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A SINGULAR HARTMAN INEQUALITY FOR EXISTENCE OF SOLUTIONS TO NONLINEAR SYSTEMS OF SINGULAR, SECOND ORDER BOUNDARY VALUE PROBLEMS

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Abstract: This paper examines the existence of solutions for nonlinear systems of singular, second order boundary value problems (BVPs). Topological techniques are applied to obtain new existence results for solutions to these problems. At the heart of the methods is the existence of novel *a priori* bounds on solutions, achieved via innovative differential inequalities. In addition, these results apply to a wide-ranging selection of singular BVPs arising in modeling of physical phenomena.

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1. Introduction

This paper presents new *a priori* bounds and existence results for systems of singular nonlinear second order boundary value problems (BVPs). The work herein builds on that of Hartman [18] and Gaines & Mawhin [15] and, more recently, the work of Tisdell & Tan [33]. The main problem investigated herein

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is the following differential equation

$$(p(t)\mathbf{y}'(t))' = p(t)q(t)\mathbf{f}(t, \mathbf{y}(t), p(t)\mathbf{y}'(t)), \quad t \in (0, T), \tag{1.1}$$

with various types of boundary conditions given by

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{g}_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t)), \tag{1.2}$$

$$\gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = \mathbf{g}_2(\mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)), \tag{1.3}$$

with $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$, $\beta, \delta \geq 0$. The vector–valued functions $\mathbf{g_i} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for $i = 1, 2$ are continuous functions and the norm chosen to be used throughout the paper for vector-valued terms is

$$\|\mathbf{u}\| := \sqrt{\sum_{k=1}^{n} |u_k|^2}, \quad \text{for} \quad \mathbf{u} \in \mathbb{R}^n.$$

Denote the spaces of continuous, continuously differentiable and twice continuously differentiable functions mapping an interval $[a, b] \subset \mathbb{R}$ to $\mathbb{R}^n$ by $C([a, b]; \mathbb{R}^n)$, $C^1([a, b]; \mathbb{R}^n)$ and $C^2([a, b]; \mathbb{R}^n)$ respectively. In addition, the function $\mathbf{f}$ is assumed to be defined and continuous on at least the set

$$\Omega_R := \{(t, \mathbf{u}, \mathbf{v}) \in [0, T] \times \mathbb{R}^{2n} \ : \ \|\mathbf{u}\| \leq R, \ \mathbf{v} \in \mathbb{R}^n\}.$$

Furthermore, we assume

$$p \in C([0, T]; \mathbb{R}) \cap C^1((0, T); \mathbb{R}) \ \text{ with } \ p > 0 \ \text{ on } \ (0, T) \tag{1.4}$$

$$q \in C((0, T); \mathbb{R}) \ \text{ with } \ q > 0 \ \text{ on } \ (0, T). \tag{1.5}$$

A solution to (1.1)–(1.3) is a $\mathbf{y} \in C([0, T]; \mathbb{R}^n) \cap C^2((0, T) : \mathbb{R}^n)$ with $p\mathbf{y}' \in C([0, T]; \mathbb{R}^n)$ such that $\mathbf{y}$ satisfies (1.1)–(1.3).

Above, the constant $R \geq 0$ will be determined by novel singular differential inequalities herein and in turn produce the existence of *a priori* bounds on the solutions. This allows for relaxed conditions regarding continuity of $\mathbf{f}$ in the classical theory for singular BVPs appearing in the works of [1], [7], [29] and [30]. For instance, the function $\mathbf{f}$ may have a singularity in the $\mathbf{y}$ variable; however, this paper proves that if appropriate *a priori* bounds on solutions (and their derivatives) can be obtained then it is possible via the results herein to yield existence of solutions to the BVP. The existence component is achieved by modifying the singular BVP and then applying topological methods, such as the Topological Transversality method of Granas, Guenther & Lee [17].

Notice that $\mathbf{f}$ is still required to be continuous at $\mathbf{y} = \mathbf{0}$. In the scalar scenario, the authors Baxely [4] and Pennline [28] have investigated this case and where f has a singularity when $y \neq 0$. More recently though, Ford & Pennline [14] considered the scalar derivative independent singular problem with a homogeneous Neumann boundary condition. The methods of [14], [28] differ from this paper. They search for solutions in finite regions where f is an increasing and differentiable function in y, and then they apply Picard iterative method to yield existence. Also, the papers [4], [14], [28], provide interesting examples of solutions to physical phenomena, such as thermal distribution in the human head, reactant concentration in a chemical reactor and oxygen diffusion in cells with Michaelis–Menten kinetics.

The differential inequalities that yield *a priori* bounds on $\mathbf{y}$ have been studied extensively in the non–singular case by many authors; who developed interesting results for *a priori* bounds on the $\mathbf{y}$ term such as [13], [15], [17], [18], [23], [31] and [33].

The derivative dependent BVP has an added element of difficultly compared to a derivative independent BVP. For these BVPs, the *a priori* bounds on the derivative of possible solutions are needed too. In the literature, this means additional conditions are placed on $\mathbf{f}$ and a common condition is the Nagumo–Bernstein condition [27]. There is much known about the Nagumo–Bernstein and other Nagumo conditions for the scalar non-singular case, see [6], [17], [21], [19], [24], [25], [27] and [32]. In the non–singular vector-valued scenario, Heinz [20] gave a discussion and an example on how further additional assumptions were even needed to yield *a priori* bounds on the derivatives. An example constructed in the singular setting is as follows:

$$(t^{1-r}\mathbf{y}')' = -t^{r-1}\|t^{1-r}\mathbf{y}'\|^2\mathbf{y}, \quad t \in (0,1), \quad 0 \leq r \leq 1; \tag{1.6}$$

$$\mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{y}(1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad n = 2. \tag{1.7}$$

Observe that the binary vector $\mathbf{y}(t) = (\cos(mt^r), \sin(mt^r))$ where $m = (2k-1)\pi$ for $k \in \mathbb{Z}$ is a solution to (1.6)–(1.7) with $\|\mathbf{y}(t)\| = 1$ and $\|t^{1-r}\mathbf{y}'(t)\| = rm$. However, there is no one bound for the derivative of solutions to (1.6)–(1.7) since the constant m is arbitrary. This paper targets the more difficult case of the vector–valued scenario for singular BVPs like (1.6)–(1.7). The results are motivated by the non–singular case and meritable results have been given by Fabry [9]; Fabry & Habets [12], [11], [10]; Gaines & Mawhin [15]; George & York [16]; Hartman [18]; Mawhin [26]; Schmitt & Thompson [31] and Tisdell & Tan [33].

In Section 2, the function $\mathbf{f}$ is assumed to be $C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$; the first novel result is presented herein for singular BVPs and uses a novel singular differential inequality motivated from the work of Hartman [18]. This result yields *a priori* bounds on all possible solutions to the BVP. The next result, Theorem 2.4 is particularly interesting since only one differential inequality is needed to prove *a priori* bounds for both the solution and their derivatives. It removes a condition in the theory of [13], [19], [18]or removing the assumption of having a Nagumo condition. This particular inequality was first introduced by Fewster–Young & Tisdell [13]. The Theorems 2.7–2.12 are novel singular results for the vector–valued Nagumo condition [18], a one–sided Nagumo condition [16] and the following inequality for some non–negative constants V_2, W_2;

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq V_2 \langle \mathbf{v}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + W_2, \qquad \text{for} \quad t \in (0,T), \ \|\mathbf{u}\| \leq R, \quad \mathbf{v} \in \mathbb{R}^n. \tag{1.8}$$

In Section 3, the continuity of $\mathbf{f}$ is relaxed to the set Ω_R and is devoted to the existence of solutions to singular BVP (1.1)–(1.3). The first novel and main result is a general existence theorem for singular systems of BVPs. This result has merit in itself by providing a framework for the existence of solutions to singular vector–valued BVPs. The result does require the assumption of the *a priori* bounds on solutions to (1.1)–(1.3) to be known. Therefore, by using this result and incorporating it with the previous novel results on *a priori* bounds in this paper, the existence of solutions to (1.1)–(1.3) can be obtained.

In the last section, an example is presented to illustrate the benefit these results have in relation to the literature.

2. A Priori Bounds on Solutions

The first main result yields *a priori* bounds to solutions to the BVP (1.1) with various boundary conditions. In this result, the boundary conditions are only assumed to be continuous and with $\beta, \delta > 0$. For the applicability of result, the aim is to first seek a $R \geq 0$, such that the differential inequality (2.1) holds then to proceed whether the other conditions hold.

Theorem 2.1. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, $\beta, \delta > 0$ *and let* (1.4), (1.5) *hold. If there exists a non–negative constant* R *such that*

$$\left\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\right\rangle + \|\mathbf{v}\|^2 > 0, \qquad \text{when } \langle \mathbf{u},\mathbf{v}\rangle = 0 \ \text{ and } \ \|\mathbf{u}\| \geq R \tag{2.1}$$

and

$$\langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s})\rangle > -\alpha\|\mathbf{r}\|^2, \quad \langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s})\rangle < \gamma\|\mathbf{r}\|^2 \quad \text{for} \quad \|\mathbf{r}\| > R, \tag{2.2}$$

then every possible solution $\mathbf{y} = \mathbf{y}(t)$ *to* (1.1), (1.2), (1.3) *satisfies*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le R.$$

Proof. Let $\mathbf{y} := \mathbf{y}(t)$ for $t \in [0,T]$ be a possible solution to (1.1)–(1.3). Suppose $r(t) := \|\mathbf{y}(t)\|^2 - R^2$ and observe that

$$(p(t)r'(t))' = \big\langle \mathbf{y}(t), (p(t)\mathbf{y}'(t))' \big\rangle + p(t)\|\mathbf{y}'(t)\|^2, \quad \text{for all} \quad t \in (0,T).$$

Suppose that r has a positive maximum at some point $s \in (0,T)$ when $r(s) \ge 0$. That is,

$$r''(s) \le 0, \quad \text{when} \quad r'(s) = 0 \quad \text{and} \quad r(s) \ge 0.$$

Every solution $\mathbf{y}$ to (1.1) implies the differential inequality (2.1) becomes

$$\begin{aligned} &\big\langle \mathbf{y}, p^2(t)q(t)\mathbf{f}(t,\mathbf{y},p\mathbf{y}') \big\rangle + p^2(t)\|\mathbf{y}'\|^2 \\ &\qquad = p(t)\left[\big\langle \mathbf{y}, (p(t)\mathbf{y}')' \big\rangle + p(t)\|\mathbf{y}'\|^2\right] > 0 \end{aligned} \tag{2.3}$$

for $t \in (0,T)$ when $\langle \mathbf{y}(t), p(t)\mathbf{y}'(t)\rangle = 0$ and $\|\mathbf{y}(t)\| \ge R$. At the point s, it now follows from (2.3) that $(p(s)r'(s))' > 0$ and in turn, $r''(s) > 0$ since $r'(s) = 2\langle \mathbf{y}(s), \mathbf{y}'(s)\rangle = 0$. This is a contradiction and means that $r(t)$ does not have a maximum for any point $t \in (0,T)$. We next proceed by cases and supposing $r(0)$ is a positive maximum and $r(0) > 0$. This means that $\lim_{t\to 0^+} p(t)r'(t) \le 0$. If $r(0) > 0$ then see that by the boundary condition (1.2) and (2.2)

$$\lim_{t\to 0^+} p(t)r'(t) = 2\left\langle \mathbf{y}(0), \frac{\alpha\mathbf{y}(0) + \mathbf{g}_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t))}{\beta} \right\rangle > 0. \tag{2.4}$$

So $r(0)$ is strictly increasing near zero and is not the maximum of $r(0)$ on $[0,T]$. The case when $s = T$ is the same argument and follows. Thus, we have $\|\mathbf{y}(t)\| \le R$ for all $t \in [0,T]$. □

Remark 2.2. Note that by using modified boundary conditions:

$$\begin{aligned} -\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) &= \mathbf{g}_1(\mathbf{y}(0), \\ \lim_{t\to 0^+} p(t)\mathbf{y}'(t)) &+ \frac{\beta\mathbf{y}(0)}{n}, \ n \in \mathbb{Z}^+, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = \mathbf{g}_2(\mathbf{y}(T), \\ \lim_{t\to T^-} p(t)\mathbf{y}'(t)) - \frac{\delta\mathbf{y}(T)}{n}, \end{aligned} \tag{2.6}$$

the strict inequalities in (2.2) can be weaken to allow equality. This is proved using the *Arzela–Ascoli* theorem [3].

The previous theorem does not allow for Dirichlet boundary conditions, that is when $\beta = 0$ or $\gamma = 0$. In this case, the condition (2.7) is introduced. Note that if condition (2.7) holds then it implies the sufficient conditions (2.2).

Theorem 2.3. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, *let* (1.4), (1.5) *and* $\alpha, \gamma \neq 0$ *hold. If there exists a non–negative constant* R *such that* (2.1) *and*

$$\max\left\{ \sup_{\mathbf{r},\mathbf{s}\in\mathbb{R}^n} \frac{\|\mathbf{g}_1(\mathbf{r},\mathbf{s})\|}{|\alpha|}, \sup_{\mathbf{r},\mathbf{s}\in\mathbb{R}^n} \frac{\|\mathbf{g}_2(\mathbf{r},\mathbf{s})\|}{|\gamma|} \right\} \leq R < \infty \tag{2.7}$$

are satisfied then every possible solution $\mathbf{y} = \mathbf{y}(t)$ *to* (1.1), (1.2), (1.3) *satisfies*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R.$$

Proof. The proof mimics that of Theorem 2.1 and so is omitted. The only difference is showing $\|\mathbf{y}(0)\|, \|\mathbf{y}(T)\| \leq R$. To see this, let $r(t) := \|\mathbf{y}(t)\|^2 - R^2$ for $t \in [0,T]$ then (2.7) implies $r(0), r(T) \leq 0$. Thus, all possible solutions (1.1), (1.2), (1.3) satisfy $\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R$. □

The next result builds on Theorem 2.1 in a paper by Fewster–Young and Tisdell [13] for the boundary conditions (1.2) and (1.3) when $\mathbf{g}_1, \mathbf{g}_2$ are vector–valued constants $\mathbf{c}, \mathbf{d}$ respectively. The case where $\mathbf{g}_i$'s are arbitrary continuous functions has a more involved proof and since the applicability to modelling is limited, the result is omitted for brevity. In addition, the majority of the literature [30], [29], [17] assumes the condition $\alpha, \beta, \gamma, \delta > 0$, the following results replaces this condition with (2.8). In that paper and in the works of Hartman [19], [18] there is a requirement that the *a priori* bound on the solutions, $R > 0$ satisfies the condition $2VR < 1$ where V is in the inequality (2.9). The forecoming results do not require this condition to yield *a priori* bounds on the derivative of solutions. result assumes that $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$ for the sole purpose of producing *a priori* bounds on the solutions and their derivatives.

Theorem 2.4. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, (1.4), (1.5) *hold. Suppose* $\mathbf{g}_1 \equiv \mathbf{c}$, $\mathbf{g}_2 \equiv \mathbf{d}$ *where* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ *and there is an* $M \geq 0$ *such that* $p^2 q \leq M$ *on* $[0,T]$,

$$K_1 := \int_0^T \frac{1}{p(s)}\, ds < \infty, \;\; K_2 := \int_0^T p(s)q(s)\, ds < \infty \text{ and } \gamma K_1 \neq \left[\frac{\beta\gamma}{\alpha} + \delta\right]. \tag{2.8}$$

If there are constants V, W *such that*

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq 2V\left(\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \frac{\|\mathbf{v}\|^2}{M}\right) + W, \quad \text{for all} \quad (t,\mathbf{u},\mathbf{v}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \tag{2.9}$$

then all possible solutions $\mathbf{y} = \mathbf{y}(t)$ *to BVP* (1.1), (1.2), (1.3) *with* $\alpha, \beta, \gamma, \delta \neq 0$ *satisfy*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq \frac{\|\mathbf{c}\|}{|\alpha|} + \left(\left|\frac{\beta}{\alpha}\right| + K_1\right)\left(\frac{\|\gamma\mathbf{c} + \alpha\mathbf{d}\| + |\gamma|(|\alpha|K_1 + |\beta|)\eta}{\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta\right|} + \eta\right)$$

and

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq \frac{\|(\gamma\mathbf{c} + \alpha\mathbf{d})\| + |\gamma|(|\alpha|K_1 + |\beta|)\eta + \eta\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta\right|}{\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta\right|}$$

where

$$\eta := V\left[\frac{\|\mathbf{d}\|^2}{|\delta||\gamma|} + \frac{\|\mathbf{c}\|^2}{|\alpha||\beta|}\right] + WK_2.$$

Proof. Let $\mathbf{y} = \mathbf{y}(t)$ for $t \in [0,T]$ be any solution to the BVP (1.1)–(1.3). From O'Regan [30][Theorem 3.3], the integral representation for all possible solutions to the BVP (1.1)–(1.3) is

$$\mathbf{y}(t) = \mathbf{B} + \mathbf{A}\int_0^t \frac{ds}{p(s)} - \int_0^t \frac{1}{p(s)}\int_s^T p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\, dxds \tag{2.10}$$

where

$$\mathbf{A} := \frac{(\gamma\mathbf{c} + \alpha\mathbf{d}) + \gamma(\alpha\int_0^T \frac{1}{p(s)}\int_s^T p(x)q(x)\mathbf{f}(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\, dxds + \beta\int_0^T pq\mathbf{f}(x,\mathbf{y},p\mathbf{y}')\, dx)}{\alpha(\gamma\int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta}$$

and

$$\mathbf{B} := \frac{\beta\mathbf{A} - \mathbf{c}}{\alpha} - \frac{\beta}{\alpha}\int_0^T p(x)q(x)\mathbf{f}(x, \mathbf{y}, p\mathbf{y}')\ dx.$$

To obtain *a priori* bounds for all possible solutions, we will estimate the term,

$$\kappa := \left\| \int_0^T p(x)q(x)\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\ dxds \right\|.$$

We now prove that

$$\kappa \leq V\left[\frac{\|\mathbf{d}\|^2}{|\delta||\gamma|} + \frac{\|\mathbf{c}\|^2}{|\alpha||\beta|}\right] + WK_2 = \eta. \tag{2.11}$$

If we employ (2.9) and $p^2q \leq M$ on $[0, T]$ then we obtain

$$\begin{aligned}
\kappa &\leq \int_0^T p(x)q(x)\|\mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\|\ dx \\
&\leq \int_0^T 2V(\langle \mathbf{y}(x), p(x)q(x)\mathbf{f}(t, \mathbf{y}(x), p(x)\mathbf{y}'(x))\rangle \\
&\quad + \frac{p^3(x)q(x)}{M}\|\mathbf{y}'(x)\|^2) + Wp(x)q(x)\ dx \\
&\leq \int_0^T 2V(\langle \mathbf{y}(x), p(x)q(x)\mathbf{f}(t, \mathbf{y}(x), p(x)\mathbf{y}'(x))\rangle + p(x)\|\mathbf{y}'(x)\|^2) \\
&\quad + Wp(x)q(x)\ dx.
\end{aligned}$$

We now use (1.1) and simplify to yield

$$\kappa \leq \int_0^T 2V(\langle \mathbf{y}(x), (p(x)\mathbf{y}'(x))'\rangle + p(x)\|\mathbf{y}'(x)\|^2)\ dx + WK_2. \tag{2.12}$$

Consider the function $r(t) := \|\mathbf{y}(t)\|^2$ and notice that

$$(p(t)r'(t))' = \langle \mathbf{y}(t), (p(t)\mathbf{y}'(t))'\rangle + p(t)\|\mathbf{y}'(t)\|^2.$$

We now may replace and integrate (2.12) to obtain

$$\kappa \leq 2V\left[p(T)r'(T) - p(0)r'(0)\right] + WK_2.$$

By substituting the boundary conditions (1.2), (1.3), we have

$$\kappa \leq 2V\left[\left\langle \frac{\mathbf{d} - \gamma\mathbf{y}(T)}{\delta}, \mathbf{y}(T)\right\rangle - \left\langle \frac{\mathbf{c} + \alpha\mathbf{y}(0)}{\beta}, \mathbf{y}(0)\right\rangle\right] + WK_2.$$

By applying the Cauchy–Schwarz inequality [8] to some terms, we have

$$\kappa \leq 2V \left[\frac{\|\mathbf{d}\|}{|\delta|} \|\mathbf{y}(T)\| - \frac{\gamma \|\mathbf{y}(T)\|^2}{\delta} + \frac{\|\mathbf{c}\|}{|\beta|} \|\mathbf{y}(0)\| - \frac{\alpha \|\mathbf{y}(0)\|^2}{\beta} \right] + W K_2.$$

We now apply the Peter–Paul inequality [8], that is

$$2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2, \quad \text{where} \quad a,\ b \in \mathbb{R} \quad \text{and} \quad \epsilon > 0,$$

such that it yields

$$\kappa \leq V \left[\frac{\|\mathbf{d}\|^2}{|\delta||\gamma|} + \frac{\|\mathbf{c}\|^2}{|\alpha||\beta|} \right] + W K_2.$$

This proves inequality (2.11), that is $\kappa \leq \eta$. We now see that by estimating $\mathbf{A}$ using (2.11), we have

$$\|\mathbf{A}\| \leq \frac{\|(\gamma \mathbf{c} + \alpha \mathbf{d})\| + |\gamma|(|\alpha| K_1 + |\beta|)\kappa}{\left| \alpha(\gamma \int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta \right|} \leq \frac{\|(\gamma \mathbf{c} + \alpha \mathbf{d})\| + |\gamma|(|\alpha| K_1 + |\beta|)\eta}{\left| \alpha(\gamma \int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta \right|}.$$

It now follows from integral representation (2.10) that all solutions to $\mathbf{y} = \mathbf{y}(t)$ for $t \in [0, T]$ to (1.1), (1.2), (1.3) satisfy

$$\|\mathbf{y}\| \leq \frac{\|\mathbf{c}\|}{|\alpha|} + \left(\left| \frac{\beta}{\alpha} \right| + K_1 \right) \left(\frac{\|(\gamma \mathbf{c} + \alpha \mathbf{d})\| + |\gamma|(|\alpha| K_1 + |\beta|)\eta}{\left| \alpha(\gamma \int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta \right|} + \eta \right).$$

Furthermore, by differentiating (2.10), we see that the equivalent integral representation for $p(t)\mathbf{y}'(t)$ for $t \in [0, T]$ is

$$p(t)\mathbf{y}'(t) = \mathbf{A} - \int_t^T p(x) q(x) \mathbf{f}(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\ dx, \quad t \in [0, T].$$

If we estimate then we obtain

$$\sup_{t \in (0,T)} \|p(t)\mathbf{y}'(t)\| \leq \|\mathbf{A}\| + \kappa \leq$$

$$\frac{\|(\gamma \mathbf{c} + \alpha \mathbf{d})\| + |\gamma|(|\alpha| K_1 + |\beta|)\eta + \eta \left| \alpha(\gamma \int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta \right|}{\left| \alpha(\gamma \int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta \right|}. \quad \square$$

This result is new for the non–singular case, that is when $p \equiv 1 \equiv q$.

Corollary 2.5. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$*. Suppose* $\mathbf{g}_1 \equiv \mathbf{c}$*,* $\mathbf{g}_2 \equiv \mathbf{d}$ *where* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$*. If there are constants* V*,* W *such that*

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq 2V(\langle \mathbf{u}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2) + W, \qquad \text{for} \quad (t,\mathbf{u},\mathbf{v}) \in [0,T] \times \mathbb{R}^{2n} \tag{2.13}$$

holds then all possible solutions $\mathbf{y} = \mathbf{y}(t)$ *for* $t \in [0,T]$ *to BVP* (1.1), (1.2), (1.3) *with* $\alpha, \beta, \gamma, \delta \neq 0$ *satisfy*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq \frac{\|\mathbf{c}\|}{|\alpha|} + \left(\left|\frac{\beta}{\alpha}\right| + T\right)\left(\frac{\|(\gamma\mathbf{c} + \alpha\mathbf{d})\| + |\gamma|(|\alpha|T + |\beta|)\eta}{|\alpha(\gamma T + \delta) + \gamma\beta|} + \eta\right)$$

and

$$\sup_{t\in(0,T)} \|\mathbf{y}'(t)\| \leq \frac{\|(\gamma\mathbf{c} + \alpha\mathbf{d})\| + |\gamma|(|\alpha|T + |\beta|)\eta + \eta\,|\alpha(\gamma T + \delta) + \gamma\beta|}{|\alpha(\gamma T + \delta) + \gamma\beta|}$$

where

$$\eta := V\left[\frac{\|\mathbf{d}\|^2}{|\delta||\gamma|} + \frac{\|\mathbf{c}\|^2}{|\alpha||\beta|}\right] + WT.$$

Furthermore, if we include the condition; suppose N is a non–negative constant such that

$$\left\langle \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)\right\rangle - \left\langle \mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t)\right\rangle \leq N \tag{2.14}$$

to Theorem 2.4 then for all various boundary conditions of (1.2), (1.3) that satisfy (2.14) there is a $S > 0$ such that

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S.$$

The condition appeared in a paper by Fabry [9] as Condition 3.1, for non–singular second order boundary value problems with homogeneous boundary conditions. Examples of boundary conditions that satisfy (2.14) are the homogeneous Dirichlet conditions; periodic conditions; homogeneous Neumann conditions; and now conditions (1.2), (1.3); conditions with a homogeneous Neumann boundary on a end point.

Theorem 2.6. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, (1.4), (1.5), (2.8), (2.14) *hold. Suppose* $\mathbf{g}_1 \equiv \mathbf{c}$, $\mathbf{g}_2 \equiv \mathbf{d}$ *where* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ *and there is* $M > 0$ *such that* $p^2 q \leq M$

on $[0,T]$. *If there are constants* V, W *such that* (2.9) *is satisfied then all possible solutions* $\mathbf{y}=\mathbf{y}(t)$ *to BVP* (1.1), (1.2), (1.3) *with* $\alpha \neq 0$ *satisfy*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le \frac{\|\mathbf{c}\|}{|\alpha|} + \left(\left|\frac{\beta}{\alpha}\right| + K_1\right)\left(\frac{\|(\gamma\mathbf{c}+\alpha\mathbf{d}\| + |\gamma|(|\alpha|K_1+|\beta|)\eta}{\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)}+\delta)+\gamma\beta\right|} + \eta\right) \tag{2.15}$$

and

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \le \frac{\|(\gamma\mathbf{c}+\alpha\mathbf{d})\| + |\gamma|(|\alpha|K_1+|\beta|)\eta + \eta\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)}+\delta)+\gamma\beta\right|}{\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)}+\delta)+\gamma\beta\right|} \tag{2.16}$$

where

$$\eta := 2VN + WK_2.$$

Proof. The proof is essentially the same as Theorem 2.4 and is omitted for brevity. The difference is

$$\kappa \le 2VN + WK_2.$$

□

Note that for the cases when $\alpha = 0$, we assume $\gamma \neq 0$ and this yields a similar integral representation and *a priori* bounds as in (2.10) and (2.15), (2.16) respectively. For the various boundary conditions mentioned above, the *a priori* bounds (2.15), (2.16) can be improved to obtain similar bounds obtained by Fewster–Young & Tisdell [13]. For example, a homogeneous Dirichlet–Neumann boundary condition ($\beta = 0,\ \gamma = 0$) yields the same bounds as in Fewster–Young & Tisdell [13].

The next result extends Hartman's [19] theorem from 1960's to systems of singular second order BVPs. Hartman's result broadened the Nagumo–Bernstein result presented in 1937 [27] to apply to systems of non–singular equations. The following result achieves a bound for all possible solutions and their derivatives'; it uses a combination of the approaches by Hartman [19], Kelley & Peterson [22] and De Coster & Habets [6]. It noteworthy to mention here that the boundary conditions do not play a role in the result like the previous results.

Theorem 2.7. *Let* $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$ *that is a solution to* (1.1) *and let* (1.4) *hold. Let* λ, M_0, M_1, K, R, S *be non–negative constants such that* $M_0 := \sup_{t\in[0,T]} p(t)$,

$$K_1 := \int_0^T \frac{ds}{p(s)} < \infty \quad \text{and} \quad \int_0^T |p'(t)|\, dt \le M_1. \tag{2.17}$$

If there is a real valued function $r(t)$ *of class* $C^2((0,T);\mathbb{R}) \cap C([0,T];\mathbb{R})$ *satisfying*

$$\|\mathbf{u}\| \le R, \qquad \|(p(t)\mathbf{y}'(t))'\| \le (pr')'(t), \quad \text{where} \quad \max_{t\in[0,T]} |r(t)| \le K \tag{2.18}$$

and a positive function $\phi(x)$ *where* $0 \le x < \infty$ *satisfying*

$$\|\mathbf{u}\| \le R, \quad \|p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \le \phi(\|\mathbf{v}\|) \quad \text{and} \int_\lambda^S \frac{x}{\phi(x)}\, dx \ge \lambda K_1 + 2K \tag{2.19}$$

where $S \ge \lambda := \frac{(2M_0+M_1)(R+K)}{T}$ *then all possible solutions to* (1.1) *satisfy*

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \le S. \tag{2.20}$$

Proof. We first will find a bound on $\|p(t)\mathbf{y}'(t)\|$ for some $t \in (0,T)$. Let $0 < \alpha < T$ and consider the integral equation,

$$\alpha p(t)\mathbf{y}'(t) = p(t+\alpha)\mathbf{y}(t+\alpha) - p(t)\mathbf{y}(t) + \int_t^{t+\alpha} p'(s)\mathbf{y}(s)\, ds - \int_t^{t+\alpha} (t+\alpha-s)(p(s)\mathbf{y}'(s))'\, ds,$$

for $0 \le t \le T-\alpha$. See that (2.18) implies

$$\alpha\|p(t)\mathbf{y}'(t)\| \le 2M_0R + M_1R + \int_t^{t+\alpha} (t+\alpha-s)(p(s)r'(s))'\, ds.$$

By completing the two integration by parts, we have

$$\alpha\|p(t)\mathbf{y}'(t)\| \le 2M_0R + M_1R + p(t+\alpha)r(t+\alpha) - p(t)r(t) - \int_t^{t+\alpha} p'(s)r(s)\, ds - \alpha p(t)r'(t).$$

Thus,

$$\alpha\|p(t)\mathbf{y}'(t)\| \le 2M_0(R+K) + M_1(R+K) - \alpha p(t)r'(t), \qquad \text{for } \ 0 \le t \le T-\alpha. \quad (2.21)$$

Similarly, consider the integral equation

$$\alpha p(t)\mathbf{y}'(t) = p(t)\mathbf{y}(t) - p(t-\alpha)\mathbf{y}(t-\alpha) - \int_{t-\alpha}^{t} p'(s)\mathbf{y}(s)\,ds + \int_{t-\alpha}^{t}(s+\alpha-t)(p(s)\mathbf{y}'(s))'\,ds$$

for $\alpha \le t \le T$. Applying the same method, we get

$$\alpha\|p(t)\mathbf{y}'(t)\| \le (2M_0+M_1)(R+K) + \alpha p(t)r'(t), \quad \text{for } \ \alpha \le t \le T. \quad (2.22)$$

If $\alpha = T/2$ then (2.21) and (2.22) imply

$$\lim_{t\to T/2}\|p(t)\mathbf{y}'(t)\| \le \frac{(2M_0+M_1)(R+K)}{T} = \lambda.$$

Suppose there is $t_0 \in [0,T/2]$ such that $\lim_{t\to t_0}\|p(t)\mathbf{y}'(t)\| > S$. The inequalities (2.19) and (2.21) imply that

$$\frac{\left|\left\langle p(t)\mathbf{y}'(t), p^2(t)q(t)\mathbf{f}(t,\mathbf{y}(t),p(t)\mathbf{y}'(t))\right\rangle\right|}{p(t)\phi(\|p(t)\mathbf{y}'(t)\|)} \le \frac{\|p(t)\mathbf{y}'(t)\|}{p(t)} \le \frac{(2M_0+M_1)(R+K)}{Tp(t)} - r'(t)$$

for $t \in [0,T/2]$. By integrating both sides from t_0 to $T/2$ and changing variables, we have

$$\left|\int_{\lim_{t\to t_0}\|p(t)\mathbf{y}'(t)\|}^{\lim_{t\to T/2}\|p(t)\mathbf{y}'(t)\|}\frac{x}{\phi(x)}\,dx\right| \le \lambda K_1 + r(t_0) - r(T/2).$$

The condition $|r(t)| \le K$ for all $t \in [0,T]$ gives

$$\int_{\lambda}^{S}\frac{x}{\phi(x)}\,dx < \int_{\lambda}^{\lim_{t\to t_0}\|p(t)\mathbf{y}'(t)\|}\frac{x}{\phi(x)}\,dx \le \lambda K_1 + 2K.$$

The argument if $t_0 \in [T/2,T]$ is the same and yields the same conclusion. However, this contradicts (2.19) and thus

$$\sup_{t\in(0,T)}\|p(t)\mathbf{y}'(t)\| \le S. \qquad \square$$

Remark 2.8. In most applications, the function has the form $p(t) = t^b$, $b \geq 0$ and $t \in [0, T]$. For example, this occurs in thermal distribution in the human head, reactant concentration in a chemical reactor and oxygen diffusion in cells with Michaelis–Menten kinetics and many more can be found in the literature of [17], [2], [14], [30]. Observe that this choice of p implies the integral condition (2.17) is satisfied;

$$\int_0^T |p'(t)|\ ds = T^b < \infty.$$

In addition, a common choice for r, ϕ are $r = \|\mathbf{y}\|^2$ and $\phi(x) = Ax^2 + B$ where A, B are positive constants. These choices appear in the literature [27], [19], [18], [9].

This naturally leads to the next theorem where if the r is a non–negative function then the previous theorem has a sharper bound.

Theorem 2.9. *Let $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$ that is a solution to (1.1) and let (1.4) hold. Let λ, M_0, M_1, K, R, S be non–negative constants such that $M_0 := \sup_{t\in[0,T]} p(t)$ and (2.17) hold. If there is a real valued non–negative function $r(t)$ of class $C^2((0,T);\mathbb{R}) \cap C([0,T];\mathbb{R}^n)$ satisfying (2.18) and there is a positive function $\phi(x)$ where $0 \leq x < \infty$ satisfying*

$$\|\mathbf{u}\| \leq R, \quad \|p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq \phi(\|\mathbf{v}\|) \quad \text{and} \quad \int_\lambda^S \frac{x}{\phi(x)}\ dx \geq \lambda K_1 + K \tag{2.23}$$

where $S \geq \lambda := \frac{M_0 R + (M_0+M_1)(R+K)}{T}$ then

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S.$$

These prior theorems are tailored for systems of singular second order BVPs. As mentioned earlier, they required additional sufficient conditions to prove the results compared to in the scalar setting. The particular conditions required were (2.17) and (2.18). The next results are presented with the focus on the scalar setting. In 1990, Bobisud [5] proved similar results for scalar singular BVPs, however they were subject to the boundary conditions; homogeneous Neumann–Dirichlet and inhomogeneous Dirichlet conditions. The next result was proven by Hartman [18] for the non–singular case in 1960 and our proof is adapted to the singular case.

Theorem 2.10. *Let* $n = 1$, $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$ *that is a solution to* (1.1) *and let* (1.4) *hold. Let* M_0 *be a non–negative constant such that* $M_0 := \sup_{t\in[0,T]} p(t)$. *If* $R \geq 0$, $S \geq 0$, $0 < \eta \leq T$ *and* $\phi(x)$ *is a positive function where* $0 \leq x < \infty$ *satisfying*

$$|y| \leq R, \quad |p^2(t)q(t)f(t,y,v)| \leq \phi(|v|) \quad \text{and} \quad \int_\lambda^S \frac{x}{\phi(x)}\, dx \geq 2R \tag{2.24}$$

where $S \geq \lambda := \frac{2RM_0}{\eta}$ *then*

$$\sup_{t\in(0,T)} |p(t)y'(t)| \leq S.$$

Proof. Suppose that $|p(t)y'(t)|$ assumes its maximum value at $t = z$ where $0 \leq z \leq T$ and $|p(z)y'(z)| > \lambda$. See that there is a point $s \in (0,T)$ such that $|p(s)y'(s)| \leq \lambda$ since

$$|p(s)y'(s)| = \left|\frac{p(s)y(T) - p(s)y(0)}{T}\right| \leq \frac{2M_0R}{T} \leq \frac{2M_0R}{\eta}.$$

Let $t = w$ be the nearest point to $t = z$ where $|p(t)y'(t)| = \lambda$. We now suppose $y'(z) > 0$ and $z > w$ otherwise y is replaced by $-y$. If we consider (2.24) and multiply by $p(t)y'(t) > 0$ then integrating from w to z gives

$$\left|\int_w^z \frac{p(t)y'(t)(p(t)y'(t))'}{\phi(py'(t))}\, dt\right| \leq \int_w^z y'(t)\, dt \leq 2R.$$

By a formal change of variables with $x = p(t)y'(t)$, this yields

$$\int_\lambda^{\lim_{s\to z} p(s)y'(s)} \frac{x}{\phi(x)}\, dt \leq 2R.$$

This implies that $|\lim_{t\to z} p(t)y'(t)| \leq S$ and thus the result follows since $\lim_{t\to z} p(t)y'(t) = \max |p(t)y'(t)|$ for $t \in [0,T]$. □

It is possible to deduce variations of the above theorem, this is motivated by the work done by Bobisud [5].

Theorem 2.11. *Let $n = 1$, $\mathbf{y} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n)$ that is a solution to* (1.1) *and let* (1.4) *hold. If $R \geq 0$, $S \geq 0$, $0 \leq r < 1$, $0 < \eta \leq T$ and $\phi(x)$ is a positive function where $0 \leq x < \infty$ satisfying*

$$|u| \leq R, \quad |f(t,u,v)| \leq \phi(|v|)$$

and

$$\int_\lambda^S \frac{x^r}{\phi(x)}\, dx \geq (2R)^r \left(\int_0^T (p^{r+1}(t)q(t))^{\frac{1}{1-r}}\, dt \right)^{1-r} \tag{2.25}$$

where $S \geq \lambda := \frac{2RM_0}{\eta}$ then

$$\sup_{t\in(0,T)} |p(t)y'(t)| \leq S.$$

Proof. The proof follows the same approach as the previous result. Suppose that $|p(t)y'(t)|$ assumes its maximum value at $t = z$ where $0 \leq z \leq T$ and $|p(z)y'(z)| > \lambda$. Let $t = w$ be the nearest point to $t = z$ where $|p(t)y'(t)| = \lambda$. If we consider (2.25) and multiply by $[p(t)y'(t)]^r > 0$ then integrating from w to z gives

$$\left| \int_w^z \frac{[p(t)y'(t)]^r (p(t)y'(t))'}{\phi(py'(t))}\, dt \right| \leq \int_w^z p^{r+1}(t)q(t)[y'(t)]^r\, dt.$$

By using *Holder's* inequality [8]

$$\int_w^z p^{r+1}(t)q(t)[y'(t)]^r\, dt \leq \left(\int_w^z (p^{r+1}(t)q(t))^{\frac{1}{r-1}}\, dt \right)^{1-r} \left(\int_w^z ([y'(t)]^r)^{1/r}\, dt \right)^r.$$

Furthermore, this gives

$$\int_w^z p^{r+1}(t)q(t)[y'(t)]^r\, dt \leq (2R)^r \left(\int_0^T (p^{r+1}(t)q(t))^{\frac{1}{1-r}}\, dt \right)^{1-r}.$$

The change of variables with $x = p(t)y'(t)$ yields

$$\int_\lambda^{\lim_{s\to z} p(s)y'(s)} \frac{x^r}{\phi(x)}\, dt \leq (2R)^r \left(\int_0^T (p^{r+1}(t)q(t))^{\frac{1}{1-r}}\, dt \right)^{1-r}.$$

This implies that $|\lim_{t\to z} p(t)y'(t)| \leq S$ and thus the result follows since $\lim_{t\to z} p(t)y'(t) = \max |p(t)y'(t)|$ for $t \in [0,T]$. □

The next result is related to a simple inequality that was used by Tisdell & Tan [33] to yield *a priori* bounds on the derivative of a non–singular system of second order BVPs. The result builds on their inequality with more relaxed conditions and the singular case investigated.

Theorem 2.12. *Let* $\mathbf{f} \in C([0,T] \times \mathbb{R}^{2n}; \mathbb{R}^n)$, $R,\ S > 0$, $\delta \neq 0$, $\mathbf{g}_1$, $\mathbf{g}_2$ *be continuous and bounded on* $\mathbb{R}^{2n}$ *by constants* Θ_1, Θ_2 *respectively, and*

$$S \geq \frac{\Theta_2 + |\gamma| R}{\delta}. \tag{2.26}$$

If $\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R$ *for all possible solutions to* (1.1), (1.2), (1.3) *and* $\mathbf{f}$ *has one of the following properties:*

$$\langle \mathbf{v}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle \geq 0, \quad \text{for} \quad t \in (0,T),\ \|\mathbf{u}\| \leq R,\ \|\mathbf{v}\| \geq S, \tag{2.27}$$

or if there exists non–negative constants $V_2,\ W_2$ *such that*

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq V_2 \langle \mathbf{v}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + W_2, \quad \text{for} \quad t \in (0,T),\ \|\mathbf{u}\| \leq R,\ \mathbf{v} \in \mathbb{R}^n, \tag{2.28}$$

then all possible solutions to (1.1), (1.2), (1.3) *satisfy*

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S. \tag{2.29}$$

Proof. Let $w(t) := \|p(t)\mathbf{y}'(t)\|^2$. By contradiction, suppose there is a point $t_0 \in (0,T)$ such that $w(t_0) > S$. By rearranging the boundary condition (1.3), we have

$$\|\lim_{t\to T^-} p(t)\mathbf{y}'(t)\| \leq \frac{\|\mathbf{g}_2(\mathbf{y}(T),\ \lim_{t\to T^-} p(t)\mathbf{y}'(t))\| + |\gamma|\|\mathbf{y}(T)\|}{\delta} \leq S.$$

This implies there is a $s \in (t_0, T)$ such that $w(s) > S^2$ and $w'(s) < 0$. However, for any $t \in (0,T)$ such that $w(t) \geq S^2$; the inequality (2.27) implies that

$$w'(t) = \left\langle p(t)\mathbf{y}'(t), (p(t)\mathbf{y}'(t))'\right\rangle = \left\langle p(t)\mathbf{y}'(t), p(t)q(t)\mathbf{f}(t,\mathbf{y}(t),p(t)\mathbf{y}'(t))\right\rangle \geq 0.$$

This gives a contradiction and (2.29) holds.

To prove the result when (2.28) is satisfied, consider the equivalent integral representation of the differential equation (1.1); that is

$$p(t)\mathbf{y}'(t) = \lim_{t\to T^-} p(t)\mathbf{y}'(t) - \int_t^T p(s)q(s)\mathbf{f}(t,\mathbf{y}(s),p(s)\mathbf{y}'(s))\,ds, \quad \text{for} \quad t \in [0,T]. \tag{2.30}$$

By estimating (2.30) and using (2.28), (1.3), we have

$$\|p(t)\mathbf{y}'(t)\| \le \frac{\Theta_2 + |\gamma|\|\mathbf{y}(T)\|}{\delta} + \int_t^T V_2 \left\langle p(s)\mathbf{y}'(s), p(s)q(s)\mathbf{f}(t,\mathbf{y}(s),p(s)\mathbf{y}'(s))\right\rangle + W_2 p(s)q(s)\ ds$$

for $t \in [0,T]$. Notice that

$$w'(t) = 2\left\langle p(s)\mathbf{y}'(s), (p(s)\mathbf{y}'(s))'\right\rangle = 2\left\langle p(s)\mathbf{y}'(s), p(s)q(s)\mathbf{f}(t,\mathbf{y}(s),p(s)\mathbf{y}'(s))\right\rangle,$$

where $w(t) = \|p(t)\mathbf{y}'(t)\|^2$. This implies that

$$\begin{aligned}\|p(t)\mathbf{y}'(t)\| &\le \frac{\Theta_2 + |\gamma|\|\mathbf{y}(T)\|}{\delta} + \int_t^T \frac{V_2}{2} w'(s) + W_2 p(s)q(s)\ ds \\ &\le \frac{\Theta_2 + |\gamma|\|\mathbf{y}(T)\|}{\delta} + \frac{V_2}{2}(w(T) - w(t)) + K_2 W_2\end{aligned}$$

for all $t \in [0,T]$. Thus, this gives the desired bound since $w \ge 0$ on $(0,T)$, that is

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \le \frac{\Theta_2 + |\gamma|R}{\delta} + \frac{V_2}{2}(\frac{\Theta_2 + |\gamma|R}{\delta})^2 + K_2 W_2. \qquad \square$$

Remark 2.13. If the inequality condition (2.27) is reversed with $\beta \neq 0$ in (1.2) or V_2 is replaced with $-V_2$ in (2.28) then it can be shown there exists an *a priori* bound on $\|p\mathbf{y}'\|$.

3. Existence Results

This section is devoted to providing existence results to the BVP (1.1), (1.2) and (1.3) using the previous theorems in Section 2. A key element compared to the previous section is the function $\mathbf{f}$ is only assumed to be continuous on the set Ω_R. The methods are based on topological transversality of Granas et al. [17]. In addition, the differential inequalities (2.1), (2.27), (2.28) introduced in the previous section to yield *a priori* bounds on solutions to (1.1) with various boundary conditions are relaxed. The main inequality (2.1) is weaken to the following;

$$\left\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\right\rangle + \|\mathbf{v}\|^2 \ge 0, \quad \text{when } \langle \mathbf{u},\mathbf{v}\rangle = 0 \text{ and } \|\mathbf{u}\| = R, \tag{3.1}$$

and the inequalities on $\mathbf{G}_i$'s are relaxed to

$$\langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s})\rangle \geq -\alpha\|\mathbf{r}\|^2, \quad \langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s})\rangle \leq \gamma\|\mathbf{r}\|^2 \quad \text{when} \quad \|\mathbf{r}\| > R. \tag{3.2}$$

By relaxing inequality (2.1), we can tackle problems with $f(t,y,py') = y(y^2 - 4)(y+2)+py'$ where as (2.1) fails for this choice of f. The only draw back is we need to introduce the original *Nagumo–Bernstein* condition instead of (2.19): let $\phi(s)$ be a positive continuous function where $0 \leq s < \infty$ satisfying $\|\mathbf{u}\| \leq R$,

$$\|\mathbf{u}\| \leq R, \quad \|p^2 q\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq \phi(\|\mathbf{v}\|) \quad \text{and} \quad \int^{\infty} \frac{x}{\phi(x)}\,dx = \infty. \tag{3.3}$$

The reason lies in the proofs when we consider a modified BVP of (1.1).

A general existence is presented next that will be used in the proceeding results. The theorem is abstract and assumes the existence of *a priori* bounds. Consider the problem:

$$L\mathbf{y} = p(t)\mathbf{f}(t,\mathbf{y},p\mathbf{y}') \tag{3.4}$$

where $L\mathbf{y} := \frac{1}{q(t)}\left((p(t)\mathbf{y}'))' + \kappa(t)p(t)\mathbf{y}' + r(t)p(t)\mathbf{y}\right)$, κ and r are continuous functions on $[0,T]$ with the general nonlinear boundary conditions

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = G_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)), \tag{3.5}$$

$$\gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = G_2(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)). \tag{3.6}$$

The constants $\alpha,\beta,\gamma,\delta$ have the restriction $\alpha^2+\beta^2 > 0$, $\gamma^2+\delta^2 > 0$ and the functions $G_i : \mathbb{R}^{2n} \to \mathbb{R}^n$ for $i = 1,2$ are continuous functions. Consider the following Banach spaces:

$$X := \{\mathbf{u} \in C^2((0,T);\mathbb{R}^n) \cap C([0,T];\mathbb{R}^n) \text{ and } p\mathbf{u}' \in C^1([0,T];\mathbb{R}^n)\} \tag{3.7}$$

and

$$X_{\mathfrak{B}} := \{\mathbf{u} \in X \ : \ -\alpha\mathbf{u}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{u}'(t) = \gamma\mathbf{u}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{u}'(t) = \mathbf{0}\} \tag{3.8}$$

which are equipped with the norm defined by

$$\|\mathbf{u}\|^* := \max\left\{ \max_{t\in[0,T]} \|\mathbf{u}(t)\|, \ \sup_{t\in(0,T)} \|p(t)\mathbf{u}'(t)\|, \ \sup_{t\in(0,T)} \left\| \frac{(p(t)\mathbf{u}'(t))'}{q(t)} \right\| \right\}. \tag{3.9}$$

Theorem 3.1. *Let* $\mathbf{f} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. If* $L : X_{\mathfrak{B}} \to C((0,T);\mathbb{R}^n)$ *is one–to–one and there are non–negative constants* R, S *independent of* $\lambda \in [0,1]$ *such that*

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R \quad \textit{and} \quad \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S$$

for all possible solutions $\mathbf{y} = \mathbf{y}(t)$ *to the problem*

$$L\mathbf{y} = \lambda p(t)\mathbf{f}(t,\mathbf{y},p(t)\mathbf{y}') \tag{3.10}$$

$$-\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \lambda G_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)), \tag{3.11}$$

$$\gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = \lambda G_2(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)). \tag{3.12}$$

then the nonlinear singular BVP (3.4)–(3.6) *has at least one solution.*

Proof. In light of the assumption

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R \quad \text{and} \quad \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S,$$

consider the compact set

$$\Omega_{R,S} := \{(t,\mathbf{y},p\mathbf{y}') \in [0,T] \times \mathbb{R}^{2n} \ : \ \max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R, \ \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S\}.$$

Since $p(t)\mathbf{f}(t,\mathbf{y},p\mathbf{y}')$ is continuous function then there is a non-negative constant M such that

$$\sup_{t\in(0,T)} \left\| \frac{(p(t)\mathbf{y}'(t))'}{q(t)} \right\| \leq \max \|p(t)\mathbf{f}(t,\mathbf{y},p\mathbf{y}') + \frac{1}{q(t)} \big(\kappa(t)p(t)\mathbf{y}' + r(t)p(t)\mathbf{y}\big) \| \leq M$$

where the maximum is computed over $\Omega_{R,S}$. This means that all possible solutions to (3.4)–(3.6) in X satisfying

$$\|\mathbf{y}\|^* < 1 + \max\{R,S,M\} =: \tilde{\mathrm{M}}.$$

Let

$$W_1(\mathbf{y}) := -\alpha\mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \quad W_2(\mathbf{y}) := \gamma\mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t),$$

$$V_i(\mathbf{y}) \quad := \quad G_i(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t), \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)), \quad \text{for} \quad i = 1, 2.$$

Consider the Banach space

$$Z := \{\mathbf{u} \in C([0,T];\mathbb{R}^n) \text{ and } p\mathbf{u}' \in C^1([0,T];\mathbb{R}^n)\}. \tag{3.13}$$

In the same nature as Granas et al. [17], define the continuous maps $\tilde{\mathrm{L}} : X \to C((0,T);\mathbb{R}^n) \times \mathbb{R}^{2n}$ and $F : Z \to C((0,T);\mathbb{R}^n) \times \mathbb{R}^{2n}$ by

$$\tilde{\mathrm{L}}\mathbf{u} = (L\mathbf{u}, W_1(\mathbf{u}), W_2(\mathbf{u}))$$

and

$$F\mathbf{v} = (p(t)\mathbf{f}(t, \mathbf{v}, p\mathbf{v}'), V_1(\mathbf{v}), V_2(\mathbf{v})).$$

Also, let $j : X \to Z$ be the natural completely continuous embedding map. See that the map $\tilde{\mathrm{L}}$ is linear, continuous and bounded and by the hypothesis is one–to–one, this implies $\tilde{\mathrm{L}}$ has a continuous, bounded inverse, $\tilde{\mathrm{L}}^{-1}$. Now, define the subspace U of X with norm (3.9) as

$$U := \{\mathbf{u} \in X \ : \ \|\mathbf{u}\|^* < \tilde{\mathrm{M}}\}.$$

Define the compact homotopy $H : \bar{U} \times [0,1] \to X$ by

$$H(\mathbf{u}, \lambda) := H_\lambda(\mathbf{u}) = \lambda\tilde{\mathrm{L}}^{-1}Fj(\mathbf{u}).$$

The verification of H being compact is a standard argument and can be found in [17], [5]. It is important to notice that the fixed points of H_λ are precisely the possible solutions to (3.10)–(3.12). Now see that on the set ∂U for all $\lambda[0,1]$, H_λ has no fixed points since a possible solution/fixed point satisfies $\|\mathbf{y}\|^* < M^*$, so there is no function $\mathbf{y} \in X$ with the property that $H_\lambda(\mathbf{y}) = \mathbf{y}$ and $\|\mathbf{y}^*\| = M^*$. This means that the compact homotopy H_λ is admissible. Now, see that the map $H_0 = \mathbf{0}$ is an essential map. This implies by the Topological Transversality theorem that $H_1 = \tilde{\mathrm{L}}^{-1}Fj$ is an essential map. This implies that the map H_1 has a fixed point $\mathbf{u} \in U$. Therefore, there is at least one solution to the nonlinear singular BVP (3.4)–(3.6). □

The previous result applies to non–invertible and invertible problems by being able to choose functions r, κ. If the BVP is invertible then the functions $r \equiv 0 \equiv \kappa$ are sufficient and this simplifies the map L. In addition, when dealing with just Sturm–Liouville boundary conditions ($\alpha \neq 0 \neq \gamma$) to secure invertibility of the map L the following condition is used:

$$K_1 \neq -\left[\frac{\beta}{\alpha} + \frac{\delta}{\gamma}\right]. \tag{3.14}$$

The next main existence result presented using the previous theorem, the relaxed differential inequality (3.1) and where $\mathbf{f}$ is assumed only to be continuous on Ω_R. Note that in all of the proceeding theorems, the boundary condition (3.2) may be replaced with the aforementioned condition (2.7) to yield the existence results.

Theorem 3.2. *Let $\mathbf{f} : \Omega_R \to \mathbb{R}^n$ be continuous where*

$$\Omega_R := \{(t, \mathbf{y}, \mathbf{v}) \in [0, T] \times \mathbb{R}^{2n} \ : \ \|\mathbf{y}\| \leq R, \ \mathbf{v} \in \mathbb{R}^n\}.$$

Let (1.4), (1.5), (2.8) *and* (3.14) *hold. If the inequalities* (3.1), (3.2) *and Theorem 2.7 with* (3.3) *are satisfied then there exists at least one solution to the BVP* (1.1) *with boundary conditions* (1.2) *and* (1.3).

Proof. Let $\epsilon > 0$ and $\tau := R\epsilon$. In view of (3.3), it is sufficient to assume that

$$\int_{\mu+2\tau}^{S} \frac{x}{\phi(x) + \tau} \, dx \geq \frac{\mu T}{2} + 2K + 4\tau, \tag{3.15}$$

where $\mu := \frac{2R(2M_0+M_1)+4K}{T}$ and $S \geq \mu + 2\tau$. Consider the family of modified BVPs;

$$\frac{1}{p}(p\mathbf{y}')' = \lambda q(\mathbf{f}^*(t, \mathbf{y}, p\mathbf{y}') + \epsilon \mathbf{y}) =: q\mathbf{g}_\lambda(t, \mathbf{y}, p\mathbf{y}'),$$
$$0 < t < T, \quad 0 < \lambda < 1; \tag{3.16}$$

with boundary conditions

$$-\alpha \mathbf{y}(0) + \beta \lim_{t \to 0^+} p(t)\mathbf{y}'(t) = \lambda \left(\mathbf{g}_1(\mathbf{y}(0), \lim_{t \to 0^+} p(t)\mathbf{y}'(t)) + \frac{\epsilon \mathbf{y}(0)}{R} \right), \tag{3.17}$$

$$\gamma \mathbf{y}(T) + \delta \lim_{t \to T^-} p(t)\mathbf{y}'(t) = \lambda \left(\mathbf{g}_2(\mathbf{y}(T), \lim_{t \to T^-} p(t)\mathbf{y}'(t)) - \frac{\epsilon \mathbf{y}(T)}{R} \right), \tag{3.18}$$

where

$$\mathbf{f}^*(t, \mathbf{u}, \mathbf{v}) := \begin{cases} \mathbf{f}(t, \mathbf{u}, \mathbf{v}), & \text{on } \ \Omega_R; \\[2ex] \dfrac{R}{\|\mathbf{u}\|} \mathbf{f}\left(t, \dfrac{R\mathbf{u}}{\|\mathbf{u}\|}, \mathbf{v}\right), & \text{for } \|\mathbf{u}\| > R. \end{cases} \tag{3.19}$$

in addition, notice that the function $\mathbf{g}$ is a continuous function on $[0.T] \times \mathbb{R}^{2n}$. The first component of the proof is to show that the constants R, S are independent of λ and ϵ such that every solution to BVP (3.16) satisfies

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R \quad \text{and} \quad \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S. \tag{3.20}$$

The second part follows naturally from the existence result, Theorem 3.1 and the *Arzela–Ascoli* theorem.
See that (3.1) implies that $\mathbf{g}$ satisfies condition (2.1) in Theorem 2.1 since

$$\begin{aligned} \left\langle \mathbf{u}, p^2 q \mathbf{g}_\lambda(t,\mathbf{u},\mathbf{v})\right\rangle + \|\mathbf{v}\|^2 &= \lambda(\left\langle \mathbf{u}, p^2 q \mathbf{f}^*(t,\mathbf{u},\mathbf{v})\right\rangle + \epsilon p^2 q\|\mathbf{u}\|^2) + \|\mathbf{v}\|^2 \\ &> \lambda\left(\left\langle \frac{R\mathbf{u}}{\|\mathbf{u}\|}, p^2 q \mathbf{f}(t, \frac{R\mathbf{u}}{\|\mathbf{u}\|}, \mathbf{v})\right\rangle + \|\mathbf{v}\|^2\right) > 0 \end{aligned}$$

when $\langle \mathbf{u}, \mathbf{v}\rangle = 0$ and $\|\mathbf{u}\| \geq R$. In addition, the boundary conditions satisfy (2.2) since

$$\left\langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s}) + \frac{\epsilon\mathbf{r}}{R}\right\rangle > \langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s})\rangle > -\alpha\|\mathbf{r}\|^2$$

and

$$\left\langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s}) - \frac{\epsilon\mathbf{r}}{R}\right\rangle < \langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s})\rangle < \gamma\|\mathbf{r}\|^2$$

when $\|\mathbf{r}\| > R$ and $\mathbf{v} \in \mathbb{R}^n$. Hence, Theorem 2.1 implies that all solutions to the BVP (3.16) with (3.17), (3.18) satisfy

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R.$$

To obtain the second part of (3.20), note that Theorem 2.7 is satisfied for (1.1). Therefore, it follows that Theorem 2.7 is satisfied for (3.16) with ρ, ϕ and K replaced by $\rho+\tau$, $\phi+\tau$ and $K+\tau$ respectively with condition (3.15). This implies that there is $S > 0$ such that (3.20) is true and independent of ϵ and λ.

To employ Theorem 3.1, it suffices to show that the map L is invertible. For brevity; in case the problem is non–invertible ($\alpha = 0 = \gamma$), let $L\mathbf{y} := \frac{1}{q(t)}\left((p(t)\mathbf{y}')\right)' - r(t)p(t)\mathbf{y})$ where $r(t) > 0$. It follows the only solution to $L\mathbf{y} = \mathbf{0}$ is the trivial solution, $\mathbf{y}(t) = \mathbf{0}$ for $t \in [0,T]$. To see this, suppose $\mathbf{y} \not\equiv \mathbf{0}$; this implies

$$\int_0^T p(t)r(t)\|\mathbf{y}(t)\|^2 + p(t)\|\mathbf{y}'(t)\|^2 \, dt > 0.$$

However, if $L\mathbf{y} = \mathbf{0}$ then this is false since

$$\int_0^T p(t)r(t)\|\mathbf{y}(t)\|^2 + p(t)\|\mathbf{y}'(t)\|^2 \, dt = \int_0^T \langle \mathbf{y}(t), (p(t)\mathbf{y}'(t))'\rangle + p(t)\|\mathbf{y}'(t)\|^2 \, dt$$

$$= \left\langle \mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t) \right\rangle - \left\langle \mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t) \right\rangle \leq 0.$$

If the problem is invertible ($\alpha \neq 0$ or $\gamma \neq 0$) then $r = 0$ on $[0, T]$ is sufficient and the conditions (2.8) and (3.14) imply $L\mathbf{y} = \mathbf{0}$ has only the trivial solution. By Theorem 3.1 there exists at least one solution $\mathbf{y} = \mathbf{y}_\epsilon(t)$ for each $\lambda \in [0, 1]$ to (3.16), (3.17) and (3.18) satisfying (3.20). The solutions we seek exist are in the scenario when $\lambda = 1$. Let $N := \max \|\mathbf{f}^*(t, \mathbf{y}, p\mathbf{y}')\| + R$ where the maximum is computed over Ω_R. Notice that the bounds on the solutions $\mathbf{y}_\epsilon(t)$ and their derivatives are independent of ϵ and satisfy (3.20). Now, see that the family of functions $\mathbf{y}_\epsilon(t)$, $p(t)\mathbf{y}'_\epsilon(t)$ are uniformly bounded and equicontinuous. By the *Arzela–Ascoli* theorem, there is a decreasing sequence $1 > \epsilon_1 > \epsilon_2 > \cdots$ such that $\epsilon_n \to 0$ as $n \to \infty$ yielding

$$\mathbf{y}(t) = \lim_{\epsilon\to 0} \mathbf{y}_\epsilon(t), \quad p(t)\mathbf{y}'(t) = \lim_{\epsilon\to 0} p(t)\mathbf{y}'_\epsilon(t)$$

for all $t \in [0, T]$ where $\mathbf{y} \in C([0, T]; \mathbb{R}^n)$, $p\mathbf{y}' \in C^1([0, T]; \mathbb{R}^n)$ and satisfying (3.20). Also, $\mathbf{y}_\epsilon$ satisfies the integral equation,

$$\mathbf{y}_\epsilon(t) = \mathbf{y}_\epsilon(0) + \mathbf{A}_\epsilon \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}^*(x, \mathbf{y}_\epsilon(x), p(x)\mathbf{y}'_\epsilon(x))\, dxds,$$

for all $t \in [0, T)$ where

$$\mathbf{A}_\epsilon := \frac{R\mathbf{g}_1(\mathbf{y}_\epsilon(0), \lim_{t\to 0^+} p(t)\mathbf{y}'_\epsilon(t)) + (\alpha R + \epsilon)\mathbf{y}_\epsilon(0)}{\beta R}.$$

For $t \in [0, T)$, $s \in [0, t]$ and $x \in [0, s]$, we have

$$\mathbf{f}^*(x, \mathbf{y}_\epsilon, p(x)\mathbf{y}'_\epsilon(x)) \to \mathbf{f}^*(x, \mathbf{y}, p(x)\mathbf{y}'(x))$$

and

$$\mathbf{g}_1(\mathbf{y}_\epsilon(0), \lim_{t\to 0^+} p(t)\mathbf{y}'_\epsilon(t)) \to \mathbf{g}_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t))$$

uniformly since $\mathbf{f}, \mathbf{g}_1$ are uniformly continuous on compact subsets of Ω_R. Thus letting $\epsilon \to 0$ obtains

$$\mathbf{y}(t) = \mathbf{y}(0) + \mathbf{A} \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_0^s p(x)q(x)\mathbf{f}^*(x, \mathbf{y}(x), p(x)\mathbf{y}'(x))\, dxds$$

for all $t \in [0, T)$ where

$$\mathbf{A} = \frac{\mathbf{g}_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t)) + \alpha\mathbf{y}(0)}{\beta}.$$

In addition, the identical ideas can be applied for $t \in (0,T]$ to show that the same $\mathbf{y}$ satisfies

$$\mathbf{y}(t) = \mathbf{y}(T) - \mathbf{B}\int_t^T \frac{ds}{p(s)} + \int_t^T \frac{1}{p(s)} \int_s^T p(x)q(x)\mathbf{f}^*(x,\mathbf{y}(x),p(x)\mathbf{y}'(x))\, dxds$$

where

$$\mathbf{B} = \frac{\mathbf{g}_2(\mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)) - \gamma\mathbf{y}(T)}{\delta}.$$

From the integral representation, we see that this particular function $\mathbf{y} \in C^2((0,T);\mathbb{R}^n)$ and is a solution to the BVP (3.16), (3.17) and (3.18) when $\lambda = 1$ and $\epsilon = 0$. We now see that all possible solutions to (3.16), (3.17), (3.18) when $\epsilon = 0$ are solutions to the following BVP since the solutions are bounded by (3.20) and (3.19),

$$(p(t)\mathbf{y}'(t))' = p(t)q(t)\mathbf{f}(t,\mathbf{y}(t),p(t)\mathbf{y}'(t)), \quad 0 < t < T \tag{3.21}$$

with (1.2), (1.3). This proves the result. □

Corollary 3.3. *Let $R > 0$, $f : [0,T] \times [-R,R] \times \mathbb{R} \to \mathbb{R}$ be continuous and* (1.4), (1.5), (3.14) *and* (2.8) *hold. If* (3.1), (3.2) *and Theorem 2.10 are satisfied with* (3.3) *when $n = 1$ then there exists at least one solution to* (1.2), (1.3).

Proof. Let $\epsilon > 0$ and $\nu := R\epsilon$. The proof is identical to the vector case, it is noteworthy to mention that in view of (3.3), it is just sufficient to have

$$\int_{\frac{2R}{\tau}}^{S} \frac{x}{\phi(x)+\nu}\, dx \geq 2R.$$ □

The next result uses the ideas of Theorem (2.12) where the inequalities (2.27) and (2.28) are relaxed to the following:

$$\langle \mathbf{v}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle \geq 0, \quad \text{for} \quad t \in (0,T),\ \|\mathbf{u}\| \leq R,\ \|\mathbf{v}\| = S, \tag{3.22}$$

or if there exists non–negative constants $V_2,\ W_2$ such that

$$\|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| \leq V_2 \langle \mathbf{v}, \mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + W_2,\ \text{for } t \in (0,T),\ \|\mathbf{u}\| \leq R,\ \mathbf{v} \in \mathbb{R}^n. \tag{3.23}$$

Theorem 3.4. *Let* $\mathbf{f} : \Omega_R \to \mathbb{R}^n$ *be continuous, let* (1.4), (1.5), (2.8), (3.2) *and* (3.14) *hold. If*

$$\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2 > 0, \quad \text{when } \langle \mathbf{u},\mathbf{v}\rangle = 0 \text{ and } \|\mathbf{u}\| = R, \tag{3.24}$$

and one of the following inequalities (3.22) *or* (3.23) *are satisfied then there exists at least one solution to the BVP* (1.1) *with boundary conditions* (1.2) *and* (1.3) *with* $\delta \neq 0$.

Proof. Consider the following family of modified BVP

$$\frac{1}{p}(p\mathbf{y}')' = q\lambda \mathbf{F}(t,\mathbf{y},p\mathbf{y}'), \quad 0 < t < T, \quad 0 < \lambda < 1, \tag{3.25}$$

$$\alpha \mathbf{y}(0) + \beta \lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{g}_1(\mathbf{y}(0), \lim_{t\to 0^+} p(t)\mathbf{y}'(t)) + \frac{\epsilon \mathbf{y}(0)}{R}, \tag{3.26}$$

$$\gamma \mathbf{y}(T) + \delta \lim_{t\to T^-} p(t)\mathbf{y}'(t) = \mathbf{g}_2(\mathbf{y}(T), \lim_{t\to T^-} p(t)\mathbf{y}'(t)) - \frac{\epsilon \mathbf{y}(T)}{R}, \quad \delta \neq 0 \tag{3.27}$$

where

$$\mathbf{F}(t,\mathbf{u},\mathbf{v}) := \begin{cases} \mathbf{f}^*(t,\mathbf{u},\mathbf{v}), & \text{for } \|\mathbf{v}\| \le S; \\ \dfrac{S}{\|\mathbf{v}\|}\mathbf{f}^*\left(t,\mathbf{u},\dfrac{S\mathbf{v}}{\|\mathbf{v}\|}\right), & \text{for } \|\mathbf{v}\| > S. \end{cases} \tag{3.28}$$

To prove that all solutions satisfy $\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le R$, it suffices to show that $\mathbf{F}$ satisfies the condition (2.1) in Theorem 2.1. In the proof of Theorem 2.1, it was proved that

$$\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2 > 0, \text{ when } \langle \mathbf{u},\mathbf{v}\rangle = 0 \quad \text{and} \quad \|\mathbf{u}\| = R. \tag{3.29}$$

For the case, when $t \in (0,T)$ and $\|\mathbf{v}\| \le S$, we have $\lambda \mathbf{F}(\mathbf{t},\mathbf{u},\mathbf{v}) = \lambda \mathbf{f}^*(t,\mathbf{u},\mathbf{v})$, so by (3.29)

$$\begin{aligned}\langle \mathbf{u}, p^2(t)q(t)\lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2 \\ = \lambda\left(\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}^*(t,\mathbf{u},\mathbf{v})\rangle + \|\mathbf{v}\|^2\right) + \|\mathbf{v}\|^2(1-\lambda) > 0\end{aligned}$$

and thus satisfies (2.1). For the other case, when $t \in (0,T)$ and $\|\mathbf{v}\| > S$,

$$\lambda \mathbf{F}(\mathbf{t},\mathbf{u},\mathbf{v}) = \frac{\lambda S}{\|\mathbf{v}\|}\mathbf{f}^*(t,\mathbf{u},\frac{S\mathbf{v}}{\|\mathbf{v}\|}).$$

Let $\mathbf{z} = S\mathbf{v}/\|\mathbf{v}\|$ and by using (3.29), we have

$$\begin{aligned}&\left\langle \mathbf{u}, p^2(t)q(t)\lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\right\rangle + \|\mathbf{v}\|^2 \\ &\qquad = \frac{\lambda S}{\|\mathbf{v}\|}\left(\left\langle \mathbf{u}, p^2(t)q(t)\mathbf{f}^*(t,\mathbf{u},\mathbf{z})\right\rangle + \|\mathbf{z}\|^2\right) + \frac{\|\mathbf{v}\|^3 - \lambda S^3}{\|\mathbf{v}\|} > 0\end{aligned}$$

for $\|\mathbf{v}\| > S$. In addition, the boundary conditions satisfy (2.2) since

$$\left\langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s}) + \frac{\epsilon\mathbf{r}}{R}\right\rangle > \langle \mathbf{r}, \mathbf{g}_1(\mathbf{r},\mathbf{s})\rangle > -\alpha\|\mathbf{r}\|^2$$

and

$$\left\langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s}) - \frac{\epsilon\mathbf{r}}{R}\right\rangle < \langle \mathbf{r}, \mathbf{g}_2(\mathbf{r},\mathbf{s})\rangle < \gamma\|\mathbf{r}\|^2$$

when $\|\mathbf{r}\| > R$ and $\mathbf{v} \in \mathbb{R}^n$. Thus, the condition (2.1) holds for $\lambda\mathbf{F}$ and hence Theorem 2.1 yields all possible solutions to (3.25)-(3.27) satisfy

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \le R.$$

To prove that all solutions to (3.25)-(3.27) satisfy

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \le S,$$

suffices to show that (2.27) or (2.28) holds for $\lambda\mathbf{F}$. To show that (2.27) is satisfied then it suffices to prove the modified function (3.28) satisfies

$$\langle \mathbf{v}, \lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\rangle \ge 0, \quad \text{for} \quad t\in[0,T],\ \|\mathbf{u}\|\le R,\ \|\mathbf{v}\|\ge S. \tag{3.30}$$

Suppose $t\in(0,T)$, $\|\mathbf{u}\|\le R$ and $\|\mathbf{v}\|\ge S$ then by (3.19) and (3.22),

$$\langle \mathbf{v}, \lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\rangle = \lambda\left\langle \mathbf{v}, \frac{S}{\|\mathbf{v}\|}\mathbf{f}^*(t,\mathbf{u},\frac{S\mathbf{v}}{\|\mathbf{v}\|})\right\rangle = \lambda\left\langle \frac{S\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{f}(t,\mathbf{u},\frac{S\mathbf{v}}{\|\mathbf{v}\|})\right\rangle \ge 0.$$

Thus, $\lambda\mathbf{F}$ satisfies (2.27). To show the other inequality (2.28) is satisfied; see that when $t\in[0,T]$, $\|\mathbf{u}\|\le R$, $\|\mathbf{v}\|\le S$, the function $\mathbf{F}\equiv\mathbf{f}$ and so $\lambda\mathbf{F}$ satisfies (2.28) since

$$\begin{aligned}\|\lambda\mathbf{F}(\mathbf{t},\mathbf{u},\mathbf{v})\| &\le V_2\langle \mathbf{v}, p(t)q(t)\lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\rangle + \lambda W_2 \\ &\le V_2\langle \mathbf{v}, p(t)q(t)\lambda\mathbf{F}(t,\mathbf{u},\mathbf{v})\rangle + W_2.\end{aligned}$$

Similarly, suppose $t\in(0,T)$, $\|\mathbf{u}\|\le R$ and $\|\mathbf{v}\|\ge S$ then by (3.19) and (3.23),

$$V_2\langle \mathbf{v}, \lambda\mathbf{F}(\mathbf{t},\mathbf{u},\mathbf{v})\rangle + W_2 \quad \ge \quad \lambda\left(V_2\left\langle \frac{S\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{f}(t,\mathbf{u},\frac{S\mathbf{v}}{\|\mathbf{v}\|})\right\rangle + W_2\right)$$

$$\geq \|\lambda \mathbf{f}(t, \mathbf{u}, \frac{S\mathbf{v}}{\|\mathbf{v}\|})\|$$
$$= \frac{\|\mathbf{v}\|}{S}\|\lambda \mathbf{F}(t, \mathbf{u}, \mathbf{v})\| \geq \|\lambda \mathbf{F}(t, \mathbf{u}, \mathbf{v})\|.$$

This implies

$$\|\mathbf{F}(t, \mathbf{u}, \mathbf{v})\| \leq V_2 \langle \mathbf{v}, \mathbf{F}(t, \mathbf{u}, \mathbf{v})\rangle + W_2, \quad \text{for} \quad t \in (0, T), \ \|\mathbf{u}\| \leq R.$$

Furthermore, by the same approach; it follows from the inequality that

$$\|\mathbf{F}(t, \mathbf{u}, \mathbf{v})\| \leq V_2 \langle \mathbf{v}, \mathbf{F}(t, \mathbf{u}, \mathbf{v})\rangle + W_2, \quad \text{for} \quad t \in (0, T). \tag{3.31}$$

Thus, Theorem 2.12 implies that all solutions to (3.25)-(3.27) satisfy

$$\sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S.$$

By choosing the same invertible map as in the proof of Theorem 3.2, we now apply Theorem 3.1 to (3.25)-(3.27). This proves that there exists at least one solution to (3.25)-(3.27) for each $\lambda \in [0, 1]$ and each $\epsilon > 0$ satisfying *a priori* bounds

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R, \qquad \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S. \tag{3.32}$$

For each $\lambda \in [0, 1]$, we see that each solution to (3.25) − (3.27) is a solution to

$$(p\mathbf{y}')' = pq\lambda \mathbf{f}(t, \mathbf{y}, p\mathbf{y}'), \quad 0 < t < T, \quad 0 \leq \lambda \leq 1, \tag{3.33}$$

with (3.26), (3.27). By taking the scenario when $\lambda = 1$, there exists at least one solution to (3.25)–(3.27) when $\lambda = 1$ and for each $\epsilon > 0$ satisfying (3.32). Employing the same argument as the previous theorem, the *Arzela–Ascoli* theorem implies that there is a solution to (3.25)–(3.27) as $\epsilon \to 0$ when $\lambda = 1$. This yields that there is at least one solution to (1.1)–(1.3) with $\delta \neq 0$ satisfying (3.32). □

The final existence result accompanies the *a priori* results to Theorems 2.4, 2.6 and Corollary 2.5. The condition (3.35) gives a value for R such that the set Ω_R is a sufficient set to yield existence of solutions. If $\mathbf{f}$ is continuous everywhere on $[0, T] \times \mathbb{R}^{2n}$ then the condition (3.35) becomes redundant.

Theorem 3.5. *Let* $\mathbf{f} : \Omega_R \to \mathbb{R}^n$ *be continuous, let* (1.4), (1.5), $p^2 q \leq M$ *on* $[0, T]$, (2.14), (2.8) *and* (3.14). *Suppose* $\mathbf{g}_1 \equiv \mathbf{c}$, $\mathbf{g}_2 \equiv \mathbf{d}$ *where* $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$. *If there are constants* V, W *such that*

$$\|\mathbf{f}(t, \mathbf{u}, \mathbf{v})\| \leq 2V(\langle \mathbf{u}, \mathbf{f}(t, \mathbf{u}, \mathbf{v})\rangle + \|\mathbf{v}\|^2) + W, \quad \text{for} \quad (t, \mathbf{u}, \mathbf{v}) \in \Omega_R \tag{3.34}$$

and

$$R \geq \frac{\|\mathbf{c}\|}{|\alpha|} + \left(\left|\frac{\beta}{\alpha}\right| + K_1\right)\left(\frac{\|(\gamma\mathbf{c} + \alpha\mathbf{d}\| + |\gamma|(|\alpha|K_1 + |\beta|)\eta}{\left|\alpha(\gamma\int_0^T \frac{ds}{p(s)} + \delta) + \gamma\beta\right|} + \eta\right) \tag{3.35}$$

where $\eta := 2VN + WK_2$ *are satisfied then the singular BVP* (1.1), (1.2), (1.3) *with* $\alpha \neq 0$ *has at least one solution satisfying* (2.15) *and* (2.16).

Proof. The proof of this result is similar to the previous existence results. The aim is to prove that all solutions to the family of BVPs,

$$(p(t)\mathbf{y}'(t))' = p(t)q(t)\lambda\mathbf{f}^*(t, \mathbf{y}(t), p(t)\mathbf{y}'(t)), \quad 0 < t < T, \quad 0 < \lambda < 1, \tag{3.36}$$

with (1.2), (1.3) satisfy

$$\max_{t\in[0,T]} \|\mathbf{y}(t)\| \leq R, \qquad \sup_{t\in(0,T)} \|p(t)\mathbf{y}'(t)\| \leq S. \tag{3.37}$$

If this is true then by Theorem 3.1, we have that there exists at least one solution to (3.36), (1.2), (1.3). By Theorem 2.6, it just suffices to show that (2.9) is satisfied for $\lambda\mathbf{f}^*$. Since (2.9) holds for $\mathbf{f}$ then we have for $(t, \mathbf{u}, \mathbf{v}) \in \Omega_R$,

$$\begin{aligned}\|\lambda\mathbf{f}(t, \mathbf{u}, \mathbf{v})\| &\leq \lambda\left[2V\left(\langle \mathbf{u}, \mathbf{f}(t, \mathbf{u}, \mathbf{v})\rangle + \frac{\|\mathbf{v}\|^2}{M}\right) + W\right] \\ &\leq 2V\left(\langle \mathbf{u}, \lambda\mathbf{f}(t, \mathbf{u}, \mathbf{v})\rangle + \frac{\|\mathbf{v}\|^2}{M}\right) + W.\end{aligned}$$

In addition, when $\|\mathbf{u}\| > R$,

$$\begin{aligned}2V(\langle \mathbf{u}, \lambda\mathbf{f}^*(t, \mathbf{u}, \mathbf{v})\rangle + \|\mathbf{v}\|^2) + W =& 2V\left(\left\langle \frac{R\mathbf{u}}{\|\mathbf{u}\|}, \lambda\mathbf{f}(t, \frac{R\mathbf{u}}{\|\mathbf{u}\|}, \mathbf{v})\right\rangle + \frac{\|\mathbf{v}\|^2}{M}\right) + W \\ \geq& 2V\left(\left\langle \frac{R\mathbf{u}}{\|\mathbf{u}\|}, \lambda\mathbf{f}(t, \frac{R\mathbf{u}}{\|\mathbf{u}\|}, \mathbf{v})\right\rangle + \frac{\|\mathbf{v}\|^2}{M}\right) + W \\ \geq& \|\lambda\mathbf{f}^*(t, \mathbf{u}, \mathbf{v})\|.\end{aligned}$$

Note that in the inequality above, we used $2V\|\mathbf{v}\|^2(1-\lambda) \geq 0$ for $\mathbf{v} \in \mathbb{R}^n$, $V \geq 0$ and $\lambda \in [0,1]$. Thus, Theorem 2.6 proves that all solutions (3.36), (1.2), (1.3) satisfy (3.37) with R, S independent of λ. By using the Theorem 3.1 where the map $L\mathbf{y} := \frac{(p\mathbf{y}')'}{q(t)}$ now proves that there exists at least one solution to (3.36), (1.2), (1.3) for each $\lambda \in [0,1]$. Note that the map L is one–to–one since the only solution to $L\mathbf{y} = \mathbf{0}$ is $\mathbf{y} = \mathbf{0}$. This is true by directly solving $L\mathbf{y} = \mathbf{0}$ with conditions (2.8) and (3.14). Considering the scenario when $\lambda = 1$ then implies that the original BVP, (1.1), (1.2), (1.3) has at least one solution satisfying (2.15) and (2.16). □

4. An Example

We conclude the paper with an example that exhibits all the properties in the introduction.

Example 4.1. Consider the following singular system;

$$(\sqrt{t}\mathbf{y}')' = \frac{1}{\sqrt{t}}\left(\frac{y_1 e^{y_2}}{2-y_1}, \frac{y_2 t y_2'^2}{2-y_1}\right)^T, \tag{4.1}$$

$$\begin{aligned} &\lim_{t\to 0^+} p(t)\mathbf{y}'(t) = \mathbf{0}, \\ &\mathbf{y}(2) + \lim_{t\to 2^-} p(t)\mathbf{y}'(t) = \begin{pmatrix} y_1(2)e^{-[y_2(2)]^2} \\ -[y_2(2)]^3[\lim_{t\to 2^-} p(t)y_2'(t)]^2 \end{pmatrix}. \end{aligned} \tag{4.2}$$

In (4.1), we have $p(t) = \sqrt{t}$, $q(t) = 1/t$,

$$\mathbf{f}(t, u_1, u_2, v_1, v_2) = \left(\frac{u_1 e^{u_2}}{2-u_1}, \frac{u_2 v_2^2}{2-u_1}\right) \quad for \quad (t, u_1, u_2, v_1, v_2) \in [0,2] \times \mathbb{R}^4$$

and

$$\mathbf{g}_1(\mathbf{r}, \mathbf{s}) = \mathbf{0}, \quad \mathbf{g}_2(\mathbf{r}, \mathbf{s}) = (r_1 e^{-[r_2]^2}, -[r_2]^3[s_2]^2).$$

The objective is to show that the BVP (4.1), (4.2) has at least one solution. The boundary conditions indicate the applicability of Theorem 3.2. See that the boundary conditions satisfy (3.2) since $\langle \mathbf{r}, \mathbf{g}_1(\mathbf{r}, \mathbf{s})\rangle = 0$ and

$$\langle \mathbf{r}, \mathbf{g}_2(\mathbf{r}, \mathbf{s})\rangle = [r_1]^2 e^{-[r_2]^2} - [r_2]^4[s_2]^2 \leq [r_1]^2 \leq \|\mathbf{r}\|^2, \quad \|\mathbf{r}\| > 0.$$

We now see that the conditions (1.4), (1.5) and (2.8) are satisfied with $K_1 = 2\sqrt{2} = K_2$. To show that (3.1) is satisfied, choose $R = \sqrt{3}$ and see that when $\|\mathbf{y}\| = \sqrt{3}$,

$$\begin{aligned}\langle \mathbf{y}, p^2 q\mathbf{f}(t,\mathbf{y},p\mathbf{y}')\rangle + \|p\mathbf{y}'\|^2 &= \left[\frac{y_1^2 e^{y_2}}{2-y_1} + \frac{y_2^2 t y_2'^2}{2-y_1}\right] + t(y_1'^2 + y_2'^2)\\ &= \left[\frac{y_1^2(e^{y_2} + t y_1'^2)}{2-y_1}\right] + t(y_1'^2 + y_2'^2) \geq 0.\end{aligned}$$

This implies that all solutions are bounded by the constant $R = \sqrt{3}$. Next, let V, W be non–negative constants and choose the function

$$r(t) := V\|\mathbf{y}(t)\|^2 + W\int_0^t \frac{1}{p(s)}\int_0^s p(x)q(x)\ dxds.$$

Furthermore, if we choose $V = \frac{1}{2}$ and $W = \frac{e^{\sqrt{3}}}{4(2-\sqrt{3})}$ then (2.18) is satisfied since

$$\begin{aligned}\|(p\mathbf{y}')'\| = \|\frac{1}{\sqrt{t}}\mathbf{f}(t,\mathbf{y},p\mathbf{y}')\| &= \frac{1}{\sqrt{t}}\left\|\left(\frac{y_1 e^{y_2}}{2-y_1}, \frac{y_2 t y_2'^2}{2-y_1}\right)\right\|\\ &\leq \frac{1}{\sqrt{t}}\left[\frac{|y_1|e^{y_2}}{|2-y_1|} + \frac{|y_2| t y_2'^2}{|2-u_1|}\right]\\ &\leq \frac{1}{\sqrt{t}}\left[\frac{|y_1|^2 e^{y_2}}{|2-y_1|} + \frac{e^{y_2}}{4|2-y_1|} + \frac{|y_2|^2 t y_2'^2}{|2-y_1|} + \frac{t y_2'^2}{4|2-y_1|}\right]\\ &\leq \frac{1}{\sqrt{t}}\left[\frac{y_1^2 e^{y_2}}{2-y_1} + \frac{y_2^2 y_2'^2}{2-y_1} + \frac{e^{\sqrt{3}}}{4|2-\sqrt{3}|} + t y_1'^2 + t y_2'^2\right]\\ &= \left\langle \mathbf{y}, \frac{1}{\sqrt{t}}\mathbf{f}(t,\mathbf{y},p\mathbf{y}')\right\rangle + \sqrt{t}\|\mathbf{y}'\|^2 + \frac{W}{\sqrt{t}}\\ &\leq 2V(\langle \mathbf{y}, (p\mathbf{y}')'\rangle + p\|\mathbf{y}'\|^2) + Wpq = (pr')'(t).\end{aligned}$$

In addition, notice that for $\|\mathbf{u}\| \leq \sqrt{3}$,

$$\begin{aligned}\|p^2(t)q(t)\mathbf{f}(t,\mathbf{u},\mathbf{v})\| = \|\mathbf{f}(t,\mathbf{u},\mathbf{v})\| &\leq \left|\frac{u_1}{2-u_1}\right| + \left|\frac{u_2 v_2^2}{2-u_1}\right|\\ &\leq \frac{\sqrt{3}}{|2-u_1|}(1+v_2^2) \leq \phi(\|\mathbf{v}\|)\end{aligned}$$

where $\phi(\|\mathbf{v}\|) = \frac{\sqrt{3}}{|2-u_1|}(\|\mathbf{v}\|^2+1)$. This choice of ϕ satisfies the integral condition (2.19) for some $S > 0$ and thus Theorem 2.7 is satisfied. See that $\mathbf{f}$ is continuous on the set

$$\Omega_R := \{(t,\mathbf{u},\mathbf{v}) \in [0,2] \times \mathbb{R}^2 \times \mathbb{R}^2 \mid \|\mathbf{u}\| \leq \sqrt{3}\}.$$

All the conditions of Theorem 3.2 are satisfied and thus there exists at least one solution to (4.1), (4.2).

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