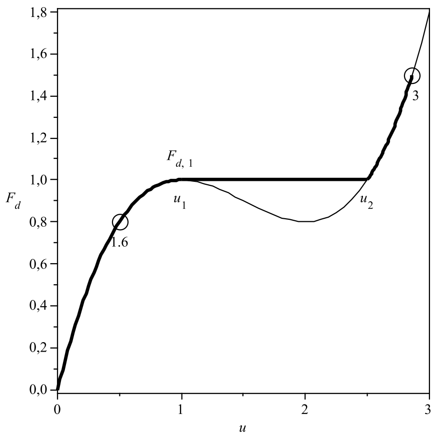

Figure 2: Direct modified function $F_d(u)$. The following numerical values are shown in the figure: $u_1 = 1, u_2 = 2.5, F_{d,1} = 1$. In the fixed points ⊙ there are values of g with coordinates $(u_s(1), g_0/2)$.

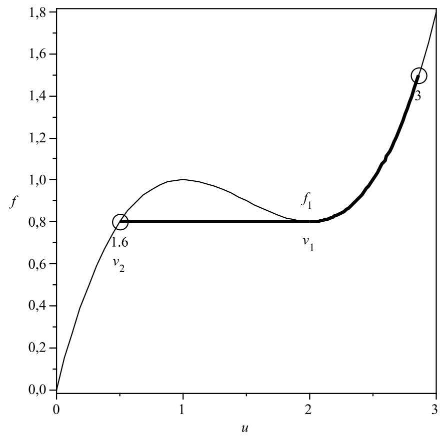

Figure 3: Reverse modified function $f(u)$. The following numerical values are shown in the figure: $v_1 = 2, v_2 = 0.5, f_1 = 0.8$. In the fixed points ⊙ there are values of g_0 with coordinates $(u_s(1), g_0/2)$.

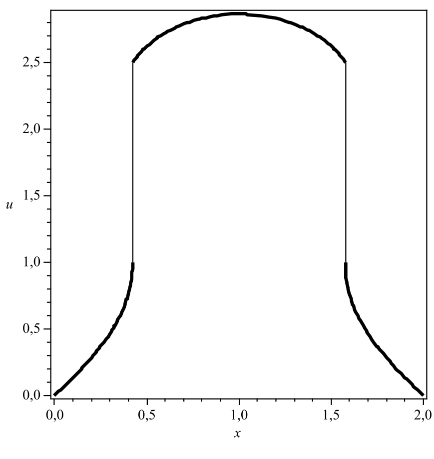

Figure 4: Stationary solution $u_s(x)$ at $g_0 = 3$ in the case of direct function.

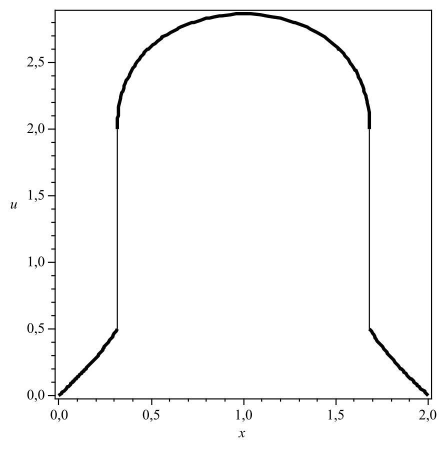

Figure 5: Stationary solution $u_s(x)$ at $g_0 = 3$ in the case of reverse function.

The **stationary solutions** with 1 jump are shown in Figure 4 (direct function) and Figure 5 (reverse function) for $g_0 = 3$. For the stationary solutions depending on the value of g_0 we can obtain one or two solutions. The modified functions F_d, f are continuous and monotonous with discontinuous first derivatives.

This stationary solutions can be also obtained with numerical simulation as the limit of the nonstationary solution of (1) with a large time moment.

4.2. Method of Lines and Finite Difference Approximations

The problems (1-5) are solved numerically using the method of lines and two way finite difference methods for the approximation of spatial derivatives: local approximation with finite differences in uniform grid (LAU) and global approximation with derivative matrices in nonuniform grid (GAN).

1. For **local approximation** LAU we consider the uniform grid in the space $x_j = jh$, $j = \overline{0, N}$, $Nh = L$.

Using the finite differences of the second order approximation for partial derivatives of the second and the fourth order with respect to x, from (1) the Cauchy problem for the system of nonlinear ODEs of the first order we obtain a problem in the following matrix form

$$\begin{cases} (E + \epsilon B)\dot{U}(t) + AF(U(t)) = G, t \in (0, T), \\ U(0) = U_0, \end{cases} \tag{17}$$

where E is the unit matrix of $N-1$ order, A is the standard 3-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^2}\{-1; 2; -1\}$ approximating the derivative of the second order $-\frac{\partial^2}{\partial x^2}$, $B = A^2$ is the 5-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^4}\{1; -4; 6; -4; 1\}$, approximating the derivative $\frac{\partial^4}{\partial x^4}$ with the second order, the first and the last elements of the matrix B are $B(1,1) = B(N-1, N-1) = 5/h^4$, using the approximation of the BCs $\frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = 0$ with the second order (we use the following finite difference expressions $u_0(t) = u_N(t) = 0, u_{-1}(t) = -u_1(t), u_{N+1}(t) = -u_{N-1}(t)$),
$U(t), \dot{U}(t), U_0, F(U), G$ are the column-vectors of $N-1$ order with the elements $u_j(t) \approx u(x_j, t)$, $\dot{u}_j(t) \approx \frac{\partial u(x_j,t)}{\partial t}, u_j(0) = u_0(x_j), f_j \approx F(u_j(t)), g_j = g_0, j = \overline{1, N-1}$.

Using the matrix form $A + \frac{h^2}{12}B$ for **approximation with the fourth**

order of derivative of the second order [16] we can obtain the following problem:

$$\begin{cases} (E+\epsilon B)\dot{U}(t) + (A+\frac{h^2}{12}B)F(U(t)) = G, \\ U(0) = U_0, \end{cases} \tag{18}$$

Concerning the R.Lattes and J.L.Lions regularization we can consider the following direct nonlinear ill-posed initial value problem:

$$\begin{cases} \dot{U}(t) + (A+\frac{h^2}{12}B)F(U(t)) + \epsilon_1 BU(t) - \epsilon_2 CU(t) = G, \\ U(0) = U_0 \end{cases} \tag{19}$$

or

$$\begin{cases} \dot{U}(t) + AF(U(t)) + \epsilon_1 BU(t) - \epsilon_2 CU(t) = G, \\ U(0) = U_0, \end{cases} \tag{20}$$

where $B = A^2, C = A^3$.

Using regularization from [6] (6) we have the following problem:

$$\begin{cases} (E-\epsilon A)\dot{U}(t) + AF(U(t)) = G, \\ U(0) = U_0. \end{cases} \tag{21}$$

where $F(U) = gU$.

2. For **global approximations** GAN we consider nonuniform grid with the grid points on the roots of the Chebyshev polynomials of the second kind

$$x_j = 0.5L(1-\cos(\pi j/N)), \quad j = \overline{0,N}. \tag{22}$$

Using this grid points we can approximate the derivatives $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^4}{\partial x^4}$, $\frac{\partial^6}{\partial x^6}$ with matrix $\bar{D}$, $\bar{D}^2$, $\bar{D}^4$, $\bar{D}^6$ of derivatives in the form [4]

$$u'_h = \bar{D}u_h, u''_h = \bar{D}^2 u_h, u^4_h = \bar{D}^4 u_h, u^6_h = \bar{D}^6 u_h, \tag{23}$$

where $u_h = (u_0, u_1, \ldots, u_N)$, $u'_h = (u'_0, u'_1, \ldots, u'_N)$, etc.

Are the column-vectors of the corresponding values u_j depending on t: $u_j \approx u(x_j,t)$, $u'_j \approx \frac{\partial u(x_j,t)}{\partial x}$, etc.

From the Lagrange interpolation follows, that the elements of matrix $\bar{D}$ are in the form

$$d_{j,k} = \frac{dl_k(x_j)}{dx}, j,k = \overline{0,N}, \tag{24}$$

where $l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)}$ are the elementary Lagrange multipliers,

$$\omega = \prod_{k=0}^{N}(x - x_k).$$

For this nonuniform grid the interpolation error is small [4].

The determinants of derivatives matrix $\bar{D}^2$ are equal to zero (this matrix is singular). Therefore we need to decrease the order of this matrix by deleting the first columns and corresponding rows. Then we have the matrix $A = -\bar{D}^2$ and $A^2 = (-\bar{D}^2)^2$ with $N-1$ order.

Using the finite differences of the second order approximation for partial derivatives of the sixth order with respect to x and $A^3 = (-\bar{D}^2)^3$, from (4) we obtain the initial value problem for the system of linear ODEs in the following form

$$\begin{cases} \dot{U}_\epsilon(t) + gAU_\epsilon(t) - \epsilon_1 A^2 U_\epsilon(t) - \epsilon_2 A^3 U_\epsilon(t) = 0, t \in (T, 0) \\ U_\epsilon(T) = U_T, \end{cases} \tag{25}$$

or

$$\begin{cases} \dot{U}_\epsilon(t) + g(A + \frac{h^2}{12}B)U_\epsilon(t) - \epsilon_1 BU_\epsilon(t) - \epsilon_2 CU_\epsilon(t) = 0, t \in (T, 0) \\ U_\epsilon(T) = U_T, \end{cases} \tag{26}$$

where $B = A^2$ and $C = A^3$ for LAU is the 7-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^6}\{-1; 6; -15; 20; -15; 6; -1\}$ approximating the derivative $-\frac{\partial^6}{\partial x^6}$ with the second order, the first and the last elements of the matrix C are $C(1,1) = C(N-1, N-1) = 14/h^4$, $C(1,2) = C(N-1, N-2) = C(2,1) = C(N-2, N-1) = -14/h^4$, the approximation of the boundary conditions $\frac{\partial^4 u_\epsilon(0,t)}{\partial x^4} = \frac{\partial^4 u_\epsilon(L,t)}{\partial x^4} = 0$ with the second order (we use following finite difference expressions $u_0(t) = u_N(t) = 0$, $u_{-1}(t) = -u_1(t)$, $u_{-2}(t) = -u_2(t)$, $u_{N+1}(t) = -u_{N-1}(t)$, $u_{N+2}(t) = -u_{N-2}(t)$),

$U_\epsilon(t), \dot{U}_\epsilon(t), U_T$ are the column-vectors of $N-1$ order with the elements $u_{\epsilon j}(t) \approx u_\epsilon(x_j, t)$, $\dot{u}_{\epsilon j}(t) \approx \frac{\partial u_\epsilon(x_j,t)}{\partial t}$, $u_{Tj} = u_T(x_j)$, $j = \overline{1, N-1}$.

For the nonlinear direct ill-posed problem (equation 15) we have the following initial value problem for the system of nonlinear ODEs:

$$\begin{cases} \dot{U}_\epsilon(t) + AF(U_\epsilon(t)) + \epsilon_1 A^2 U_\epsilon(t) - \epsilon_2 A^3 U_\epsilon(t) = G, t \in (0, T) \\ U_\epsilon(0) = U_0. \end{cases} \tag{27}$$

For the **bounded solution** of (25) the estimate can be obtained (7).

∇ For the discrete functions v, u we define the scalar product

$$(v,u) = h\sum_{j=1}^{N-1} v_j u_j, \quad (v,u] = h\sum_{j=1}^{N} v_j u_j$$

and difference operators

$$\Lambda u_j = u_{x,\bar{x}_j} = \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}), \Lambda u = -Au$$

(for the approximation of the second order derivatives) and

$$u_{\bar{x}_j} = \frac{1}{h}(u_j - u_{j-1}), \quad u_{x_j} = \frac{1}{h}(u_{j+1} - u_j)$$

(for the approximation of the first order derivatives). Then for $u_0 = u_N = 0$ follows $(\Lambda u, u) = -(u_{\bar{x}}, u_{\bar{x}}]$ [16].

Similarly $(\Lambda^2 u, u) = (\Lambda u_{\bar{x}}, \Lambda u_{\bar{x}}]$, $(\Lambda^3 u, u) = (\Lambda^2 u_{\bar{x}}, \Lambda^2 u_{\bar{x}}]$ if $\Lambda u_0 = \Lambda u_N = \Lambda^2 u_0 = \Lambda^2 u_N = u_0 = u_N = 0$, $u_{-1} = -u_1$, $u_{-2} = -u_2$, $u_{N-1} = -u_{N+1}$, $u_{N-2} = -u_{N+2}$.

Then the ODEs (25) can be rewritten in the following form:

$$\dot{u} = g\Lambda u + \epsilon_1 \Lambda^2 u - \epsilon_2 \Lambda^3 u.$$

For the squared norm $f(t) = ||u(t)||^2 = (u(t), u(t))$ we can prove that the logarithm $K(t) = \ln(f(t))$ is a convex function or $K''(t) \geq 0$.

By using the differential equation we obtain

$$f' = 2(u,\dot{u}) = 2(u, g\Lambda u + \epsilon_1\Lambda^2 u - \epsilon_2\Lambda^3 u)$$
$$= -2(g(u_{\bar{x}}, u_{\bar{x}}] + \epsilon_1(\Lambda u_{\bar{x}}, \Lambda u_{\bar{x}}] - \epsilon_2(\Lambda^2 u_{\bar{x}}, \Lambda^2 u_{\bar{x}}]).$$

Further,

$$f'' = -4(g(u_{\bar{x}}, \dot{u}_{\bar{x}}] + \epsilon_1(\Lambda u_{\bar{x}}, \Lambda \dot{u}_{\bar{x}}] - \epsilon_2(\Lambda^2 u_{\bar{x}}, \Lambda^2 \dot{u}_{\bar{x}}])$$
$$= 4(g(\Lambda u, \dot{u}) + \epsilon_1(\Lambda^2 u, \dot{u}) - \epsilon_2(\Lambda^3 u, \dot{u}))$$
$$= 4(g\Lambda u + \epsilon_1\Lambda^2 u - \epsilon_2\Lambda^3 u, \dot{u}) = 4(\dot{u}, \dot{u}) = 4||\dot{u}||^2.$$

Now, we have $f''f - (f')^2 = 4||u||^2||||\dot{u}||^2 - 4(u,\dot{u})^2 \geq 0$ according to the discrete Schwarz inequality.

Therefore,

$$K(t) \leq (1 - t/T)K(0) + t/TK(T),$$

$$f(t) \leq f(0)^{1-t/T} f(T)^{t/T}$$

and we obtain the estimate (7).

$\triangle$**.** The solution of (25) can be obtained in the following matrix form $U_\epsilon(t) = exp(C_\epsilon(t-T)U_T$ and

$$U_\epsilon(0) = exp(-TC_\epsilon)U_T = exp(-T(C_\epsilon + A))U_0, \quad (28)$$

where $C_\epsilon = -gA + \epsilon_1 A^2 + \epsilon_2 A^3$. Similarly from (5) we obtain the initial value problem for the system of linear ODEs

$$\begin{cases} (E + \epsilon_2 A^2)\dot{U}_\epsilon(t) + gAU_\epsilon(t) - \epsilon_1 A^2 U_\epsilon(t) = 0, t \in (T, 0) \\ U_\epsilon(T) = U_T. \end{cases} \quad (29)$$

In this case the matrix $C_\epsilon = (E + \epsilon_2 A^2)^{-1}(-gA + \epsilon_1 A^2)$.

5. The Spectral Representation for LAU and Discrete Fourier Methods

The solution of the corresponding discrete spectral problem $Aw^k = \mu_k w^k$, $k = \overline{1, N-1}$ for matrix A obtained with LAU is orthonoral eigenvectors w^k $((w^k, w^m) = \sum_{j=1}^{N-1} w_j^k w_j^m = \delta_{k,m})$, with the elements

$$w_j^k = \sqrt{\frac{2}{N}} \sin \frac{\pi j k}{N}, \quad j = \overline{1, N-1}$$

(elements of the symmetric matrix W), and eigenvalues

$$\mu_k = \frac{4}{h^2} \sin^2 \frac{k\pi}{2N}, k = \overline{1, N-1},$$

see [16].

In the matrix form we get

$$AW = WD, WW = E, W^{-1} = W, A = WDW,$$

where the elements of the diagonal matrix D is $d_k = \mu_k$.

The representations for matrix $A = WDW$ is equivalent with the standard 3-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^2}\{-1; 2; -1\}$ approximating the derivative of the second order $-\frac{\partial^2}{\partial x^2}$.

For **the difference scheme with exact spectrum** (FDSES) [1] the matrix A is replaced with the matrix WDW, where the diagonal matrix D contains the first $N-1$ eigenvalues $\lambda_k = (\frac{k\pi}{L})^2$ of the differential operator $(-\frac{\partial^2}{\partial x^2})$. For FDS the elements of the diagonal matrix D is $d_k = \mu_k$.

For any matrix function $p(A)$ follows that $p(A) = Wp(D)W$. Then

$$A^2 = WD^2W, A^3 = WD^3W$$

and from (28) we get $U_\epsilon(0) = Wexp(-T(D_\epsilon + gD))WU_0$, where $D_\epsilon = -gD + \epsilon_1 D^2 + \epsilon_2 D^3$ for the problem (25) and $D_\epsilon = (E + \epsilon_2 D^2)^{-1}(-gD + \epsilon_1 D^2)$ for the problem (29).

In the limit case $((\epsilon_1, \epsilon_2) \to 0)$ follows that $U_\epsilon(0) \to U_0$. Currently this is only theoretical result without error numerical calculations.

Using the transformation $V = WU$, $V_\epsilon = WU_\epsilon$ $(U = WV$, $U_\epsilon = WV_\epsilon)$ or

$$U_\epsilon(t) = \sum_{k=1}^{N-1} v_{\epsilon k}(t) w^k$$

we obtain

$$V_\epsilon(0) = exp(-T(D_\epsilon + gD))V_0$$

or $v_{\epsilon k}(0) = exp(-T(d_{\epsilon k} + gd_k))v_k(0)$, $k = \overline{1, N-1}$, where v_k, $v_{\epsilon k}$ are the elements of vectors V, V_ϵ, $d_{\epsilon k} = -gd_k + \epsilon_1 d_k^2 + \epsilon_2 d_k^3$ for the problem (25) and $d_{\epsilon k} = (1 + \epsilon_2 d_k^2)^{-1}(-gd_k + \epsilon_1 d_k^2)$ for the problem (29).

Therefore for the problem (25) $v_{\epsilon k}(0) = exp(-T(\epsilon_1 d_k^2 + \epsilon_2 d_k^3))v_k(0)$ and for the problem (29) $v_{\epsilon k}(0) = exp(-T((1 + \epsilon_2 d_k^2)^{-1}(\epsilon_1 d_k^2 + \epsilon_2 d_k^3))v_k(0)$.

Similarly the analytical solutions of the problems (25,29) are $(U_\epsilon = WV_\epsilon)$:

$$v_{\epsilon k}(t) = exp(d_{\epsilon k}(t - T))v_{Tk}, k = \overline{1, N-1}, \tag{30}$$

or

$$V_\epsilon(t) = exp(D_\epsilon(t - T))V_T, \quad U_\epsilon(t) = Wexp(D_\epsilon(t - T))WU_T,$$

where v_{Tk} is the element of vector $V_T = WU_T$ or $V_T = \sum_{k=1}^{N-1} u_{Tk} w^k$.

For the error $\delta = ||U_\epsilon(0) - U_0|| = max|u_{\epsilon k}(0) - u_k(0)|$ we have the following estimate:

$$\delta \leq ||exp(-T(D_\epsilon + gD)) - E||||U_0||.$$

For minimal error

$$|exp(-T \min_k(d_{\epsilon k} + gd_k)) - 1|, (\min_k(d_{\epsilon k} + d_k) = d_{\epsilon 1} + d_1),$$

we need to choose the parameters $\epsilon_1^2 + \epsilon_2^2 > 0$ that the numerical process is stable. Here

$$d_{\epsilon 1} + gd_1 = \begin{cases} d_1^2(\epsilon_1 + \epsilon_2 d_1), & \text{for (4.9)} \\ d_1^2(\epsilon_1 + \epsilon_2 d_1)/(1 + \epsilon_2 d_1^2), & \text{for (4.13).} \end{cases}$$

For the finite difference scheme with the exact spectrum (FDSES) or discrete Fourier method the diagonal matrix D contains the first $N-1$ eigenvalues $d_k = \lambda_k = (k\pi/L)^2$, $k=\overline{1, N-1}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ correspondingly (the eigenvectors remain).

For fixed eigenvector $U_0 = w^m$, $1 \le m \le N-1$ we have $U_T = exp(-d_m T)w^m$, $U_\epsilon(0) = exp(-T(d_{\epsilon m} + gd_m))U_0$ and the error

$$\delta_m = ||U_\epsilon(0) - U_0|| = ||U_0(exp(-T(d_{\epsilon m} + gd_m)) - 1)||.$$

For the time reverse discrete problems (25, 26) with $g = const > 0$ we analogously obtain that:
$\epsilon_1 \ge \frac{g}{\mu_1}$ or $\epsilon_1 \ge \frac{g(1+\mu_1 h^2/12)}{\mu_1}$, if $\epsilon_2 = 0$
and
$\epsilon_2 \ge \frac{g}{\mu_2}$ or $\epsilon_2 \ge \frac{g(1+\mu_1 h^2/12)}{\mu_2}$, if $\epsilon_1 = 0$.

6. Some Numerical Results

In the following examples we show the accuracy of the 4 way finite difference approximations (FDA) for boundary-value problems (BVP) of simple ODEs by $L = 2$, $N = 10$:

1) BVP: $u''(x) = 1$, $u(0) = u(L) = 0$, exact solution – $u(x) = x^2/2 - x$, FDA: $AU = -e_1$ with maximal errors – $8.3 * 10^{-16}$ (GAN), $3.3 * 10^{-17}$ (LAU),

2) BVP: $u^{(4)}(x) = 1$, $u(0) = u(L) = 0$, $u^{(2)}(0) = u^{(2)}(2) = 0$, exact solution – $u(x) = x^4/24 - x^3/6 + x/3$,

FDA: $A^2U = e_1$ with maximal errors – $7.5*10^{-15}$ (GAN), $4.2*10^{-4}$ (LAU),

3) BVP: $u^{(6)}(x) = 1$, $u(0) = u(L) = 0$, $u^{(2)}(0) = u^{(2)}(2) = 0$, $u^{(4)}(0) = u^{(4)}(2) = 0$, exact solution – $u(x) = x^6/720 - x^5/120 + x^3/18 - 2 * x/15$,

FDA: $A^3U = -e_1$ with maximal errors – $1.5 * 10^{-13}$ (GAN), $1.4 * 10^{-3}$ (LAU).

Here e_1 is the column-vector of the $N-1$ order with ones.

The discrete problems (17-29) are solved numerically by the MATLAB with relative error 10^{-7} (RelTol=10^{-7}),

6.1. Some Numerical Results for Ill-Posed Problem for Nonlinear Heat Transfer Equation

Constructed direct ($F_d(u)$) and reverse($f(u)$ see Figs. 2, 3) functions allow us to calculate the solution corresponding to the direct path with increasing the source g_0 starting from the initial condition $u_0 = 0$ and the reverse path with decreasing g_0 starting from the initial condition $u_0 = u_s(x)$, where $u_s(x)$ is the stationary solution calculated in the direct path. The numerical simulations are carried out by integrating the system of ODEs (17) in two ways:

1) $\epsilon \neq 0$ (an example $\epsilon = 10^{-4}$),

2) $\epsilon = 0$ by using the modified functions $F_d(u)$, $f(u)$.

The obtained results in either case are consistent. Some numerical results obtained for different g_0 are shown in Figure 6. Here for $\epsilon = 10^{-4}$ the first stationary solution $u_s(x)$ is obtained for $g_0 = 1.6$ with decreasing the source from $g_0 = 3$ ($u_0(x)$ is equal to the $u_s(x)$ obtained in the direct path by $g_0 = 3$, $u_0(x) = 0$, $T = 6$), the second $u_s(x)$ is obtained in the direct path by $g_0 = 1.6$, $u_0(x) = 0$, $T = 6$. The stationary solutions at $g_0 = 1.6$ are obtained by direct and reverse paths are different.

Using the direct and reverse functions two stationary solutions at $g_0 = 1.6$, $T = 6$ are obtained in the following way:

1) we use the direct function $F_d(u)$ with $u_0(x) = 0$ and obtain the $u_s(x)$ by $g_0 = 1.6$ and $g_0 = 3$,

2) we use the reverse function $f(u)$ with $u_0(x)$ equal to stationary solution obtained with direct function by $g_0 = 3$.

See also the dynamic of the solution $u(x,t)$ obtained in the reverse path from $g_0 = 3$ to $g_0 = 1.6$ by $t = 0$ (this is $u_s(x)$ obtained in the direct path for $g_0 = 3$, $u_0(x) = 0$, $T = 6$), $t = 0.47$, $t = 0.98$, $t = 2.6$, $t = 6.0$ ($u_s(x)$), (Figure 7).

The stationary solutions for $g_0 = 3$, $u(1) = 2.861$ are shown in the Figs. 4, 5.

Using (20, 21) and (17) by decreasing the source from $g_0 = 2$ to $g_0 = 1.6$ (the reverse path) the following results are obtained for max value $M_s = u_s(1)$ of the stationary solutions (the exact value is $M_s = 2$):

1) for (20) $\epsilon_2 = 0 :$ FDSES: $M_s = 2.074, \epsilon_1 = 3{*}10^{-5}$; FDS: $M_s = 1.995, \epsilon_1 = 4.6 * 10^{-5}$,

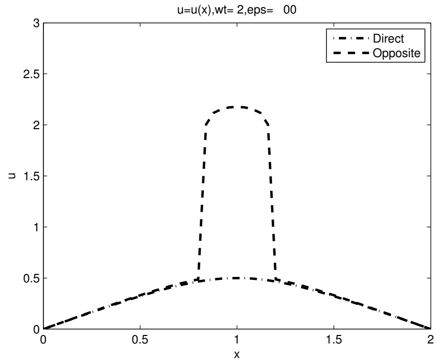

Figure 6: Two different $u_s(x)$ by $g_0 = 1.6$, $T = 6$ obtained in the direct path $(- \cdot - \cdot)$ at $u_0(x) = 0$ and in the reverse path $(- - -)$ at $u_0(x) = u_s(x)$, where $u_s(x)$ obtained in the direct path at $g_0 = 3, u_0(x) = 0$.

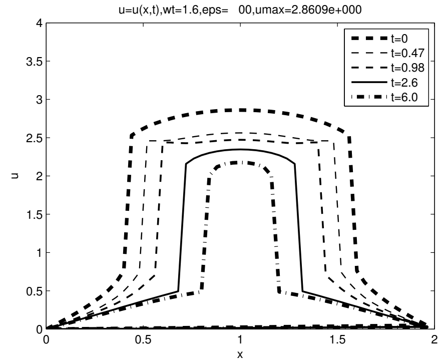

Figure 7: Dynamic of solution $u = u(x,t)$ obtained in the reverse path at $g_0 = 1.6$, $u_0(x) = u_s(x)$, where $u_s(x)$ $(---)$ obtained in the direct path at $g_0 = 3$, $u_0(x) = 0$.

2) for (20) $\epsilon_1 = 0$: FDSES: $M_s = 2.012, \epsilon_2 = 4.8 * 10^{-9}$; FDS: $M_s = 2.052, \epsilon_2 = 1.2 * 10^{-8}$,

3) for (17) – FDSES and FDS: $M_s = 1.990, \epsilon = 6.0 * 10^{-3}$,

4) for (21) – FDSES and FDS: $M_s = 2.224, \epsilon = 7.0 * 10^{-1}$,

For (18, 19) we obtain similar results.

We can use the transformation with the inverse function $u = F^{-1}(v)$, where $F(u)$ is the modified function. Then we have the following initial-boundary-value problem

$$\begin{cases} \frac{\partial v}{\partial t} = F'(u)(\frac{\partial^2 v}{\partial x^2} + g_0, \\ v(0,t) = v(2,t) = 0, t \in (0,T), v(x,0) = 0, x \in [0,2], \end{cases} \tag{31}$$

where $F'(u) = 1.2u^2 - 3.6u + 2.4$ and $u = u(x,t)$ is the solution of the cubic equation $u^3 - 4.5u^2 + 6u - 2.5v = 0$. Using Cardano formula we obtain for $v \in [0, 0.8] \bigcup [1, \infty)$ one real root in the form $u = 1.5 + v1 + v2, v1 = ((10v - 9)/8 + 0.625\sqrt{v3})^{1/3}, v2 = ((10v-9)/8 - 0.625\sqrt{v3})^{1/3}, v3 = 4v^2 - 7.2v + 3.2$. We can obtain the maximal value of the stationary solution from this root, where $v = g_0/2$. The nonstationary solutions $v(x,t)$ are obtained at $g_0 = 1.6$ and by $t = 0 (v = 0)$, $t = 0.003$, $t = 0.027$, $t = 0.45$, $t = 2.8 (v_s(x))$ (see Figure 8).

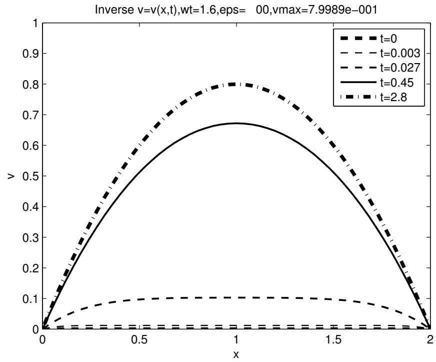

Figure 8: Dynamic of solution $v = v(x,t)$ obtained from (31) at $g_0 = 1.6$ (stationary solution $v_s(x)$ $(- \cdot - \cdot)$ is obtained by $t = T = 2.8$).

6.2. Some Numerical Results for the Ill-Posed Linear Time-Dependent Problem

For retrospective problems we consider 3 following functions for the initial conditions:

$$u_0(x) = sin(\pi x/L), \quad sin(\pi x/L)(1 - x/L), \quad sin(2\pi x/L)(1 - x/L)$$

with $L = 2, g = 1$.

The minimal value depending on ϵ_2 of maximal error δ for $\epsilon_1 = 0$ is obtained. The Lagrange interpolation method with uniform grid is unstable by increasing N and can not be applied for solving retrospective problem.

1. **Some results for initial condition with one eigenfunction** $u_0(x) = sin(\pi x/2)$.

We use following expressions for maximal error:

1) $\delta_1 = (1-\exp(-T\frac{d_1^2(\epsilon_1+\epsilon_2 d_1)}{1+\epsilon_2 d_1^2}) \max |u_0|$ for equation (29) with mixed derivative,

2) $\delta_2 = (1-\exp(-Td_1^2(\epsilon_1+\epsilon_2 d_1)) \max |u_0|$ for equation (25) with two epsilon, where d_1 is the first eigenvalue ($d_1 = \mu_1 = \frac{4}{h^2}\sin^2\frac{\pi}{2N}$ for FDS, $d_1 = \lambda_1 = (\frac{\pi}{L})^2$ for FDSES).

In the table 1 the following results for calculation of the problem (29) by using 3 finite approximations methods (GAN, LAU-FDS, LAU-FDSES) are shown:

N_m – the maximal obtained value of grid number N with $\epsilon_1 = \epsilon_2 = 0$ and maximal error $\delta < 0.01$; $\epsilon_2(T_1)$, $\delta_1(T_1)$ – the minimal values of error δ_1 obtained for $N \in [20, 80]$ with corresponding $\epsilon_2 \neq 0$, $\epsilon_1 = 0, T_1 = T = 0.1$ (similar for $\epsilon_1 \neq 0$, $\epsilon_2 = 0$) and $\epsilon_2 \neq 0$, $\epsilon_1 = 0$, $T_2 = T = 1$.

We can see that the precision for the three considered methods is equivalent. We have the following values for the first eigenvalue $d_1 = 2.4674$ for FDSES,

Table 1: The values of N_m, ϵ_1, δ_1, ϵ_2, obtained with $T_1 = 0.1$, $T_2 = 1$

$Method$	N_m	$\epsilon_2(T_1)$	$\delta_1(T_1)$	$\epsilon_2(T_2)$	$\delta_1(T_2)$	$\epsilon_1(T_1)$	$\delta_1(T_1)$
GAN	9	8.10^{-6}	1.10^{-5}	4.10^{-4}	.0062	.0020	.0014
FDS	18	5.10^{-6}	8.10^{-6}	3.10^{-4}	.0045	0018	.0013
$FDSES$	12	6.10^{-6}	9.10^{-6}	4.10^{-4}	.0062	.0019	.0014

$d_1 = 2.4671$ for FDS by N=80.

The calculated maximal error δ is equal to theoretical error δ_1.

The functions in figs. 9-14 are obtained by $N = 80$. In figs. 9,10 are represented the results of calculation by $T = 1$, $\epsilon_1 = 0$, $\epsilon_2 = 0.0004$, $\delta_1 = 0.0062$ and $T = 5$, $\epsilon_1 = 0$, $\epsilon_2 = 0.004$, $\delta_1 = 0.259$. 2. **Some results for initial**

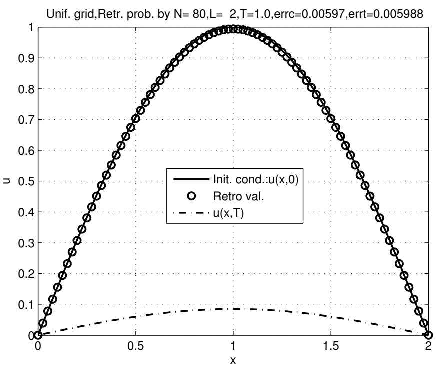

Figure 9: Functions $u(x, 0) = sin(\pi x/2)$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.0062$ for $T = 1$ with FDSES.

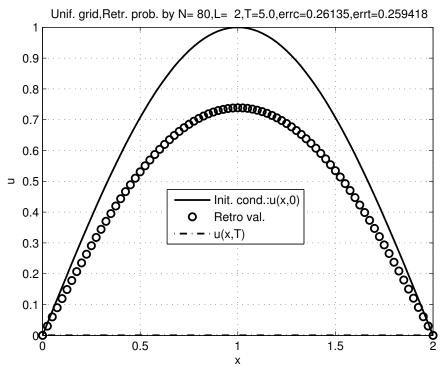

Figure 10: Functions $u(x, 0) = sin(\pi x/2)$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.259$ for $T = 5$ with FDSES.

condition $u_0(x) = (1 - x/2)sin(\pi x/2)$.

In table 2 similar to table 1 the following results for calculation of the problem (29) are shown.

In figs. 11, 12 the results of calculation by $T = 0.1$, $\epsilon_1 = 0$, $\epsilon_2 = 4.10^{-6}$, $\delta = 0.0042$ with GAN and $T = 1$, $\epsilon_1 = 0$, $\epsilon_2 = 0.004$, $\delta = 0.052$ with FDSES are represented.

Table 2: The values of N_m, ϵ_1, δ_1, ϵ_2, obtained with $T_1 = 0.1$, $T_2 = 1$

$Method$	N_m	$\epsilon_2(T_1)$	$\delta_1(T_1)$	$\epsilon_2(T_2)$	$\delta_1(T_2)$	$\epsilon_1(T_1)$	$\delta_1(T_1)$
GAN	9	4.10^{-6}	.0042	4.10^{-4}	.060	.0015	.008
FDS	18	3.10^{-6}	.0034	3.10^{-4}	.050	.001	.0066
$FDSES$	12	3.10^{-6}	.0035	3.10^{-4}	.052	.001	.0066

3. **Some results for initial condition** $u_0(x) = (1 - x/2)sin(\pi x)$.
In table 3 the following results for calculation of the problem (29) are shown.

Table 3: The values of N_m, ϵ_1, δ_1, ϵ_2, obtained with $T_1 = 0.1$, $T_2 = 1$

$Method$	N_m	$\epsilon_2(T_1)$	$\delta_1(T_1)$	$\epsilon_2(T_2)$	$\delta_1(T_2)$	$\epsilon_1(T_1)$	$\delta_1(T_1)$
GAN	5	4.10^{-6}	.0085	3.10^{-4}	.298	.0015	.027
FDS	18	3.10^{-6}	.0074	2.10^{-4}	.251	.0009	.018
$FDSES$	13	3.10^{-6}	.0076	2.10^{-4}	.290	.001	.018

In figs. 13, 14 the results of calculation with GAN by $T = 0.1, \epsilon_1 = 0, \epsilon_2 = 4.10^{-6}, \delta = 0.00845$ and $T = 1$, $\epsilon_1 = 0$, $\epsilon_2 = 3.10^{-4}$, $\delta = 0.298$ are represented.

For (25) we have similar results with FDS and FDSES methods, but using GAN the error is 2-3 times larger.

7. Conclusions

- Lattes-Lions technique for regularization to linear ill-posed problems with two small parameters ϵ_1, ϵ_2 is used.
- The effect of the regularization is investigated by numerical calculation with Matlab solver "ode15" at different values of two ϵ values and the solution of ill-posed problem with 3 different methods is obtained.
- The stability of bounded solutions for the continuous and discrete problems and the solvability in the Sobolev space are investigated and some theoretical estimations are obtained.

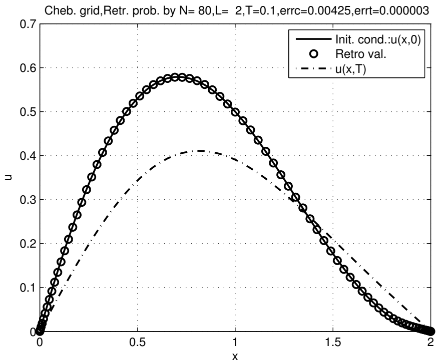

Figure 11: Functions $u(x, 0 = (1 - x/2)sin(\pi x/2))$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.0042$ for $T = 0.1$ with GAN.

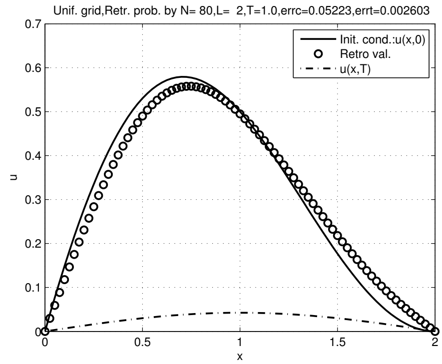

Figure 12: Functions $u(x, 0) = (1 - x/2)sin(\pi x/2)$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.0522$ for $T = 1$ with FDSES.

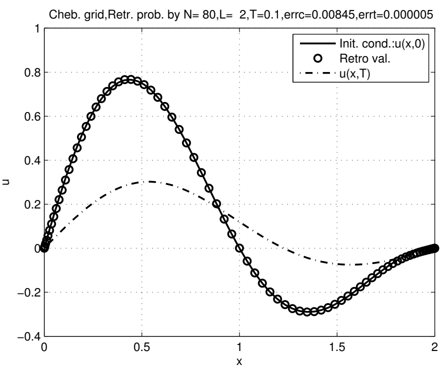

Figure 13: Functions $u(x, 0) = (1 - x/2)sin(\pi x)$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.00845$ for $T = 0.1$ with GAN.

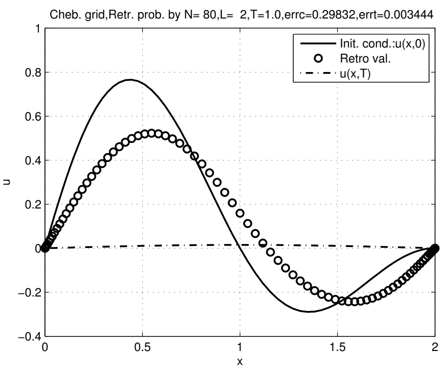

Figure 14: Functions $u(x, 0) = (1 - x/2)sin(\pi x)$, $u(x, T)$, $u_\epsilon(x, 0)$, $\delta = 0.298$ for $T = 1$ with GAN.

- To determine the parameters ϵ some theoretical estimations are obtained.
- The methods FDSES and GAN are considered for comparison of the results of calculation by different numerical methods with higher order of approximation.
- The ill-posed problem for non-linear parabolic PDE may be regularized by introducing higher order differential operators and construction of monotonous continuous functions. By numerical simulation it is found that both approaches are equivalent.
- It is shown that increasing the accuracy of approximation for the discrete problem allows to decrease the number of time steps to obtain the same results.
- Two different solutions of non-linear PDE are obtained for fixed value of the source term depending on the change of this term (this leads to hysteresis phenomena in the solutions).

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